MTH

GS-2016 (Mathematics)

TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Written Test in **MATHEMATICS - December 13, 2015**
For the Ph.D. Programs at TIFR (Mumbai, and CAM and ICTS, Bangalore)
and for the Int. Ph.D. Programs at TIFR (CAM, Bangalore and Mumbai)

Duration : Two hours (2 hours)

Name : ____ Ref. Code : ____________

Please read all instructions carefully before you attempt the questions.

1. Please fill in details about name, reference code etc. on the answer sheet. The Answer Sheet is machine-readable. Use only Black/Blue ball point pen to fill in the answer sheet.

2. There are thirty (30) multiple choice questions divided into two parts. Part I consists of 20 questions and Part II consists of 10 questions.

3. Bachelors students who have applied only for the Integrated Ph.D. program at TIFR CAM, Bangalore will only be evaluated on Part I. All other students (including Bachelors students applying for the Integrated Ph.D. programs at TIFR, Mumbai) will be evaluated on both Parts I and II.

4. Indicate your answer ON THE ANSWER SHEET by blackening the appropriate circle for each question. **Each corect answer will get 1 mark. There is no negative marking for wrong answers. A question not answered will not get you any mark.** Do not mark more than one circle for any question : this will be treated as a wrong answer.

5. We advise you to first mark the correct answers on the QUESTION PAPER and then to TRANSFER these to the ANSWER SHEET only when you are sure of your choice.

6. Rough work may be done on blank pages of the question paper. If needed, you may ask for extra rough sheets from an invigilator.

7. **Use of calculators, mobile phones, laptops, tablets (or other electronic devices, including those connecting to the internet) is NOT permitted.**

8. Do NOT ask for clarifications from the invigilators regarding the questions. They have been instructed not to respond to any such inquiries from candidates. In case a correction/clarification is deemed necessary, the invigilators will announce it publicly.

9. Notation and Conventions used in this test are given on page 2 of the question paper.

NOTATION AND CONVENTIONS

$\mathbb{N}$:= Set of natural numbers = $\{1, 2, 3, \ldots\}$
$\mathbb{Z}$:= Set of integers
$\mathbb{Q}$:= Set of rational numbers
$\mathbb{R}$:= Set of real numbers
$\mathbb{C}$:= Set of complex numbers
$\mathbb{R}^n$:= n-dimensional vector space over $\mathbb{R}$

$(a, b) := \{x \in \mathbb{R} | a < x < b\}$
$(a, b] := \{x \in \mathbb{R} | a < x \leq b\}$
$[a, b) := \{x \in \mathbb{R} | a \leq x < b\}$
$[a, b] := \{x \in \mathbb{R} | a \leq x \leq b\}$

A sequence is always indexed by the set of natural numbers.
The cyclic group with n elements is denoted by $\mathbb{Z}/n\mathbb{Z}$.
Unless stated otherwise, subsets of $\mathbb{R}^n$ carry the induced topology.
For any set S, the cardinality of S is denoted by $|S|$.

Part I

1. The value of the product $\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots\right)\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots\right)$ is

   A. 1
   B. e^2
   C. 0
   D. $\log_e 2$.

2. Which of the following is false ?

   A. $\sum_{n=1}^{\infty} \sin\dfrac{1}{n}$ diverges
   B. $\sum_{n=1}^{\infty} \sin\dfrac{1}{n^2}$ converges
   C. $\sum_{n=1}^{\infty} \cos\dfrac{1}{n}$ diverges
   D. $\sum_{n=1}^{\infty} \cos\dfrac{1}{n^2}$ converges.

3. The value of the series $\sum_{n=1}^{\infty} \dfrac{n}{2^n}$ is

   A. 1
   B. 2
   C. 3
   D. 4.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = \dfrac{\sin x}{|x| + \cos x}$. Then

   A. f is differentiable at all $x \in \mathbb{R}$
   B. f is not differentiable at $x = 0$
   C. f is differentiable at $x = 0$ but f' is not continuous at $x = 0$
   D. f is not differentiable at $x = \frac{\pi}{2}$.

5. Which of the following continuous functions $f : (0, \infty) \to \mathbb{R}$ can be extended to a continuous function on $[0, \infty)$?

A. $f(x) = \sin \frac{1}{x}$
B. $f(x) = \frac{1 - \cos x}{x^2}$
C. $f(x) = \cos \frac{1}{x}$
D. $f(x) = \frac{1}{x}$.

6. Let V be the vector space over $\mathbb{R}$ consisting of polynomials $p(t)$ over $\mathbb{R}$ of degree less than or equal to 4. Let $D : V \to V$ be the linear operator that takes any polynomial $p(t)$ to its derivative $p'(t)$. Then the characteristic polynomial $f(x)$ of D is

A. x^4
B. x^5
C. $x^3(x - 1)$
D. $x^4(x - 1)$.

7. Let $A = \{\sum_{i=1}^{\infty} \frac{a_i}{5^i} : a_i = 0, 1, 2, 3 \text{ or } 4\} \subset \mathbb{R}$. Then

A. A is a finite set
B. A is countably infinite
C. A is uncountable but does not contain an open interval
D. A contains an open interval.

8. The number of group homomorphisms from $\mathbb{Z}/20\mathbb{Z}$ to $\mathbb{Z}/29\mathbb{Z}$ is

A. 1
B. 20
C. 29
D. 580.

9. Let $p(x)$ be a polynomial of degree 3 with real coefficients. Which of the following is possible ?

   A. $p(x)$ has no real roots
   B. $p(x)$ has exactly 2 real roots
   C. $p(1) = -1$, $p(2) = 1$, $p(3) = 11$ and $p(4) = 35$
   D. $i - 1$ and $i + 1$ are roots of $p(x)$, where i is the square root of -1

10. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of real numbers such that the series $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converge. Then the series $\sum_{n=1}^{\infty} a_n b_n$

   A. is absolutely convergent
   B. may not converge
   C. is always convergent, but may not converge absolutely
   D. converges to 0.

11. Let $v_i = (v_i^{(1)}, v_i^{(2)}, v_i^{(3)}, v_i^{(4)})$, for $i = 1, 2, 3, 4$, be four vectors in $\mathbb{R}^4$ such that $\sum_{i=1}^{4} v_i^{(j)} = 0$, for each $j = 1, 2, 3, 4$. Let W be the subspace of $\mathbb{R}^4$ spanned by $\{v_1, v_2, v_3, v_4\}$. Then the dimension of W over $\mathbb{R}$ is always

   A. either equal to 1 or equal to 4
   B. less than or equal to 3
   C. greater than or equal to 2
   D. either equal to 0 or equal to 4.

12. Let A be a subset of $[0, 1]$ with non-empty interior, and let $\mathbb{Q} + A = \{q + a : q \in \mathbb{Q},\ a \in A\}$. Which of the following is true ?

   A. $\mathbb{Q} + A = \mathbb{R}$
   B. $\mathbb{Q} + A$ can be a proper subset of $\mathbb{R}$
   C. $\mathbb{Q} + A$ need not be closed in $\mathbb{R}$
   D. $\mathbb{Q} + A$ need not be open in $\mathbb{R}$.

13. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that $|f(x) - f(y)| \geq |x - y|$, for all $x, y \in \mathbb{R}$. Then the equation $f'(x) = \frac{1}{2}$

A. has exactly one solution
B. has no solution
C. has a countably infinite number of solutions
D. has uncountably many solutions.

14. Let $f : \mathbb{R} \to [0, \infty)$ be a continuous function such that $g(x) = (f(x))^2$ is uniformly continuous. Which of the following statements is always true ?

A. f is bounded
B. f may not be uniformly continuous
C. f is uniformly continuous
D. f is unbounded.

15. Which of the following sequences of functions $\{f_n\}_{n=1}^{\infty}$ converges uniformly ?

A. $f_n(x) = x^n$ on $[0, 1]$
B. $f_n(x) = 1 - x^n$ on $[\frac{1}{2}, 1]$
C. $f_n(x) = \dfrac{1}{1 + nx^2}$ on $[0, \frac{1}{2}]$
D. $f_n(x) = \dfrac{1}{1 + nx^2}$ on $[\frac{1}{2}, 1]$.

16. Let S be a collection of subsets of $\{1, 2, \ldots, 100\}$ such that the intersection of any two sets in S is non-empty. What is the maximum possible cardinality $|S|$ of S ?

A. 100
B. 2^{100}
C. 2^{99}
D. 2^{98}.

17. Let S be the set of all 3×3 matrices A with integer entries such that the product AA^t is the identity matrix. Here A^t denotes the transpose of A. Then $|S| =$

A. 12
B. 24
C. 48
D. 60.

18. Let A be a 3×3 matrix with integer entries such that $\det(A) = 1$. What is the maximum possible number of entries of A that are even ?

A. 2
B. 3
C. 6
D. 8.

19. The limit

$$\lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} \right) =$$

A. e
B. 2
C. $\log_e 2$
D. e^2.

20. Let $G = \mathbb{Z}/100\mathbb{Z}$ and let $S = \{h \in G : \text{Order}(h) = 50\}$. Then $|S|$ equals

A. 20
B. 25
C. 30
D. 50.

Part II

21. Let $A_1 \supset A_2 \supset \cdots \supset A_n \supset A_{n+1} \supset \cdots$ be an infinite sequence of non-empty subsets of $\mathbb{R}^3$. Which of the following conditions ensures that their intersection is non-empty ?

    A. Each A_i is uncountable
    B. Each A_i is open
    C. Each A_i is connected
    D. Each A_i is compact. ✓

22. Let (X, d) be a metric space. Which of the following is possible ?

    A. X has exactly 3 dense subsets
    B. X has exactly 4 dense subsets ✓
    C. X has exactly 5 dense subsets
    D. X has exactly 6 dense subsets.

23. Let $\{f_n\}_{n=1}^{\infty}$ be the sequence of functions on $\mathbb{R}$ defined by $f_n(x) = n^2 x^n$. Let A be the set of all points a in $\mathbb{R}$ such that the sequence $\{f_n(a)\}_{n=1}^{\infty}$ converges. Then

    A. $A = \{0\}$
    B. $A = [0, 1)$
    C. $A = \mathbb{R} \setminus \{-1, 1\}$
    D. $A = (-1, 1)$. ✓

24. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(i) = 0$, for all $i \in \mathbb{Z}$. Which of the following statements is always true ?

A. Image(f) is closed in $\mathbb{R}$
B. Image(f) is open in $\mathbb{R}$
C. f is uniformly continuous
D. None of the above. ✔

25. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. Which of the following is false ? Any continuous function from S^1 to $\mathbb{R}$

A. is bounded
B. is uniformly continuous
C. has image containing a non-empty open subset of $\mathbb{R}$ ✔
D. has a point $z \in S^1$ such that $f(z) = f(-z)$.

26. Which of the following is false ?

A. Any continuous function from $[0, 1]$ to $[0, 1]$ has a fixed point
B. Any homeomorphism from $[0, 1)$ to $[0, 1)$ has a fixed point
C. Any bounded continuous function from $[0, \infty)$ to $[0, \infty)$ has a fixed point
D. Any continuous function from $(0, 1)$ to $(0, 1)$ has a fixed point. ✔

27. For $n \geq 1$, let S_n denote the group of all permutations on n symbols. Which of the following statements is true ?

A. S_3 has an element of order 4
B. S_4 has an element of order 6
C. S_4 has an element of order 5
D. S_5 has an element of order 6. ✔

28. Which of the following rings is an integral domain ?

A. $\mathbb{R}[x]/(x^2 + x + 1)$
B. $\mathbb{R}[x]/(x^2 + 5x + 6)$
C. $\mathbb{R}[x]/(x^3 - 2)$
D. $\mathbb{R}[x]/(x^7 + 1)$.

29. Let $f : \mathbb{R} \to (0, \infty)$ be a twice differentiable function such that $f(0) = 1$ and $\int_a^b f(x)\ dx = \int_a^b f'(x)\ dx$, for all a, $b \in \mathbb{R}$, with $a \leq b$. Which of the following statements is false ?

A. f is one to one
B. The image of f is compact
C. f is unbounded
D. There is only one such function.

30. For $X \subset \mathbb{R}^n$, consider X as a metric space with metric induced by the usual Euclidean metric on $\mathbb{R}^n$. Which of the following metric spaces X is complete?

A. $X = \mathbb{Z} \times \mathbb{Z} \subset \mathbb{R} \times \mathbb{R}$
B. $X = \mathbb{Q} \times \mathbb{R} \subset \mathbb{R} \times \mathbb{R}$
C. $X = (-\pi, \pi) \cap \mathbb{Q} \subset \mathbb{R}$
D. $X = [-\pi, \pi] \cap (\mathbb{R} \setminus \mathbb{Q}) \subset \mathbb{R}$.