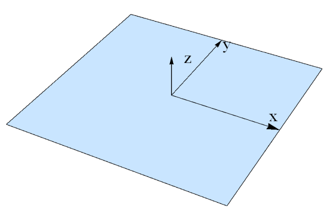
$\boxed{\text{Calculus II (part 6a): Vectors and 3D Geometry}}$ (by Evan Dummit, 2012, v. 1.50)

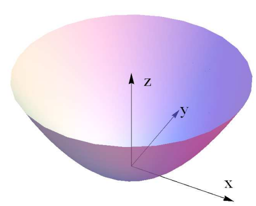
1 Vectors and 3-Dimensional Geometry

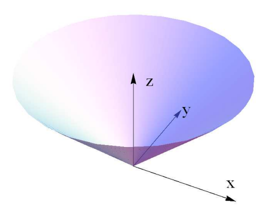
- We are used to graphing curves in the plane – given a function $f(x)$, we know how to produce the graph of the curve $y = f(x)$. We also know how to graph parametrically-defined functions, like the cycloid $x = t + \sin(t)$, $y = 1 + \cos(t)$. And we even know, more or less, how to graph implicitly-defined functions, like the circle $x^2 + y^2 = 1$.
- Now we will talk about how to graph functions in three-dimensional space, which is often called 3-space for short. We will then discuss some features of 3-dimensional geometry and introduce vectors, which clarify a great deal of the concepts.

1.1 Surfaces in 3-Space

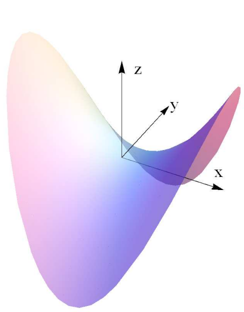
- Points in 3-space are represented by a triplet of numbers (x, y, z).
- We have a distance formula, which is just the Pythagorean Theorem applied twice, which says that the distance between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.
- The simplest type of functions to graph is one of the form $z = f(x, y)$, where now f is a function of the two variables x and y. At the point (x, y) in the plane, this graph has the height $z = f(x, y)$; so we see that as (x, y) varies through the plane, the function $z = f(x, y)$ will trace out a surface.
- Some example of graphs are given below.
  - Example: The graph $z = 0$ is the xy-plane.
    - Note more generally that any equation of the form $a\,x + b\,y + c\,z = d$ for some constants a, b, c, d will give a plane.
  - Example: The graph $z = x^2 + y^2$ is a paraboloid (i.e., a parabolic dish).
  - Example: The graph $z = \sqrt{x^2 + y^2}$ is a right circular cone opening upward, with vertex at the origin.

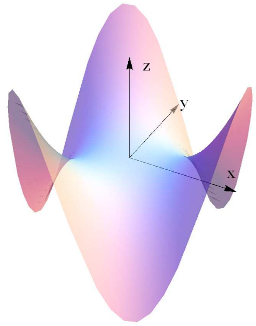


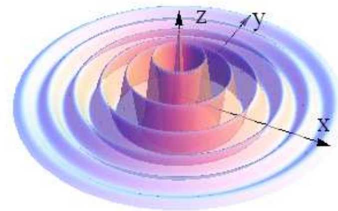




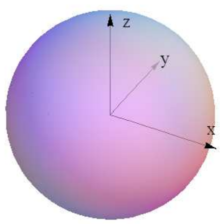
- Here are a few more unusual-looking graphs. You don't need to know any of this – it's just here so I have an excuse to make more pretty graphs.
  - Example: The graph $z = x^2 - y^2$ is called a hyperbolic paraboloid, or more colloquially, a saddle, since it curves upward along the x-direction but downward along the y-direction. [The hyperbolic paraboloid is called that because it looks like a hyperbola in one cross-section, and a parabola in two others.]
  - Example: The graph $z = x^3 - 3xy^2$ is called the "monkey saddle", as it has three depressions rather than the two for the regular saddle (one for each leg, and one for the tail).
  - Example: The graph $z = e^{3-\sqrt{x^2+y^2}/12} \cdot \cos\left(\sqrt{x^2 + y^2}\right)$ produces a surface that looks like ripples in a pool of water.

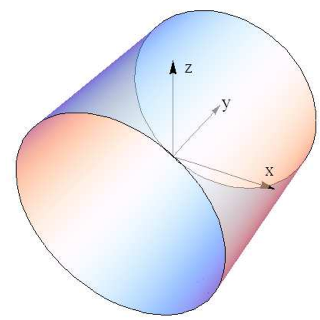






- We also can graph some functions defined implicitly.
  - Example: From the distance formula, we can see that the set of points satisfying $x^2 + y^2 + z^2 = 1$ are precisely those which are at a distance of 1 from the origin. But this is just another way of describing the sphere of radius 1 centered at $(0, 0, 0)$. A graph of the sphere is below.





  - Example: Again from the distance formula, we can see that the set of points satisfying $x^2 + z^2 = 1$ are those which are at a distance of 1 from the y-axis. This describes a right circular cylinder of radius 1, oriented along the y-axis. A graph of the cylinder is above.
- Surfaces can also be defined parametrically – however, since a surface is 2-dimensional, one must use 2 parameters rather than 1. We won't cover parametric definitions of surfaces in this course – that is a topic for multivariable calculus.

1.2 Vectors and Vector-Valued Functions

- A vector is a quantity which has both a magnitude and a direction.
  - This is in contrast to a scalar, which carries only a magnitude.
- We denote the n-dimensional vector from the origin to the point $(a_1, a_2, \cdots, a_n)$ as $\mathbf{v} = \langle a_1, a_2, \cdots, a_n \rangle$, where the a_i are scalars.
  - We use the angle brackets $\langle \cdot \rangle$ rather than parentheses $(\cdot)$ so as to underscore the difference between a vector and the coordinates of a point in space. We will, however, view coordinates of vectors and coordinates of points as essentially interchangeable.
  - We also write vectors in boldface ($\mathbf{v}$, not v), so that we can tell them apart from scalars. When writing by hand, it is hard to differentiate boldface, so the notation $\vec{v}$ is also sometimes used.
- The typical way to think of vectors is as "directed line segments": the length of the line segment gives the magnitude of the vector, and the direction the segment is pointing gives the direction of the vector.

- ◦ Note/Warning: Vectors are a little bit different from directed line segments, because we don't care where a vector starts: we only care about the difference between the starting and ending positions. Thus: the directed segment whose start is $(0,0)$ and end is $(1,1)$ and the segment starting at $(1,1)$ and ending at $(2,2)$ represent the same vector, $\langle 1,1\rangle$. This distinction is rarely necessary in most applications, however.

- We can add vectors (provided they are of the same dimension!) in the obvious way, one component at a time: if $\mathbf{v} = \langle a_1, \cdots, a_n\rangle$ and $\mathbf{w} = \langle b_1, \cdots, b_n\rangle$ then $\mathbf{v}+\mathbf{w} = \langle a_1+b_1, \cdots, a_n+b_n\rangle$.
  - ◦ Similarly we can 'scale' a vector by a scalar, one component at a time: if r is a scalar, then we have $r\,\mathbf{v} = \langle ra_1, \cdots, ra_n\rangle$.
  - ◦ Scaling a vector by a factor of $1/2$, for example, produces a new vector in the same direction, but with half the length as the original.
  - ◦ Scaling a vector by -1 produces a new vector with the same length but pointing in the opposite direction from the original vector.

- Example: If $\mathbf{v} = \langle -1,2,2\rangle$ and $\mathbf{w} = \langle 3,0,-4\rangle$ then $2\mathbf{w} = \boxed{\langle 6,0,-8\rangle}$, and $\mathbf{v}+\mathbf{w} = \boxed{\langle 2,2,-2\rangle}$. Furthermore, $\mathbf{v}-2\mathbf{w} = \boxed{\langle -7,2,10\rangle}$.

- We can also define functions that involve vectors:

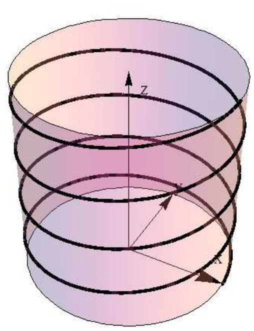
- Definition: A vector-valued function $\mathbf{r}(t)$ is a function whose output is a vector, each of whose components is a function of the parameter t.
  - ◦ Example: $\mathbf{r}(t) = \langle t^2, 2t\rangle$.

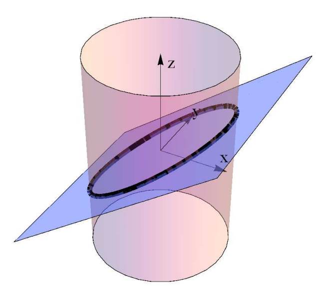
- We add and scalar-multiply vector-valued functions in the same manner as normal vectors.
  - ◦ Example: For $\mathbf{r}_1(t) = \langle e^t, \cos(t), t^2-1\rangle$ and $\mathbf{r}_2(t) = \langle t, 0, -t^2\rangle$ we have $\mathbf{r}_1(t)+\mathbf{r}_2(t) = \langle e^t+t, \cos(t), -1\rangle$ and $2\mathbf{r}_2(t) = \langle 2t, 0, -2t^2\rangle$.

- We will primarily be interested in vector functions of the form $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ and $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, which have a single input parameter t and output a vector with 2 or 3 coordinates. These functions trace out parametric curves in 2 or 3-dimensional space (respectively).

- Example: The curve given by $(x(t), y(t), z(t)) = (t,t,t)$ is a line passing through the origin.

- Example: The curve given by $(x(t), y(t), z(t)) = (\sin(t), \cos(t), t)$ is a helix wrapping around the cylinder $x^2+y^2=1$. We can see that as t increases, the x and y parts just trace around a unit circle at constant speed, while z increases at constant speed. Here is a plot of the curve winding around the cylinder, for $0 \le t \le 8\pi$:





- Example: The curve given by $(x(t), y(t), z(t)) = (\cos(t), \sin(t), \cos(t))$ is an ellipse. We can see that this curve is an ellipse by observing that it is the intersection of the plane $z = x$ with the cylinder $x^2+y^2=1$ (above).

1.3 The Dot Product

- One thing we might naturally want to know about a vector is its length (or norm, or magnitude). If we think of a vector as just a directed line segment in n-dimensional space, we can just use the distance formula (which is just the Pythagorean Theorem applied a few times) to see that the length of the line segment from the origin to $(a_1, \ldots, a_n)$ is just $\sqrt{(a_1)^2 + \cdots + (a_n)^2}$.

- Definition: We define the norm (length, magnitude) of the vector $\mathbf{v} = \langle a_1, \ldots, a_n \rangle$ as $\boxed{||\mathbf{v}|| = \sqrt{(a_1)^2 + \cdots + (a_n)^2}}$.

  - This is just an application of the distance formula: the norm of the vector $\langle a_1, \ldots, a_n \rangle$ is just the length of the line segment joining the origin $(0, \ldots, 0)$ to the point $(a_1, \ldots, a_n)$.
  - Example: For $\mathbf{v} = \langle -1, 2, 2 \rangle$ and $\mathbf{w} = \langle 3, 0, -4 \rangle$, we have $||\mathbf{v}|| = \sqrt{(-1)^2 + 2^2 + 2^2} = \boxed{3}$, and $||\mathbf{w}|| = \sqrt{3^2 + 0^2 + (-4)^2} = \boxed{5}$.
  - If r is a scalar, we can see immediately from the definition that $||r\,\mathbf{v}|| = |r|\;||\mathbf{v}||$, since we can just factor out a $\sqrt{r^2} = |r|$ from each term under the square root.

- From any nonzero vector we can find a unit vector (that is, a vector of norm 1) in the same direction of $\mathbf{v}$ just by scaling $\mathbf{v}$ by 1 over its norm. In other words, the vector $\boxed{\vec{u} = \dfrac{\mathbf{v}}{||\mathbf{v}||}}$ is a unit vector in the same direction as $\mathbf{v}$. This vector $\vec{u}$ is sometimes called the normalization of $\mathbf{v}$.

  - Example: For $\mathbf{v} = \langle -1, 2, 2 \rangle$, we see that $\vec{u}_1 = \boxed{\left\langle -\frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle}$ is a unit vector in the same direction as $\mathbf{v}$, and for $\mathbf{w} = \langle 3, 0, -4 \rangle$ we see that $\vec{u}_2 = \boxed{\left\langle \frac{3}{5}, 0, -\frac{4}{5} \right\rangle}$ is a unit vector in the direction of $\mathbf{w}$.

- If we have two vectors, we now know how to find their lengths. But another thing we might want to know about two vectors is the angle θ between them. This motivates the definition of the dot product:

- Definition: The dot product of two vectors $\mathbf{v}_1 = \langle a_1, \ldots, a_n \rangle$ and $\mathbf{v}_2 = \langle b_1, \ldots, b_n \rangle$ is defined to be the scalar $\boxed{\mathbf{v}_1 \cdot \mathbf{v}_2 = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}$.

  - Example: The dot product $\langle 1, 2 \rangle \cdot \langle 3, 4 \rangle$ is $(1)(2) + (3)(4) = \boxed{14}$.
  - Example: The dot product $\langle -1, 2, 2 \rangle \cdot \langle 3, 0, -4 \rangle$ is $(-1)(3) + (2)(0) + (2)(-4) = \boxed{-11}$.
  - Remark: The dot product obeys several very nice properties reminiscent of standard multiplication. For any vectors $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{w}$, and any scalar r, we can verify the following properties directly from the definition:
    * Dot product is commutative: $\boxed{\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}}$.
    * Dot product distributes over addition: $\boxed{(\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{w} = (\mathbf{v}_1 \cdot \mathbf{w}) + (\mathbf{v}_2 \cdot \mathbf{w})}$.
    * Dot product is "sort of" associative with scalar multiplication: $\boxed{(r\,\mathbf{v}) \cdot \mathbf{w} = r\,(\mathbf{v} \cdot \mathbf{w})}$.
    * Dot product of a vector with itself is the square of the norm: $\boxed{\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2}$.

- Theorem: For vectors $\vec{v}_1$ and $\vec{v}_2$ forming an angle θ between them, we have $\boxed{\mathbf{v}_1 \cdot \mathbf{v}_2 = ||\mathbf{v}_1||\;||\mathbf{v}_2||\,\cos(\theta)}$.

  - Proof: To prove this statement, we use the Law of Cosines in the triangle formed by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_2 - \mathbf{v}_1$, which states that $||\mathbf{v}_2 - \mathbf{v}_1||^2 = ||\mathbf{v}_1||^2 + ||\mathbf{v}_2||^2 - 2\,||\mathbf{v}_1||\;||\mathbf{v}_2||\cos(\theta)$. Now we know that the square of the norm is the dot product of a vector with itself so we can apply this and the other dot product properties to see that

$$\begin{aligned} ||\mathbf{v}_2 - \mathbf{v}_1||^2 &= (\mathbf{v}_2 - \mathbf{v}_1) \cdot (\mathbf{v}_2 - \mathbf{v}_1) \\ &= (\mathbf{v}_2 \cdot \mathbf{v}_2) - (\mathbf{v}_1 \cdot \mathbf{v}_2) - (\mathbf{v}_2 \cdot \mathbf{v}_1) + (\mathbf{v}_1 \cdot \mathbf{v}_1) \\ &= ||\mathbf{v}_2||^2 - 2(\mathbf{v}_1 \cdot \mathbf{v}_2) + ||\mathbf{v}_1||^2 . \end{aligned}$$

Now by comparing to the Law of Cosines expression, we see that everything cancels and leaves us precisely with the result we wanted.

- Definition: We say two vectors are orthogonal if their dot product is zero.
  - From the Dot Product Theorem, since $\cos\left(\frac{\pi}{2}\right) = 0$, we see that two nonzero vectors are orthogonal if the angle between them is $\frac{\pi}{2}$, which is to say, if they are perpendicular.
  - Example: The vectors $\langle 2, -1, 4\rangle$ and $\langle 3, 2, -1\rangle$ are orthogonal, since their dot product is $(2)(3)+(-1)(2)+(4)(-1) = 0$.
  - Example: The vectors $\langle 2, 2, -1\rangle$ and $\langle 3, 0, -4\rangle$ have a dot product of 10, as we computed earlier, and norms of 3 and 5 respectively. Therefore we see that the angle θ between them satisfies $10 = 3 \cdot 5 \cdot \cos(\theta)$, hence $\theta = \boxed{\cos^{-1}\left(\frac{2}{3}\right)}$.

1.4 3-Space: Lines, Planes, and The Cross Product

- At this point, we will restrict ourselves to talking just about 3-dimensional space. Our primary reason for this is that most of the immediate applications of vectors (e.g., to physics) happen in 3-dimensional space.
- It will be useful to have a way to denote the "unit coordinate" vectors of 3-dimensional space. So we denote $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, and $\mathbf{k} = \langle 0, 0, 1\rangle$.
- Before getting to vectors, we will take a brief excursion to talk about lines and planes.

1.4.1 Lines and Planes in 3-Space

- Proposition: Given distinct points $\mathbf{P}_0 = \langle x_0, y_0, z_0\rangle$ and $\mathbf{P}_1 = \langle x_1, y_1, z_1\rangle$, the points $\langle x, y, z\rangle$ on the line l through $\mathbf{P}_0$ and $\mathbf{P}_1$ are given parametrically by $\langle x, y, z\rangle = \boxed{\mathbf{P}_0 + t\,(\mathbf{P}_1 - \mathbf{P}_0)}$, as t varies through the real numbers.
  - Proof: There is a unique line between two points, by the axioms of geometry. So we just need to check that this is a line, and that it goes through $\mathbf{P}_0$ and $\mathbf{P}_1$.
    - The parametric equation for l explicitly is tells us that $x = x_0 + t(x_1 - x_0)$, $y = y_0 + t(y_1 - y_0)$, and $z = z_0 + t(z_1 - z_0)$, and these are all linear equations. So it's a line.
    - We see l goes through $\mathbf{P}_0$ because at $t = 0$ we get $\mathbf{P}_0$. Similarly, at $t = 1$ we get $\mathbf{P}_1$. So we're done.
  - Note: This procedure works to find the parametrization of a line in any space, not just 3-space.
  - Remark: We call the vector $\mathbf{v} = \mathbf{P}_1 - \mathbf{P}_0$ the "direction vector" for the line l: it tells us in which direction the line is moving. The term $\mathbf{P}_0$ in the sum $\mathbf{P}_0 + t\,(\mathbf{P}_1 - \mathbf{P}_0)$ specifies which, of all possible lines in that direction, is the line we want.
  - Example: To find the line through the points $(1, 2, 3)$ and $(-1, 2, -1)$ we just need to find the direction vector, which is $\mathbf{v} = \langle (-1) - 1, 2 - 2, (-1) - 3\rangle = \langle -2, 0, -4\rangle$. Then the line is given parametrically by $\boxed{\langle x, y, z\rangle = \langle 1 - 2t,\ 2,\ 3 - 4t\rangle}$.
- Proposition: The plane defined by $a\,x + b\,y + c\,z = d$ is orthogonal to its normal vector $\mathbf{n} = \langle a, b, c\rangle$. In other words, every line lying in this plane is orthogonal to $\langle a, b, c\rangle$.
  - Proof: Suppose l is a line in the plane. All we need to show is that its direction vector is orthogonal to $\mathbf{n}$.
    - So say the direction vector is $\mathbf{v} = \mathbf{P}_2 - \mathbf{P}_1$, where both of the points $\mathbf{P}_2 = \langle x_2, y_2, z_2\rangle$ and $\mathbf{P}_1 = \langle x_1, y_1, z_1\rangle$ lie in the plane.
    - Then $\mathbf{P}_1 \cdot \mathbf{n} = a\,x_1 + b\,y_1 + c\,z_1 = d$ since $\mathbf{P}_1$ lies in the plane, and similarly $\mathbf{P}_2 \cdot \mathbf{n} = d$.
    - But then we have $\mathbf{v} \cdot \mathbf{n} = \mathbf{P}_2 \cdot \mathbf{n} - \mathbf{P}_1 \cdot \mathbf{n} = d - d = 0$, which is exactly what we wanted.

- Proposition: Given a vector $\mathbf{n} = \langle a, b, c\rangle$, there is a unique plane normal to that vector passing through a given point (x_0, y_0, z_0).
  - Proof: For the converse statement of the proof, clearly if $\mathbf{n} = \langle a, b, c\rangle$ then the equation of the plane must be $a\,x + b\,y + c\,z = \Box$ for some value of $\Box$, by the previous proposition. But if we are given a point that lies in the plane, we can plug in to see that $\Box = ax_0 + by_0 + cz_0$, and so we have uniquely determined the equation of the plane, and hence the plane.

- Now we have some basic facts about lines and planes. We know how to find the line passing through 2 points $\mathbf{P}_2$ and $\mathbf{P}_1$, but if we're given 3 points $\mathbf{P}_3$, $\mathbf{P}_2$, and $\mathbf{P}_1$ (not on a single line), how do we find the plane passing through all 3?
  - We know how to produce two direction vectors $\mathbf{P}_2 - \mathbf{P}_1$ and $\mathbf{P}_3 - \mathbf{P}_1$ lying in the plane. (Conversely, it will turn out, any two nonparallel vectors will span a plane.)
  - We also know a normal vector to the plane, along with any point in the plane (like $\mathbf{P}_1$), will specify the plane.
  - Therefore, what we need to know to solve the problem is how to find a vector orthogonal to the two vectors $\mathbf{P}_2 - \mathbf{P}_1$ and $\mathbf{P}_3 - \mathbf{P}_1$.

1.4.2 The Cross Product

- Definition: The cross product of $\mathbf{v}_1 = \langle x_1, y_1, z_1\rangle$ and $\mathbf{v}_2 = \langle x_2, y_2, z_2\rangle$ is defined to be the vector $\boxed{\mathbf{v}_1 \times \mathbf{v}_2 = \langle y_1 z_2 - y_2 z_1,\ z_1 x_2 - z_2 x_1,\ x_1 y_2 - x_2 y_2\rangle}$. It is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$.
  - Important Note: The cross product is only defined for vectors with 3 components, and outputs another vector with 3 components. Contrast with the dot product, which is defined for vectors of any length, and outputs a scalar.
  - A way to remember the cross product formula (aside from memorization) is the "determinant formula" $\boxed{\mathbf{v}_1 \times \mathbf{v}_2 = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \mathbf{k}}$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the standard unit vectors: $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, and $\mathbf{k} = \langle 0, 0, 1\rangle$.
    - It's a little unusual to have vectors inside a determinant, but it works out to the correct answer. Don't forget the minus sign on the middle term.
  - We claim that this vector $\mathbf{v}_1 \times \mathbf{v}_2$ is orthogonal to $\mathbf{v}_1$ and to $\mathbf{v}_2$. To verify this, we can just evaluate the dot products $\mathbf{v}_1 \cdot (\mathbf{v}_1 \times \mathbf{v}_2)$ and $\mathbf{v}_2 \cdot (\mathbf{v}_1 \times \mathbf{v}_2)$ and check that they are both zero. For example we have $\mathbf{v}_1 \cdot \mathbf{w} = x_1(y_1 z_2 - y_2 z_1) + y_1(z_1 x_2 - z_2 x_1) + z_1(x_1 y_2 - x_2 y_2)$, which is zero because each term appears once with a $+$and once with a $-$.
  - Unlike the dot product, the cross product is NOT commutative! Indeed, we can see from the definition that $\boxed{\mathbf{v}_1 \times \mathbf{v}_2 = -(\mathbf{v}_2 \times \mathbf{v}_1)}$. In particular, we see that $\mathbf{v} \times \mathbf{v} = 0$ for any vector $\mathbf{v}$.
  - We still do have a distributivity property, like with the dot product: it is fairly easy to check from the definition that $\boxed{(\mathbf{v}_1 + \mathbf{v}_2) \times \mathbf{w} = (\mathbf{v}_1 \times \mathbf{w}) + (\mathbf{v}_2 \times \mathbf{w})}$.
  - We also have the same scalar multiplication "sort of associativity": $\boxed{(r\,\mathbf{v}) \times \mathbf{w} = r\,(\mathbf{v} \times \mathbf{w})}$.

- Theorem: If θ is the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$, then $\boxed{||\mathbf{v}_1 \times \mathbf{v}_2|| = ||\mathbf{v}_1||\,||\mathbf{v}_2||\,\sin(\theta) = A}$, where A is the area of the parallelogram formed by $\mathbf{v}_1$ and $\mathbf{v}_2$.
  - Proof: We just need to show that $||\mathbf{v}_1 \times \mathbf{v}_2||^2 + (\mathbf{v}_1 \cdot \mathbf{v}_2)^2 = ||\mathbf{v}_1||^2\,||\mathbf{v}_2||^2$, because we know that $\mathbf{v}_1 \cdot \mathbf{v}_2 = ||\mathbf{v}_1||\,||\mathbf{v}_2||\,\cos(\theta)$ from the Dot Product Theorem.
    - To check this we multiply everything out. So we need to see that $(y_1 z_2 - y_2 z_1)^2 + (z_1 x_2 - z_2 x_1)^2 + (x_1 y_2 - x_2 y_2)^2 + (x_1 x_2 + y_1 y_2 + z_1 z_2)^2$ is equal to $\left[(x_1)^2 + (y_1)^2 + (z_1)^2\right] \cdot \left[(x_2)^2 + (y_2)^2 + (z_2)^2\right]$.

        - When we expand the first thing, we get each of the 9 possible square terms $(\Box_1 \triangle_2)^2$ where $\Box$ and $\triangle$ are each one of x, y, or z, and the "cross" terms like $2x_1 x_2 y_1 y_2$ will all cancel out.
        - We get exactly the same sum of 9 square terms when we expand the second thing. So they are equal and we're done.
        - For the statement about the area, we can just use geometry to see that the area of the triangle with sides $\vec{v}_1$ and $\vec{v}_2$ is $\frac{1}{2}\,||\mathbf{v}_1||\,||\mathbf{v}_2||\,\sin(\theta)$. The parallelogram's area is twice this.
    - Remark: This quite nice property is one reason we chose the definition we did for the cross product.

- Example: Let us find an equation for the plane passing through $\mathbf{P}_1 = (3, 0, -1)$, $\mathbf{P}_2 = (1, 2, 2)$ and $\mathbf{P}_3 = (-2, 1, 4)$.
    - We have $\mathbf{v}_2 = \mathbf{P}_3 - \mathbf{P}_1 = \langle -5, 1, 5\rangle$ and $\mathbf{v}_1 = \mathbf{P}_2 - \mathbf{P}_1 = \langle -2, 2, 3\rangle$.
    - Then we can compute the normal vector to the plane, which will be given by the cross product. This gives us $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -5 & 5 \\ -2 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -5 & 1 \\ -2 & 2 \end{vmatrix}\mathbf{k} = \langle -7, 5, -8\rangle$. For a sanity check, we compute $\mathbf{n} \cdot \mathbf{v}_1 = (-7)(-5) + (5)(1) + (-8)(5) = 0$ and $\mathbf{n} \cdot \mathbf{v}_2 = (-7)(-2) + (5)(2) + (-8)(3) = 0$.
    - Now we get that the plane's equation is $-7x + 5y - 8z = d$, for some d.
    - To find the constant we plug in the point $\mathbf{P}_1$ to see $d = (-7)(3) + 5(0) - 8(-1) = -29$.
    - Therefore the equation of the plane is $\boxed{-7x + 5y - 8z = -29}$.
    - For an additional error check, we could plug in all three points to ensure they really do lie on this plane, and they do.

- Theorem: The volume of the parallelepiped whose edges are the three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ is given by the "scalar triple product" $\boxed{V = |\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)|}$.
    - Proof: The volume of the solid is its height times the area of its base.
        - The area of the base (whose sides are $\mathbf{v}_2$ and $\mathbf{v}_3$) is given by the magnitude of the cross product $\mathbf{v}_2 \times \mathbf{v}_3$, while the height is equal to $||\mathbf{v}_1||\sin(\phi)$ where ϕ is the angle between $\mathbf{v}_1$ and the plane that the base lies in.
        - We can check with a diagram that $\sin(\phi) = \cos(\theta)$ where θ is the angle between $\mathbf{v}_1$ and the normal $\mathbf{n}$ to the plane of the base, since $\phi = \frac{\pi}{2} - \theta$.
        - Now applying the Dot Product Theorem shows that $V = ||\mathbf{v}_1||\,||\mathbf{v}_2 \times \mathbf{v}_3||\cos(\theta) = |\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)|$, as we claimed.

Well, you're at the end of my handout. Hope it was helpful.