Subject Code: 10CS661 **I.A. Marks : 25**
Hours/Week : 04 **Exam Hours: 03**
Total Hours : 52 **Exam Marks: 100**

PART - A

UNIT – 1 6 Hours

Introduction, Linear Programming – 1: Introduction: The origin, nature and impact of OR; Defining the problem and gathering data; Formulating a mathematical model; Deriving solutions from the model; Testing the model; Preparing to apply the model; Implementation .

Introduction to Linear Programming: Prototype example; The linear programming (LP) model.

UNIT – 2 7 Hours

LP – 2, Simplex Method – 1: Assumptions of LP; Additional examples. The essence of the simplex method; Setting up the simplex method; Algebra of the simplex method; the simplex method in tabular form; Tie breaking in the simplex method

UNIT – 3 6 Hours

Simplex Method – 2: Adapting to other model forms; Post optimality analysis; Computer implementation Foundation of the simplex method.

UNIT – 4 7 Hours

Simplex Method – 2, Duality Theory: The revised simplex method, a fundamental insight.

The essence of duality theory; Economic interpretation of duality, Primal dual relationship; Adapting to other primal forms

PART - B

UNIT – 5 7 Hours

Duality Theory and Sensitivity Analysis, Other Algorithms for LP : The role of duality in sensitive analysis; The essence of sensitivity analysis; Applying sensitivity analysis. The dual simplex method; parametric linear programming; the upper bound technique.

UNIT – 6 7 Hours

Transportation and Assignment Problems: The transportation problem; A streamlined simplex method for the transportation problem; The assignment problem; A special algorithm for the assignment problem.

UNIT – 7 6 Hours

Game Theory, Decision Analysis: Game Theory: The formulation of two persons, zero sum games; Solving simple games- a prototype example; Games with mixed strategies; Graphical solution procedure; Solving by linear programming, Extensions.

Decision Analysis: A prototype example; Decision making without experimentation; Decision making with experimentation; Decision trees.

UNIT – 8 6 Hours

Metaheuristics: The nature of Metaheuristics, Tabu Search, Simulated Annealing, Genetic Algorithms.

Text Books:

1. Frederick S. Hillier and Gerald J. Lieberman: Introduction to Operations Research: Concepts and Cases, 8th Edition, Tata McGraw Hill, 2005. (Chapters: 1, 2, 3.1 to 3.4, 4.1 to 4.8, 5, 6.1 to 6.7, 7.1 to 7.3, 8, 13, 14, 15.1 to 15.4)

Reference Books:

1. Wayne L. Winston: Operations Research Applications and Algorithms, 4th Edition, Cengage Learning, 2003.

2. Hamdy A Taha: Operations Research: An Introduction, 8th Edition, Pearson Education, 2007.

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UNIT -1

Introduction, Linear programming-1

1.1 The origin of operations research

OR is a relatively new discipline. Whereas 70 years ago it would have been possible to study mathematics, physics or engineering (for example) at university it would not have been possible to study OR, indeed the term OR did not exist then. It was only really in the late 1930's that operational research began in a systematic fashion, and it started in the UK.

Early in 1936 the British Air Ministry established Bawdsey Research Station, on the east coast, near Felixstowe, Suffolk, as the centre where all pre-war radar experiments for both the Air Force and the Army would be carried out. Experimental radar equipment was brought up to a high state of reliability and ranges of over 100 miles on aircraft were obtained.

It was also in 1936 that Royal Air Force (RAF) Fighter Command, charged specifically with the air defense of Britain, was first created. It lacked however any effective fighter aircraft - no Hurricanes or Spitfires had come into service - and no radar data was yet fed into its very elementary warning and control system.

It had become clear that radar would create a whole new series of problems in fighter direction and control so in late 1936 some experiments started at Biggin Hill in Kent into the effective use of such data. This early work, attempting to integrate radar data with ground based observer data for fighter interception, was the start of OR.

The first of three major pre-war air-defense exercises was carried out in the summer of 1937. The experimental radar station at Bawdsey Research Station was brought into operation and the information derived from it was fed into the general air-defense warning and control system. From the early warning point of view this exercise was encouraging, but the tracking information obtained from radar, after filtering and transmission through the control and display network, was not very satisfactory.

In July 1938 a second major air-defense exercise was carried out. Four additional radar stations had been installed along the coast and it was hoped that Britain now had an aircraft location and control system greatly improved both in coverage and effectiveness.

Not so! The exercise revealed, rather, that a new and serious problem had arisen. This was the need to coordinate and correlate the additional, and often conflicting, information received from the additional radar stations. With the out-break of war apparently imminent, it was obvious that something new - drastic if necessary - had to be attempted.

Some new approach was needed. Accordingly, on the termination of the exercise, the Superintendent of Bawdsey Research Station, A.P. Rowe, announced that although the exercise had again demonstrated the technical feasibility of the radar system for detecting aircraft, its operational achievements still fell far short of requirements. He therefore proposed that a crash program of research into the operational - as opposed to the technical - aspects of the system should begin immediately. The term "operational research" [RESEARCH into (military) OPERATIONS] was coined as a suitable description of this new branch of applied science. The first team was selected from amongst the scientists of the radar research group the same day.

In the summer of 1939 Britain held what was to be its last pre-war air defense exercise. It involved some 33,000 men, 1,300 aircraft, 110 antiaircraft guns, 700 searchlights, and 100 barrage balloons. This exercise showed a great improvement in the operation of the air defense warning and control system. The contribution made by the OR teams was so apparent that the Air Officer Commander-in-Chief RAF Fighter Command (Air Chief Marshal Sir Hugh Dowding) requested that, on the outbreak of war, they should be attached to his headquarters at Stanmore.

On May 15th 1940, with German forces advancing rapidly in France, Stanmore Research Section was asked to analyze a French request for ten additional fighter squadrons (12 aircraft a squadron) when losses were running at some three squadrons every two days. They prepared graphs for Winston Churchill (the British Prime Minister of the time), based upon a study of current daily losses and replacement rates, indicating how rapidly such a move would deplete fighter strength.

No aircraft were sent and most of those currently in France were recalled. This is held by some to be the most strategic contribution to the course of the war made by OR (as the aircraft and pilots saved were consequently available for the successful air defense of Britain, the Battle of Britain).

In 1941 an Operational Research Section (ORS) was established in Coastal Command which was to carry out some of the most well-known OR work in World War II. Although scientists had (plainly) been involved in the hardware side of warfare (designing better planes, bombs, tanks, etc) scientific analysis of the operational use of military resources had never taken place in a systematic fashion before the Second World War. Military personnel, often by no means stupid, were simply not trained to undertake such analysis.

These early OR workers came from many different disciplines, one group consisted of a physicist, two physiologists, two mathematical physicists and a surveyor. What such people brought to their work were "scientifically trained" minds, used to querying assumptions, logic, exploring hypotheses, devising experiments, collecting data, analyzing numbers, etc. Many too were of high intellectual caliber (at least four wartime OR personnel were later to win Nobel prizes when they returned to their peacetime disciplines).

By the end of the war OR was well established in the armed services both in the UK and in the USA. OR started just before World War II in Britain with the establishment of teams of scientists to study the strategic and tactical problems involved in military operations. The objective was to find the most effective utilization of limited military resources by the use of quantitative techniques.

Following the end of the war OR spread, although it spread in different ways in the UK and USA.

You should be clear that the growth of OR since it began (and especially in the last 30 years) is, to a large extent, the result of the increasing power and widespread availability of computers. Most (though not all) OR involves carrying out a large number of numeric calculations. Without computers this would simply not be possible

1.2 THE METHODOLOGY OF OR

When OR is used to solve a problem of an organization, the following seven step procedure should be followed:

Step 1. Formulate the Problem: OR analyst first defines the organization's problem. Defining the problem includes specifying the organization's objectives and the parts of the organization (or system) that must be studied before the problem can be solved.

Step 2. Observe the System: Next, the analyst collects data to estimate the values of parameters that affect the organization's problem. These estimates are used to develop (in Step 3) and evaluate (in Step 4) a mathematical model of the organization's problem.

Step 3. Formulate a Mathematical Model of the Problem: The analyst, then, develops a mathematical model (in other words an idealized representation) of the problem. In this class, we describe many mathematical techniques that can be used to model systems.

Step 4. Verify the Model and Use the Model for Prediction: The analyst now tries to determine if the mathematical model developed in Step 3 is an accurate representation of reality. To determine how well the model fits reality, one determines how valid the model is for the current situation.

Step 5. Select a Suitable Alternative: Given a model and a set of alternatives, the analyst chooses the alternative (if there is one) that best meets the organization's objectives. Sometimes the set of alternatives is subject to certain restrictions and constraints. In many situations, the best alternative may be impossible or too costly to determine.

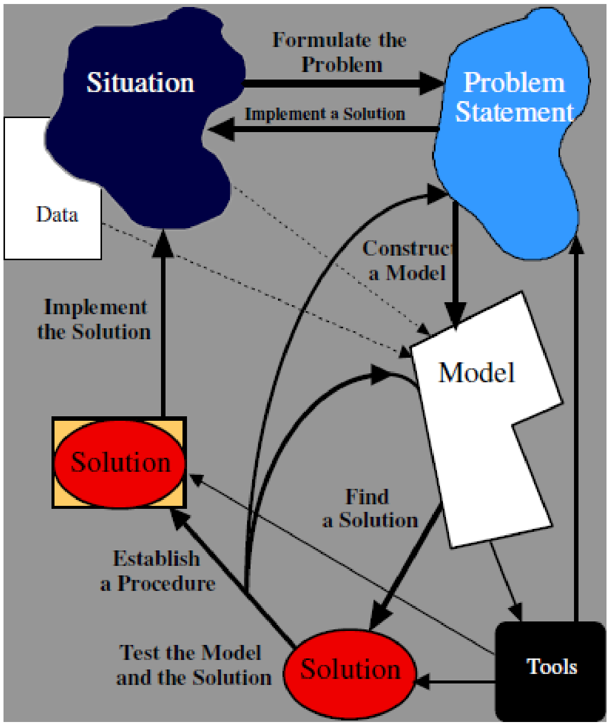
Step 6. Present the Results and Conclusions of the Study: In this step, the analyst presents the model and the recommendations from Step 5 to the decision making individual or group. In some situations, one might present several alternatives and let the 2 organization choose the decision maker(s) choose the one that best meets her/his/their needs. After presenting the results of the OR study to the decision maker(s), the analyst may find that s/he does not (or they do not) approve of the recommendations. This may result from incorrect definition of the problem on hand or from failure to involve decision maker(s) from the start of the project. In this case, the analyst should return to Step 1, 2, or 3.

Step 7. Implement and Evaluate Recommendation: If the decision maker(s) has accepted the study, the analyst aids in implementing the recommendations. The system must be constantly monitored (and updated dynamically as the environment changes) to ensure that the recommendations are enabling decision maker(s) to meet her/his/their objectives.

Defining the problem and gathering data:

Goal: solve a problem

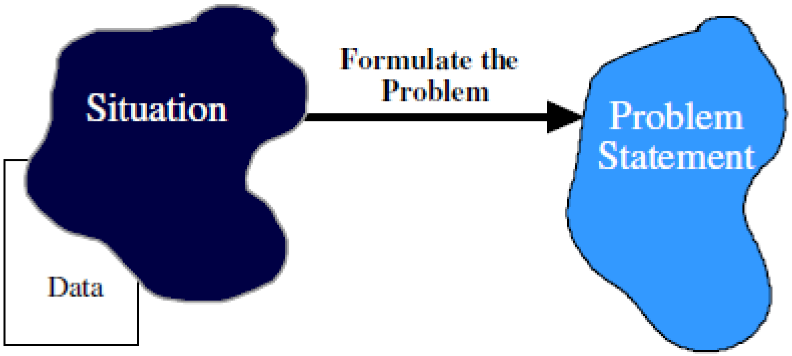
- Model must be valid
- Model must be tractable
- Solution must be useful



The Situation

- May involve current operations or proposed expansions due to expected market shifts
- May become apparent through consumer complaints or through employee suggestions
- May be a conscious effort to improve efficiency or response to an unexpected crisis.

Problem Formulation

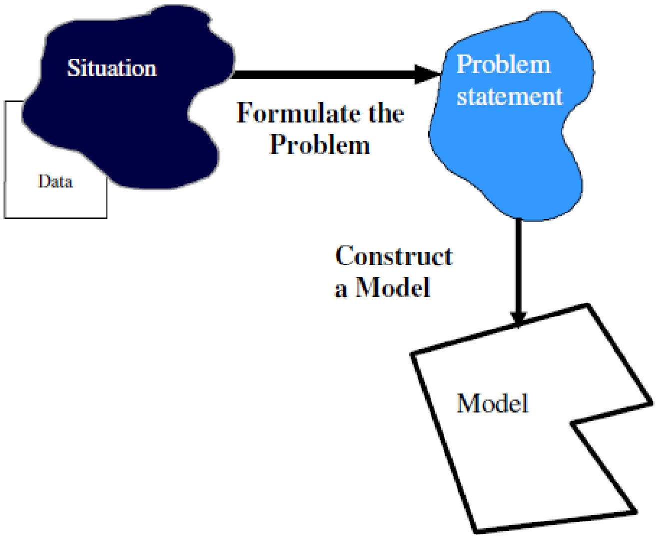


- Describe system
- Define boundaries
- State assumptions
- Select performance measures
- Define variables
- Define constraints
- Data requirements

Example: Maximize individual nurse preferences subject to demand requirements.

Preparing to apply the model:

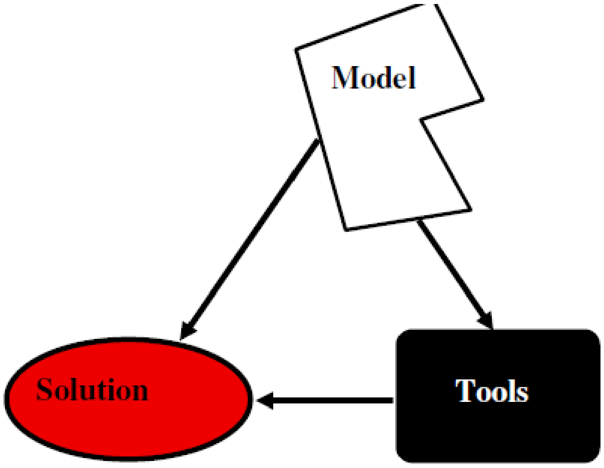
- Problem must be translated from verbal, qualitative terms to logical, quantitative terms
- A logical model is a series of rules, usually embodied in a computer program
- A mathematical model is a collection of functional relationships by which allowable actions are delimited and evaluated.



Example: Define relationships between individual nurse assignments and preference violations; define tradeoffs between the use of internal and external nursing resources.

Formulating the Mathematical Model:

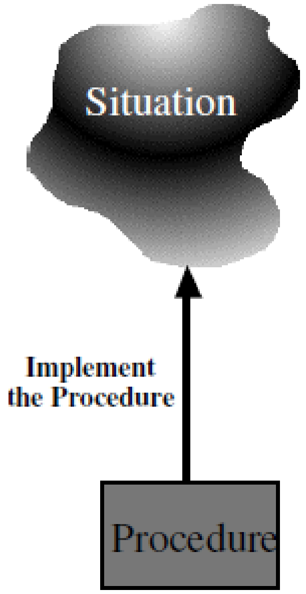
- Many tools are available as will be discussed in this course
- Some lead to “optimal” solutions
- Others only evaluate candidate’s -> trial and error to find “best” course of action



Example: Read nurse profiles and demand requirements, apply algorithm, post-processes results to get monthly schedules.

Implementation

- A solution to a problem usually implies changes for some individuals in the organization
- Often there is resistance to change, making the implementation difficult
- User-friendly system needed
- Those affected should go through training



Example: Implement nurse scheduling system in one unit at a time. Integrate with existing HR and T&A systems. Provide training sessions during the workday.

Introduction to Linear Programming

Linear programming problem arises whenever two or more candidates or activities are competing for limited resources.Linear programming applies to optimization technique in which the objective and constraints functions are strictly linear.

Application of Linear Programming

Agriculture, industry, transportation, economics, health Systems, behavioral and social sciences and the military. It can be computerized for 10000 of constraints and variables.

Mathematical formulation:

A mathematical program is an optimization problem in which the objective and constraints are given as mathematical functions and functional relationships.

The procedure for mathematical formulation of a LPP consists of the following steps

Step1: write down the decision variables (Products) of the problem

Step2: formulate the objective function to be optimized (maximized or minimized) as linear function of the decision variables

Step3: formulate the other conditions of the problem such as resource limitation, market, constraints, and interrelations between variables etc., linear in equations or equations in terms of the decision variables.

Step4: add non-negativity constraints

Linear Programming Problem.

The general formulations of the LPP can be stated as follows:

In order to find the values of n decision variables x1, x2, x3---------- xn to MAX or MIN the objective function.

$$\text{Max } Z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n \qquad (a) \longrightarrow \text{Objective Function}$$

Also satisfy m – constraints or Subject to Constraint

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2 \\ | \ \ | \\ | \ \ | \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = b_n \\ x_1 >= 0 \ , \ x_2 >= 0, \ \ldots\ldots + x_n >= 0 \end{array}\right\} \begin{array}{l} (b) \longrightarrow \text{Constant} \\ \\ \\ (c) \longrightarrow \text{Non Negative Restriction} \end{array}$$

cj (j = 1,2----n) is the objective function in equations (a) are called cost coefficient (max profit or min cost) bi (i= 1,2,----m) defining the constraint requirements or available in equation(b) or available in equation (b) is called stipulations and the constants aij (i=1,2,----m; j= 1,2,---n)are called structural co-efficient in equation (c) are known as non-negative restriction

Matrix form

$$\underset{(mxn)}{A =} \left\{\begin{array}{llll} a_{11} & a_{12} & \ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots\ldots & a_{2n} \\ & & & \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{array}\right\} \qquad \underset{(mxn)}{X =} \begin{pmatrix} x_1 \\ x_2 \\ - \\ - \\ x_n \end{pmatrix} \qquad b= \begin{pmatrix} b_1 \\ b_2 \\ -- \\ - \\ b_n \end{pmatrix}$$

$$\text{and } C = (c_{1n} \, , c_{2n} \ \ldots\ldots\ldots c_n)$$

A is called the coefficient matrix X is the decisions Vector

B is the requirement Vector and c is the profit (cost) vector of the linear program.

The LPP can be expressed in the matrix as follows

Max or Min Z= CX Objective Function

Subject to Constraint

AX=b Structural coefficient

X>=0 Non negativity

Problem 1

A Manufacture produces two types of models M1 and M2 each model of the type M1 requires 4 hrs of grinding and 2 hours of polishing, whereas each model of the type M2 requires 2 hours of grinding and 5 hours of polishing. The manufactures have 2 grinders and 3 polishers. Each grinder works 40 hours a week and each polisher's works for 60 hours a week. Profit on M1 model is Rs. 3.00 And on Model M2 is Rs 4.00. Whatever produced in week is sold in the market. How should the manufacturer allocate is production capacity to the two types models, so that he may make max in profit in week?

Solutions:

Decision variables. Let X and X be the numbers of units of M1 and M2 Model

Objective function: since, the profit on M1 and M2 is Rs. 3.0 and Rs 4.

Max Z =3x1 + 4x2

Constraint: there are two constraints one for grinding and other is polishing

No of grinders are 2 and the hours available in grinding machine is 40 hrs per week, therefore, total no of hours available of grinders is 2 X 40 = 80 hours

No of polishers are 3 and the hours available in polishing machine is 60 hrs per week, therefore, total no of hours available of polishers is 3 X 60 = 180 hours

The grinding constraint is given by:

4x1 + 2x2 <= 80

The Polishing Constraint is given by:

2x1 + 5x2 <= 180

Non negativity restrictions are

x1 , x2 >= 0 if the company is not manufacturing any products

The LPP of the given problem

Max Z =3x1 + 4x2

STC

4x1 + 2x2 <= 80

2x1 + 5x2 <= 180

x1 , x2 >= 0

Problem 2:

Egg contains 6 units of vitamin A per gram and 7 units of vitamin B per gram and cost 12 paise per gram. Milk contains 8 units of vitamin A per gram and 12 units of vitamin B per gram and costs 20 paise per gram. The daily requirements of vitamin A and vitamin B are 100 units and 120 units respectively. Find the optimal product mix.

EGG MILK Min

Requirements

Vitamin A 6 8 100

Vitamin B 7 12 120

Cost 12 20

The LPP of the given Problem

Min Z =12x1 + 20x2

STC

6x1 + 8x2 >= 100

7x1 + 12x2 >= 120

x1 , x2 >= 0

Problem 3: A farmer has 100 acre. He can sell all tomatoes. Lettuce or radishes he raise the price. The price he can obtain is Re 1 per kg of tomatoes, Rs 0.75 a head for lettuce and Rs 2 per kg of radishes. The average yield per acre is 2000kg tomatoes, 3000 heads of lettuce and 1000 kgs of radishes. Fertilizer is available at Rs 0.5 per kg and the amount required per acre 100 kgs each for tomatoes and lettuce, and 50 kgs for radishes. Labor required for sowing, cultivating and harvesting per acre is 5 man-days for tomatoes and radishes, 6 man-days for lettuce. A total of 400 man days of labor available at Rs 20 per man day formulate the problem as linear programming problem model to maximize the farmers' total profit.

Formulation:

Farmer's problem is to decide how much area should be allotted to each type of crop. He wants to grow to maximize his total profit.

Let the farmer decide to allot X1, X2 and X3 acre of his land to grow tomatoes, lettuce and radishes respectively.

So the farmer will produce 2000 X1 kgs of tomatoes, 3000 X2head of lettuce and 1000 X3 kgs of radishes.

Profit= sales – cost

= sales – (Labor cost +fertilizer cost)

Sales = 1x 2000 X1 + 0.75 x 3000 X2 + 2 x 1000 X3

Labor cost = 5x 20 X1 + 6 x 20 X2 + 5 x 20 X3

Fertilizer cost = 100x0.5 X1 + 0.5x 100 X2 + 0.5x50 X3

Max Z= 1850 X1 + 2080 X2 + 1875 X3

STC

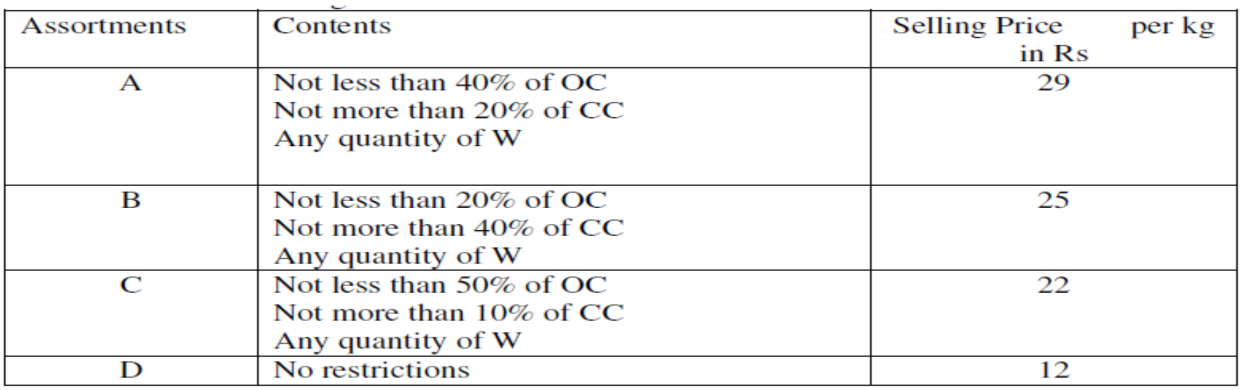
X1 + X2 + X3 <= 100

5X1 + 6X2 + 5X3 <= 400

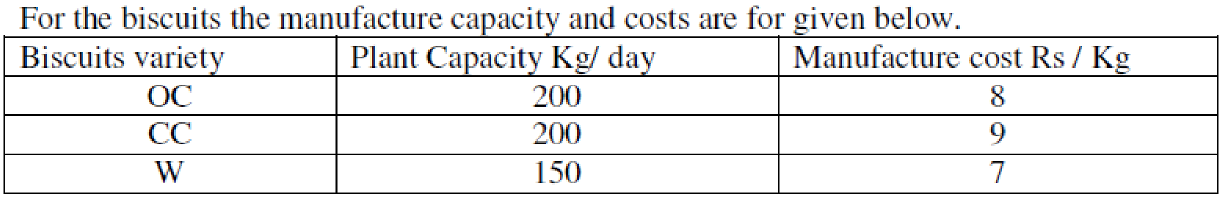
X1 , X2 , X3 >= 0

Problem 4:

A Manufacturer of biscuits is considering 4 types of gift packs containing 3 types of biscuits, orange cream (oc), chocolate cream (cc) and wafer's(w) market research study conducted recently to assess the preferences of the consumers shows the following types of assortments to be in good demand.



Assortments	Contents	Selling Price per kg in Rs
A	Not less than 40% of OC Not more than 20% of CC Any quantity of W	29
B	Not less than 20% of OC Not more than 40% of CC Any quantity of W	25
C	Not less than 50% of OC Not more than 10% of CC Any quantity of W	22
D	No restrictions	12



For the biscuits the manufacture capacity and costs are for given below.

Biscuits variety	Plant Capacity Kg/ day	Manufacture cost Rs / Kg
OC	200	8
CC	200	9
W	150	7

Formulate a LP model to find the production schedule which maximizes the profit assuming that there are no market restrictions.

Formulation: the company manufacturer 4 gift packs which oc, cc and w. the quantity of ingredients in each pack is not known.

Let x11 denotes the quantities OC of gift pack A

x12 denotes the quantities CC of gift pack A x13 denotes the quantities W of gift pack A

x21 denotes the quantities OC of gift pack B

x22 denotes the quantities CC of gift pack B

x23 denotes the quantities W of gift pack B

x31 denotes the quantities OC of gift pack C

x32 denotes the quantities CC of gift pack C

x33 denotes the quantities W of gift pack C

x41 denotes the quantities OC of gift pack D

x42 denotes the quantities CC of gift pack D

x43 denotes the quantities W of gift pack D

The objective Function is to max total profit

Max Z = 20(x11 +x12 +x13) + 25(x21 +x22 +x23)+ 22(x31 +x32 +x33)

+ 12(x41 +x42 +x43) – 8(x11+x21+x31+ x41) - 9(x12+x22+x32 +x42) - 7(x13

+x23+x33+x43)

STC

Gift pack A

x11 >= 0.4 (x11 +x12 +x13)

x12 <= 0.2 (x11 +x12 +x13)

Gift pack B

x21 >= 0.2 (x21 +x22 +x23)

x12 <= 0.2 (x21 +x22 +x23)

Gift pack C

x31 >= 0.2 (x31 +x32 +x33)

x32 <= 0.2 (x31 +x32 +x33)

$\sum Xij \ \sum Xij \ \sum Xij \ \sum Xij \ \geq 0$

Graphical Method:

The graphical procedure includes two steps

1. Determination of the solution space that defines all feasible solutions of the model.

2. Determination of the optimum solution from among all the feasible points in the solution space.

There are two methods in the solutions for graphical method

- Extreme point method
- Objective function line method

Steps involved in graphical method are as follows:

- Consider each inequality constraint as equation
- Plot each equation on the graph as each will geometrically represent a straight line.
- Mark the region. If the constraint is <= type then region below line lying in the first quadrant (due to non negativity variables) is shaded.
- If the constraint is >= type then region above line lying in the first quadrant is shaded.
- Assign an arbitrary value say zero for the objective function.
- Draw the straight line to represent the objective function with the arbitrary value
- Stretch the objective function line till the extreme points of the feasible region. In the maximization case this line will stop farthest from the origin and passing through at least one corner of the feasible region. In the minimization case, this line will stop nearest to the origin and passing through at least one corner of the feasible region.
- Find the co-ordination of the extreme points selected in step 6 and find the maximum or minimum value of Z.

Problem 1

Max Z =3x1 + 5x2

STC

x1 <= 4

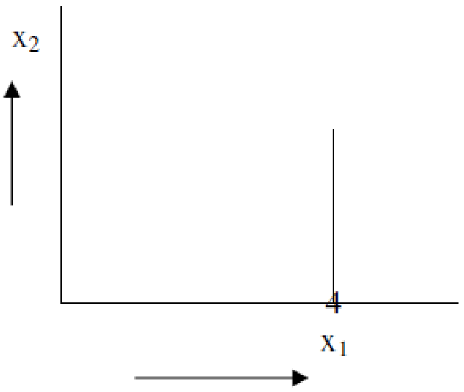
2x2 <=12

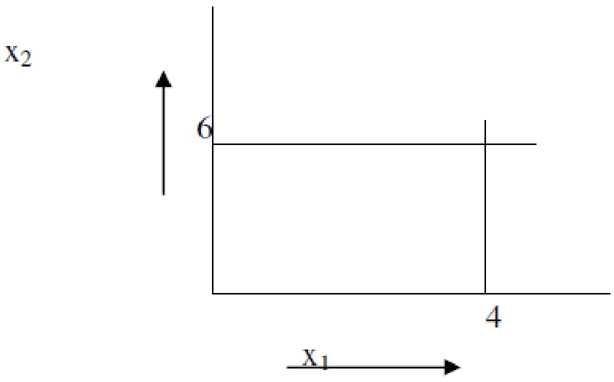
3x1 + 2x2 <= 18

x1, x2 >= 0

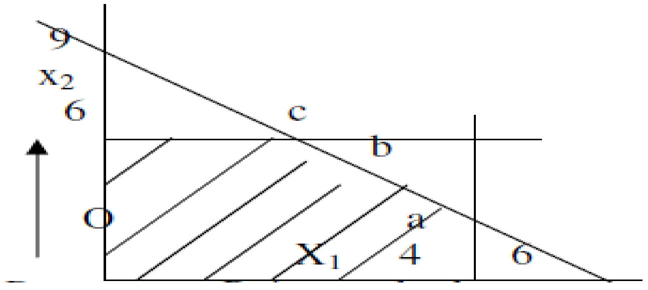
Solution

x1 <= 4, x1 = 4, 2x2 <=12, 2x2=12, x2 =6



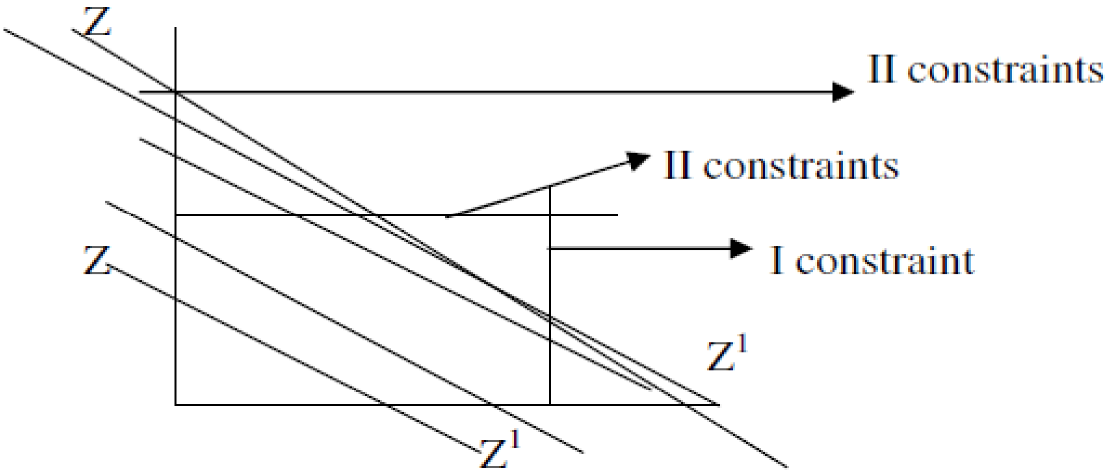


3x1 + 2x2 <= 18 put x1 =0, x1 = 6, put x2 = 0, x2 = 9



By extreme Point method

O=(0,0)	Z=0
A=(4,0)	Z=12
B=(4,3)	Z=12+15=27
C=(2,6)	Z= 6+30 =36
D= (0,6)	Z= 30



By objective function line method,

To find the point of Max value of Z in the feasible region we use objective function line as ZZ1 , same type of lines are used for different assumed Z value to find the Max Z in the solution space as shown in the above figure.

Let us start = 10

Max Z=10 =3x1 + 5x2 this will show the value as (3.33,2) by plotting this points in the solution space it explains that Z must be large as 10 we can see many points above this line and within the region.

When Z=20

Max Z=20 =3x1 + 5x2 this will show the value as (6.66, 4) by plotting this points in the solution space z must be at least 20. The trial and error procedure involves nothing more than drawing a family of parallel lines containing at least one point in the permissible region and selecting the distance from the origin. This lines passes through the points (2,6) or Z=36

Max Z = 36= 3x1 + 5x2 the points are (12, 7.2) this points lies at the intersection of the 2 lines 2x2 =12 and 3x1 + 2x2 = 18. so, this point can be calculated algebraically as the simultaneous solutions of these 2 equation.

Conclusions:

The solution indicates that the company should produce products 1 & 2 at the rate of 2 per minute and d6 / minute respectively with resulting profitable of 36 / minute.

No other mix of 2 products would be profitable according to the model.

Problem 1: find the max Value of the given LPP

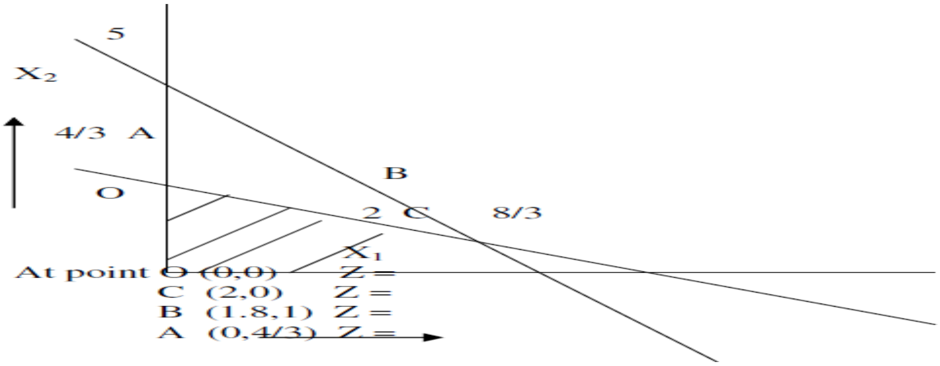
Max Z =x1 + 3x2

STC

3x1 + 6x2 <= 100 ----→(8/3, 4/3)

5x1 + 2x2 <= 120 ----→(2,5)

x1 , x2 >= 0



Problem 2: find the max Value of the given LPP

Max Z =5x1 + 2x2

STC

x 1 + x2 <= 4

3x1 + 8x2 <= 24

10x1 +78x2 <= 35

x1 , x2 >= 0

Problem 3: find the max Value of the given LPP

Max Z = - x1 + 2x2

STC

-x1 +3x2 <= 10

x1 +x2 <= 6

x1 - x2 <= 2

x1 , x2 >= 0

Problem 4: find the max Value of the given LPP

Max Z =7x1 + 3x2

STC

x1 + 2x2 <= 3

x1 + x2 <= 4

0<= x1 <=5/2

0 <= x2 <= 3/2

x1 , x2 >= 0

Problem 5: find the max Value of the given LPP

Max Z =20x1 + 10x2

STC

x1 +2x2 <= 40

3x1 + x2 <= 30

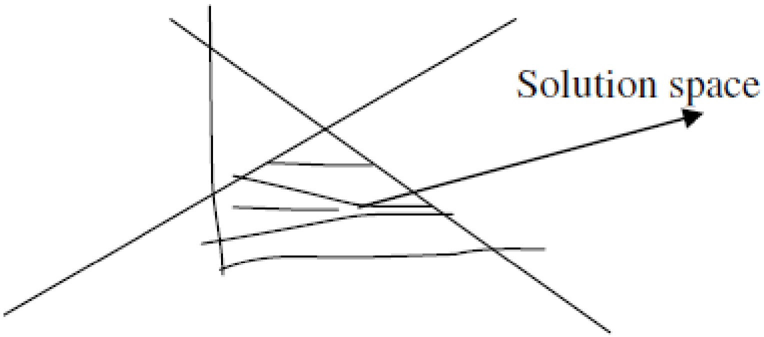
4x1 + 3x2 <= 30

x1 , x2 >= 0

Solution space

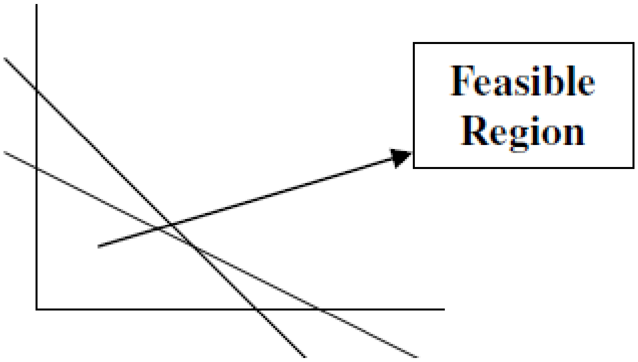
Solutions mean the final answer to a problem

Solutions space to a LPP is the space containing such points. The co-ordinates of which satisfy all the constraints simultaneously. The region of feasibility of all the constraints including non-negativity requirements.



Feasible:

The feasible region for an LP is the set of all points that satisfies all the LPs constraints and sign restrictions.



Basic feasible

A basic feasible solution is a basic solution which also satisfies that is all basic variables are non-negative.

Example:

4x1 + 2x2 <= 80 we add x3 as slack variable

2x1 + 5x2 <= 180 we add x4 as slack variable

$\begin{bmatrix} 4 & 2 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is unit matrix is called basic feasible solution

Optimal

Any feasible solution which optimizes (Min or Max) the objective function of the LPP is called its optimum solution.

Infeasible / Inconsistency in LPP

Inconsistency also known as infeasibility

The constraint system is such that one constraint opposes one or more. It is not possible to find one common solutions to satisfy all the constraints in the system.

Ex:-

2x1 + x2 <= 20 ----→(10,20)

2x1 + x2 <= 40 ----→(20,40)

If both the constraint cannot be satisfied simultaneously. Such constraint system is said to be raise to inconsistency or infeasibility

Redundancy;

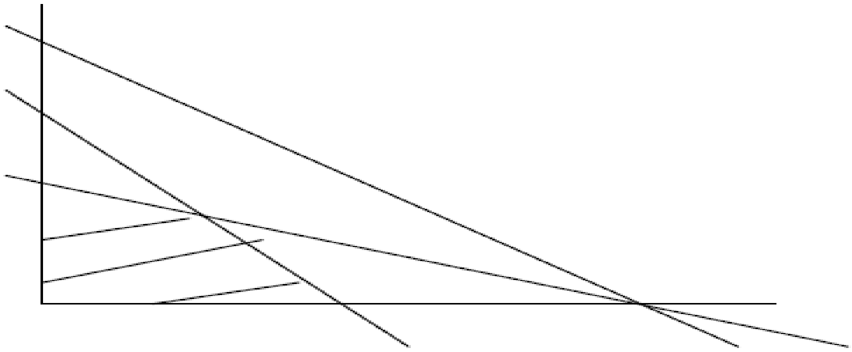
A set of constraint is said to be redundant if one or more of them are automatically satisfied on the basis of the requirement of the others.

Ex:

2x1 + x2 <= 20 (10,20)

2x1 + x2 <= 35 (17.5, 35)

x1 + 2x2 <= 20 (20,10)



A redundant constraint system is one in which deletion of at least one of the constraint will not alter the solution space.

Degeneracy

A basic feasible solution is said to degenerate if one or more basic variable are zero.

The values of the variables may be increases indefinitely without violating any of the constraint i.e., the solution space is unbounded in at least one direction.

As result the objective function value may increase or decrease indefinitely.

Ex;

x1 - x2 <= 10 (10,-10)

2x1 <=40 (20,0)

I constraint

Unbounded solutions

II constraint Space

Standard form

The standard form of a linear programming problem with m constraints and n variables can be represented as follows:

Max Z =c1x1 + c2x2 + ……. + cnxn

Also satisfy m – constraints or Subject to Constraint

a11x1 + a12x2 + ……. + a1nxn = b1

a21x1 + a22x2 + ……. + a2nxn = b2

| |

| |

am1x1 + am2x2 + ……. + amnxn = bn

x1>=0 , x2 >=0, ……. + xn >=0

The main features of the standard form

1. The objective function is of the maximization or minimization type
2. All constraints are expressed as equations
3. All variables are restricted to be nonnegative
4. The right-hand side constant of each constraint is nonnegative.

Now, considering how a LPP can be formulated in the standard form will be as follows:

Case (a): if a problem aims at minimizing the objective function. Then it can be converted into a maximization problem simply by multiplying the objective by (-1)

Case (b): if a constraint is of <= type , we add a non negative variable called slack variables is added to the LHS of the constraint on the other hand if the constraint is of >= type, we subtract a non-negative variable called the surplus variable from the LHS.

Case (c) when the variables are unrestricted in sign it can be represented as

Xj = X1 j – X11 j or X1 = X1 1 – X11 1

It may become necessary to introduce a variable that can assume both +ve and –ve values. Generally, unrestricted variable is generally replaced by the difference of 2 non - ve variables.

Problem 1:

Rewrite in standard form the following linear programming problem

Min Z =12x1 + 5x2

STC

5x1 + 3x2 >= 15

7x1 + 2x2 <= 14

x1 , x2 >= 0

Solution:

Since, the given problem is minimization then it should be converted to maximization by just multiply by (-1) and the first constraint is >= type it is standard by adding by surplus variable as x3>=0 and 2nd constraint is <= type it is standard by adding slack variable and then the given problem is reformulated as follows:

Max Z = -12x1 - 5x2 - 0x3+ 0x4

STC

5x1 + 3x2 - x3 = 15

7x1 + 2x2 + x4 = 14

x1 , x2 , x3 , x4 >= 0

The matrix form

Max Z = (-12, -5,0, 0) (-x1, - x2 ,- x3, x4)

STC

$$\left\{\begin{matrix} 5 & 3 & -1 & 0 \\ 7 & 2 & 0 & 1 \end{matrix}\right\} \left\{\begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix}\right\} = \left\{\begin{matrix} 15 \\ 14 \end{matrix}\right\}$$

$$x_1 , x_2 , x_3 , x_4 >= 0$$

Problem 2:

Rewrite in standard form the following linear programming problem

Max Z =2x1 + 5x2 + 4x3

STC

-2x1 + 4x2 <= 4

x1 + 2x2+x3 >=5

2x1 + 3x3 <= 2

x1 , x2 >= 0 x3 is unrestricted in sign

Solution:

In the given problem it is maximization problem and the constraint are of in equations.

The first constraint is <= type we introduce slack variable as x4 >=0, 2nd constraint is of

>= type, we introduce surplus variable as x5 >=0 and third constraint is <= type we introduce slack variable as x6 >=0. the x3 variable is un restricted in sign. So, this can be written as

X3 = X1 3 – X11 3

Then, the given LPP is rewritten as

Max Z =2x1 + 5x2 + 4X1 3 – 4X11 3 + 0x4 - 0x5+ 0x6

STC

-2x1 + 4x2 + x4 = 4

x1 + 2x2+ X1 3 – X11 3 - x5 =5

2x1 + 3 X1 3 – 3X11 3 + 0x6= 2

x1 , x2 , x1 3, x11 3, x4 ,x5, x6 >= 0

The matrix form

Max Z = (2,5,4,-4,0,0,0) (x1 , x2 , x1 3, x11 3, x4 ,x5, x6)

STC

$$\left\{\begin{matrix} -2 & 4 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & -1 & 0 & -1 & 0 \\ 2 & 0 & 3 & -3 & 0 & 0 & 1 \end{matrix}\right\} \left\{\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}\right\} = \left\{\begin{matrix} 4 \\ 5 \\ 2 \end{matrix}\right\}$$

$$x_1 , x_2 , x^1{}_{3,} x^{11}{}_{3,} x_4 ,x_{5,} x_6 >= 0$$

UNIT 2

LP – 2, Simplex Method – 1

Assumption of Linear Programming:

Proportionality assumption: The contribution of each activity to the value of the objective function Z is proportional to the level of the activity xj, as represented by the cjxj term in the objective function. Similarly, the contribution of each activity to the left-hand side of each functional constraint is proportional to the level of the activity xj, as represented by the aijxj term in the constraint.

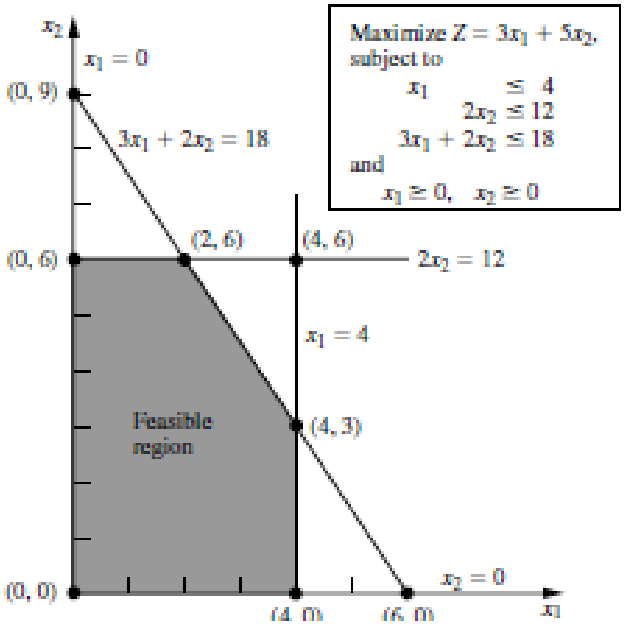
Additivity assumption: Every function in a linear programming model (whether the objective function or the function on the left-hand side of a functional constraint) is the sum of the individual contributions of the respective activities.

Divisibility assumption: Decision variables in a linear programming model are allowed to have any values, including non-integer values that satisfy the functional and non-negativity constraints. Thus, these variables are not restricted to just integer values. Since each decision variable represents the level of some activity, it is being assumed that the activities can be run at fractional levels.

Certainty assumption: The value assigned to each parameter of a linear programming model is assumed to be a known constant.

Essence of the SIMPLEX METHOD:

Simplex method d is an algebraic procedure with the underlying geometric concept. To illustrate the general geometric concepts consider the wyndor class CO. Problem as shown in below figure. The five constraint boundaries and their points of intersection are highlighted in this figure because they are the keys to the analysis. Here, each **constraint boundary** is a line that forms the boundary of what is permitted by the corresponding constraint.



The points of intersection are the **corner-point solutions** of the problem. The five that lie on the corners of the *feasible region*—(0, 0), (0, 6), (2, 6), (4, 3), and (4, 0)—are the *corner-point feasible solutions* (**CPF solutions**). The other three—(0, 9), (4, 6), and (6, 0)—are called *corner-point infeasible solutions*.

In this example, each corner-point solution lies at the intersection of *two* constraint boundaries. (For a linear programming problem with *n* decision variables, each of its corner-point solutions lies at the intersection of *n* constraint boundaries.1) Certain pairs of the CPF solutions in Fig. 4.1 share a constraint boundary, and other pairs do not. It will be important to distinguish between these cases by using the following general definitions.

For any linear programming problem with *n* decision variables, two CPF solutions are **adjacent** to each other if they share *n* _ 1 constraint boundaries. The two adjacent CPF solutions are connected by a line segment that lies on these same shared constraint boundaries. Such a line segment is referred to as an **edge** of the feasible region.

Since *n* =2 in the example, two of its CPF solutions are adjacent if they share *one* constraint boundary; for example, (0, 0) and (0, 6) are adjacent because they share the *x*1 _ 0 constraint boundary. The feasible region in Fig. has five edges, consisting of the five line segments forming the boundary of this region. Note that two edges emanate from each CPF solution. Thus, each CPF solution has two adjacent CPF solutions (each lying at the other end of one of the two edges), as enumerated in Table. (In each row of this table, the CPF solution in the first column is adjacent to each of the two CPF solutions in the second column, but the two CPF solutions in the second column are not adjacent to each other.)

One reason for our interest in adjacent CPF solutions is the following general property about such solutions, which provides a very useful way of checking whether a CPF solution is an optimal solution.

Optimality test: Consider any linear programming problem that possesses at least one optimal solution. If a CPF solution has no adjacent CPF solutions that are better (as measured by Z), then it must be an optimal solution.

Thus, for the example, (2, 6) must be optimal simply because it's Z =36 is larger than Z =30 for (0, 6) and Z =27 for (4, 3). This optimality test is the one used by the simplex method for determining when an optimal solution has been reached.

Now we are ready to apply the simplex method to the example.

Solving the Example

Here is an outline of what the simplex method does (from a geometric viewpoint) to solve the Wyndor Glass Co. problem. At each step, first the conclusion is stated and then the reason is given in parentheses. (Refer to Fig. 4.1 for a visualization.)

Initialization: Choose (0, 0) as the *initial* CPF solution to examine. (This is a convenient choice because no calculations are required to identify this CPF solution.)

Optimality Test: Conclude that (0, 0) is *not* an optimal solution. (Adjacent CPF solutions are better.)

Iteration 1: Move to a better *adjacent* CPF solution, (0, 6), by performing the following three steps.

1. Considering the two edges of the feasible region that emanate from (0, 0), choose to move along the edge that leads up the $x2$ axis. (With an objective function of $Z = 3x1 + 5x2$, moving up the $x2$ axis increases Z at a faster rate than moving along the $x1$ axis.)

2. Stop at the first new constraint boundary: $2x2$ _ 12. [Moving farther in the direction selected in step 1 leaves the feasible region; e.g., moving to the second new constraint boundary hit when moving in that direction gives (0, 9), which is a corner-point *infeasible* solution.]

3. Solve for the intersection of the new set of constraint boundaries: (0, 6). (The equations for these constraint boundaries, $x1 = 0$ and $2x2 = 12$, immediately yield this solution.)

Optimality Test: Conclude that (0, 6) is *not* an optimal solution. (An adjacent CPF solution is better.)

Iteration 2: Move to a better adjacent CPF solution, (2, 6), by performing the following three steps.

1. Considering the two edges of the feasible region that emanate from (0, 6), choose to move along the edge that leads to the right. (Moving along this edge increases Z, whereas backtracking to move back down the $x2$ axis decreases Z.)

2. Stop at the first new constraint boundary encountered when moving in that direction: $3x1 + 2x2 = 12$. (Moving farther in the direction selected in step 1 leaves the feasible region.)

3. Solve for the intersection of the new set of constraint boundaries: (2, 6). (The equations for these constraint boundaries, $3x1 + 2x2 = 18$ and $2x2 = 12$, immediately yield this solution.)

Optimality Test: Conclude that (2, 6) *is* an optimal solution, so stop. (None of the adjacent

CPF solutions are better.)

This sequence of CPF solutions examined is shown in Fig., where each circled number identifies which iteration obtained that solution.

Now let us look at the six key solution concepts of the simplex method that provide the rationale behind the above steps. (Keep in mind that these concepts also apply for solving problems with more than two decision variables where a graph like Fig. is not available to help quickly find an optimal solution.)

The Key Solution Concepts

The first solution concept is based directly on the relationship between optimal solutions and CPF solutions.

Solution concept 1: The simplex method focuses solely on CPF solutions. For any problem with at least one optimal solution, finding one requires only finding a best CPF solution.1

Since the number of feasible solutions generally is infinite, reducing the number of solutions that need to be examined to a small finite number (just three in Fig. 4.2) is a tremendous simplification. The next solution concept defines the flow of the simplex method.

Solution concept 2: The simplex method is an *iterative algorithm* (a systematic solution procedure that keeps repeating a fixed series of steps, called an *iteration,* until a desired result has been obtained) with the following structure.

When the example was solved, note how this flow diagram was followed through two iterations until an optimal solution was found.

We next focus on how to get started.

Solution concept 3: Whenever possible, the initialization of the simplex method chooses the *origin* (all decision variables equal to zero) to be the initial CPF solution. When there are too many decision variables to find an initial CPF solution graphically, this choice eliminates the need to use algebraic procedures to find and solve for an initial CPF solution.

Choosing the origin commonly is possible when all the decision variables have nonnegativity constraints, because the intersection of these constraint boundaries yields the origin as a corner-point solution. This solution then is a CPF solution *unless* it is *infeasible* because it violates one or more of the functional constraints. If it is infeasible, special procedures described are needed to find the initial CPF solution.

The next solution concept concerns the choice of a better CPF solution at each iteration.

Solution concept 4: Given a CPF solution, it is much quicker computationally to gather information about its *adjacent* CPF solutions than about other CPF solutions. Therefore, each time the simplex method performs an iteration to move from the current CPF solution to a better one, it *always* chooses a CPF solution that is *adjacent* to the current one. No other CPF solutions are considered. Consequently, the entire path followed to eventually reach an optimal solution is along the *edges* of the feasible region.

The next focus is on which adjacent CPF solution to choose at each iteration.

Solution concept 5: After the current CPF solution is identified, the simplex method examines each of the edges of the feasible region that emanate from this CPF solution. Each of these edges leads to an *adjacent* CPF solution at the other end, but the simplex method does not even take the time to solve for the adjacent

CPF solution. Instead, it simply identifies the *rate of improvement in Z* that would be obtained by moving along the edge. Among the edges with a *positive* rate of improvement in *Z*, it then chooses to move along the one with the *largest* rate of improvement in *Z*. The iteration is completed by first solving for the adjacent

CPF solution at the other end of this one edge and then relabeling this adjacent CPF solution as the *current* CPF solution for the optimality test and (if needed) the next iteration.

At the first iteration of the example, moving from (0, 0) along the edge on the *x*1 axis would give a rate of improvement in *Z* of 3 (*Z* increases by 3 per unit increase in *x*1), whereas moving along the edge on the *x*2 axis would give a rate of improvement in *Z* of 5 (*Z* increases by 5 per unit increase in *x*2), so the decision is made to move along the latter edge. At the second iteration, the only edge emanating from (0, 6) that would yield a *positive* rate of improvement in *Z* is the edge leading to (2, 6), so the decision is made to move next along this edge.

The final solution concept clarifies how the optimality test is performed efficiently.

Solution concept 6: Solution concept 5 describes how the simplex method examines each of the edges of the feasible region that emanate from the current CPF solution. This examination of an edge leads to quickly identifying the rate of improvement in *Z* that would be obtained by moving along the edge toward the adjacent CPF solution at the other end. A *positive* rate of improvement in *Z* implies that the adjacent CPF solution is *better* than the current CPF solution, whereas a *negative* rate of improvement in *Z* implies that the adjacent CPF solution is *worse*. Therefore, the optimality test consists simply of checking whether *any* of the edges give a *positive* rate of improvement in *Z*. If *none* do, then the current CPF solution is optimal.

In the example, moving along *either* edge from (2, 6) decreases *Z*. Since we want to maximize

Z, this fact immediately gives the conclusion that (2, 6) is optimal.

Setting up the simplex method.

The simplex method is developed by G.B Dantzig is an iterative procedure for solving linear programming problem expressed in standard form. In addition to this simplex method requires constraint equations to be expressed as a canonical system form which a basic feasible solution can be readily obtained

The solutions of any LPP by simplex algorithm the existence of an IBFS is always assumed, the following steps help to reach an optimum solution of an LPP.

Procedure for Maximization

Step 1: write the given LPP in standard form

Step 2: check whether the objective function of the given LPP is to be Maximized or minimized if its to be minimized then we convert it into a problem of maximizing using the result.

Min z = - Max Z or (-Z)

Step 3: check whether all bi (i=123…..m0 are non-negative. if anyone bi is –VE, then multiply the corresponding in equation of the constraint by -1.

Step 4: Convert all the in equations of the constraint into equations by introducing slack or surplus variable in the constraints. Put these costs equal to zero in objective cost.

Step 5: Obtain an IBFS to problem in the form identity matrix form in canonical form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and put it in the 1st column of simples table.

Step 6: compute the net evaluation row (Zj – Cj) (j=1,2,3….n)

Zj – Cj = P/U (Profit / unit) x Xj -Cj (j=1,2,3…n) examine the sign of Zj – Cj

- if all (Zj – Cj) >= 0 then the IBFS solutions column is an optimum basic feasible solutions.
- if at least one (Zj – Cj) <=0 proceed to the next step

Step 7: if there are more than one –ve (Zj – Cj) then choose the most –ve of them then it will become key column

- if all the no's in the key column is –ve then there is an unbounded solutions to the given problem
- if at least one Xm >0 (m =123…n) then the corresponding vector Xm >=0 (m=1,2,3…n) then the corresponding vector Xm entry the basis of solution column

Step 8: compute the ratios = solutions column no / key column no. And choose the minimum of them. Let the minimum of these ratios be the key row these variable in the basic variable column of the key row will become the leaving element or variable.

Step 9: using the below relation to find new no of other than key row and new no for key row also New no for pivot row = current pivot row / pivot element
Other than key row

New element = old element – (PCE * NPRE)
PCE= Pivot column element, NPRE=new pivot row element
New no= old no – (corresponding Key column / Key element) x (corresponding key row)

Step 10: go to step5 and repeat the computational procedure until either an optimum solutions is obtained or there is an indication of an unbounded solution.

Note: case 1 in case of a tie for entering basis vector. i.e., there are 2 or more Zj – Cj which are equal and at the same time the highest –ve values then arbitrary selection of any one of them will not alter optimality.

Case 2 in case of a tie for the leaving variable i.e., there are 2 or more min ratio column
i.e., (solution no / key column no) which are equal and greater than zero then arbitrary select any one of them will not alter optimality. But, if the tied ratios are zeros then charnes method of penalty should be followed.

Problems

1. Use simplex method to solve the given LPP

Max Z =5x1 + 3x2
STC
x1 + x2 <= 2
5x1 + 2x2 <= 10

3x1 + 8x2 <= 12

x1, x2 >= 0

Solution:

Step 1: Since the problem is maximization problem all the constraint are <= type and the requirements are +ve. This satisfies the simplex method procedure.

Step 2: since all the constraints are <= type we introduce the slack variables for all the constraints as x3 >=0, x4 >=0, x5 >=0 for the I II and III constraint

Step 3: the given LPP can be put in standard form

Max Z =5x1 + 3x2 + (0) x3+ (0) x4 + (0)x5
STC
x1 + x2 + x3 <= 2
5x1 + 2x2+ x4 <= 10
3x1 + 8x2 + x5 <= 12
x1, x2 ,x3,x4 ,x5 >= 0

Step 4: matrix form

Max Z = (5,3,0,0,0) (x1, x2 ,x3,x4 ,x5)

STC

$$\left\{\begin{matrix} 1 & 1 & 1 & 0 & 0 \\ 5 & 2 & 0 & 1 & 0 \\ 3 & 8 & 0 & 0 & 1 \end{matrix}\right\} \left\{\begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix}\right\} = \left\{\begin{matrix} 2 \\ 10 \\ 12 \end{matrix}\right\}$$

$$x_1, x_2, x_3, x_4, x_5 >= 0$$

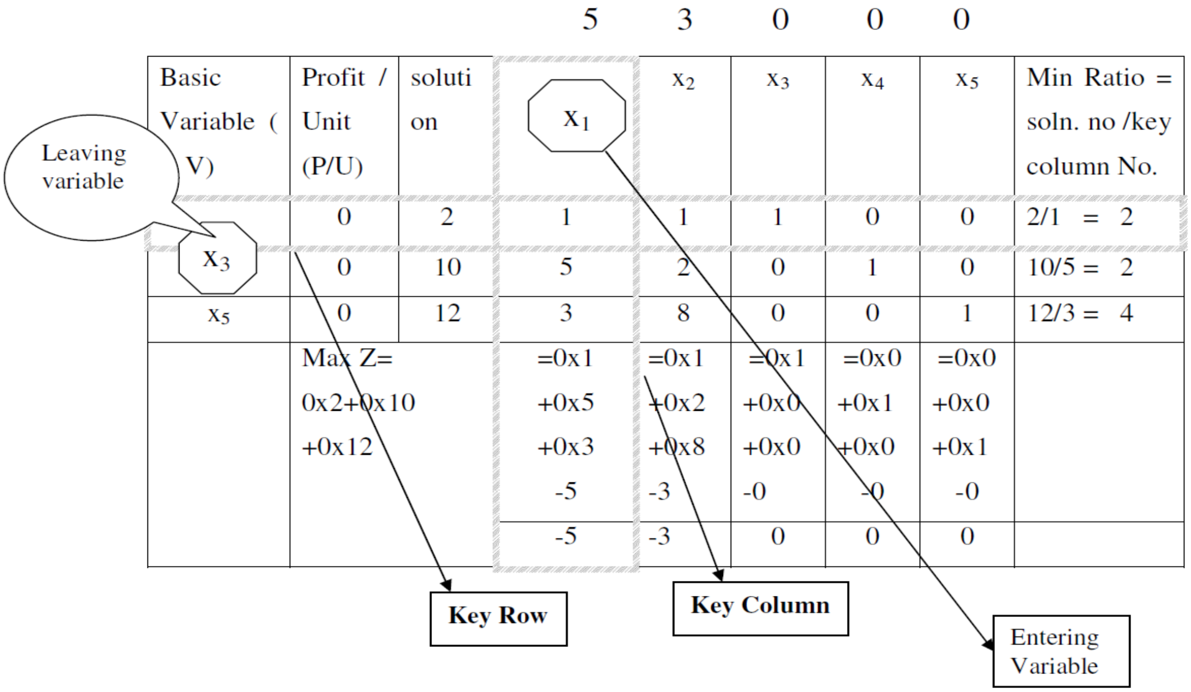
Step 5: since, considering sub-matrix from the matrix are which form basic variables for the starting table of simplex

(1 0 0) (0 1 0) (0 0 1) are linearly independent column vectors of A.

Therefore, the sub Matrix is

$$B= \left\{ \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right\}$$

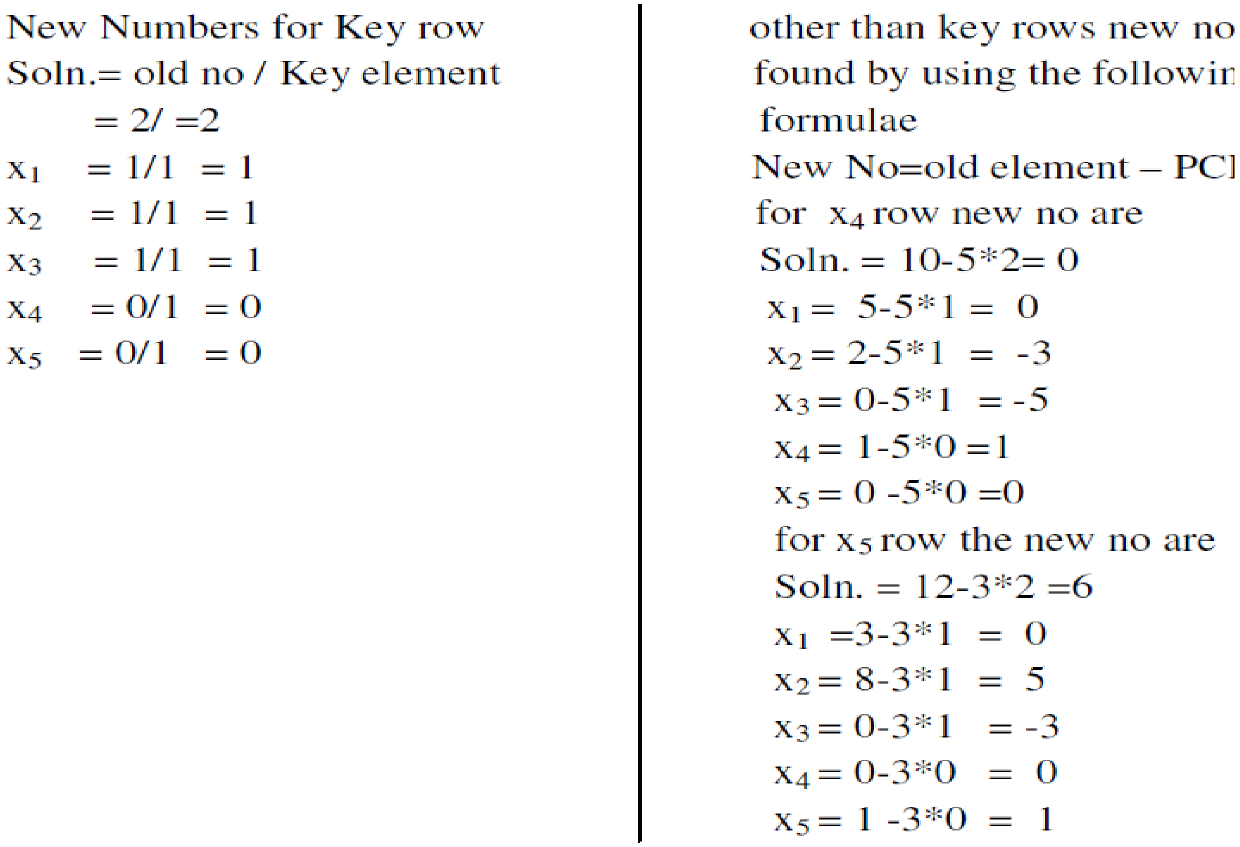
The corresponding variables of the sub matrix is (x3, x4 ,x5) and these variables are the basic variables for the starting iteration of the simplex problem and there obvious initial basic feasible solutions are (x3, x4 ,x5) = (2, 10, 12)



			5	3	0	0	0	
Basic Variable (V)	Profit / Unit (P/U)	solution	x_1	x_2	x_3	x_4	x_5	Min Ratio = soln. no /key column No.
	0	2	1	1	1	0	0	2/1 = 2
x_3	0	10	5	2	0	1	0	10/5 = 2
x_5	0	12	3	8	0	0	1	12/3 = 4
	Max Z= 0x2+0x10 +0x12		=0x1 +0x5 +0x3 -5	=0x1 +0x2 +0x8 -3	=0x1 +0x0 +0x0 -0	=0x0 +0x1 +0x0 -0	=0x0 +0x0 +0x1 -0	
			-5	-3	0	0	0	

5 3 0 0 0

Basic Variable (B V)	Profit / Unit (P/U)	solution	x_1	x_2	x_3	x_4	x_5	Min Ratio = soln. no /key column No.
x_1	5	2	1	1	1	0	0	
x_4	0	0	0	-3	-5	1	0	
x_5	0	6	0	5	-3	0	1	
	Max Z= 5x2+0x0+0x6 =10		=5x1+ 0x0 +0x0- 5	=5x1+0 x-3 +0x5 -3	=5x1 +0x-5 + 0x-3 - 0	=5x0+ 0x1+0 x0-0	=5x0+ 0x0+0 x1-0	
			0	2	5	0	0	



New Numbers for Key row

Soln.= old no / Key element

= 2/ =2

x_1 = 1/1 = 1

x_2 = 1/1 = 1

x_3 = 1/1 = 1

x_4 = 0/1 = 0

x_5 = 0/1 = 0

other than key rows new no

found by using the followir

formulae

New No=old element – PCI

for x_4 row new no are

Soln. = 10-5*2= 0

x_1 = 5-5*1 = 0

x_2 = 2-5*1 = -3

x_3 = 0-5*1 = -5

x_4 = 1-5*0 =1

x_5 = 0 -5*0 =0

for x_5 row the new no are

Soln. = 12-3*2 =6

x_1 =3-3*1 = 0

x_2 = 8-3*1 = 5

x_3 = 0-3*1 = -3

x_4 = 0-3*0 = 0

x_5 = 1 -3*0 = 1

Since, the given problems net evaluation row is +ve, then given problem as attained the optimum

Therefore, x1 =2, x2 =0, x3 =0, x4 =0, x5 =6,

Substitute in the objective function

Max Z =5x1 + 3x2 + (0) x3+ (0) x4 + (0)x5

Max Z =5x2+3x0+0x0+0x0+0x6

Max Z = 10

2. Solve the given problem by simplex method

Max Z =107x1 + x2 +2 x3
STC
14x1 + x2 - 6x3 +3x4 = 7
16x1 + 1/2x2 -6x3 <= 5
16x1 - 8x2 - x3 <= 0
x1, x2, x3, x4 >= 0

Solutions:

In the given problem the objective function is MaxZ and it has only three variables.

The I constraint is of standard form already slack variable is introduced as x4 >= 0 and the value should be one but, it is having 3 due this it should be divided by three for enter equation on both sides and II & III constraint are of <= type so we introduce x5 >= 0 x6>= 0 as slack variable.

Then the given problem can be rewritten as

Max Z =107x1 + x2 +2 x3+ 0x4 +0 x5 +0 x6
STC
14/3x1 + 1/3 x2 – 6/3x3 +3/3x4 = 7/3
16x1 + 1/2x2 - 6x3+ x5 = 5
16x1 - 8x2 - x3 + x6 = 0
x1,x2,x3,x4 x5 x6 >= 0

The matrix form

Max Z= (107, 1, 2, 0, 0, 0) (x1, x2, x3, x4 ,x5, x6)

STC

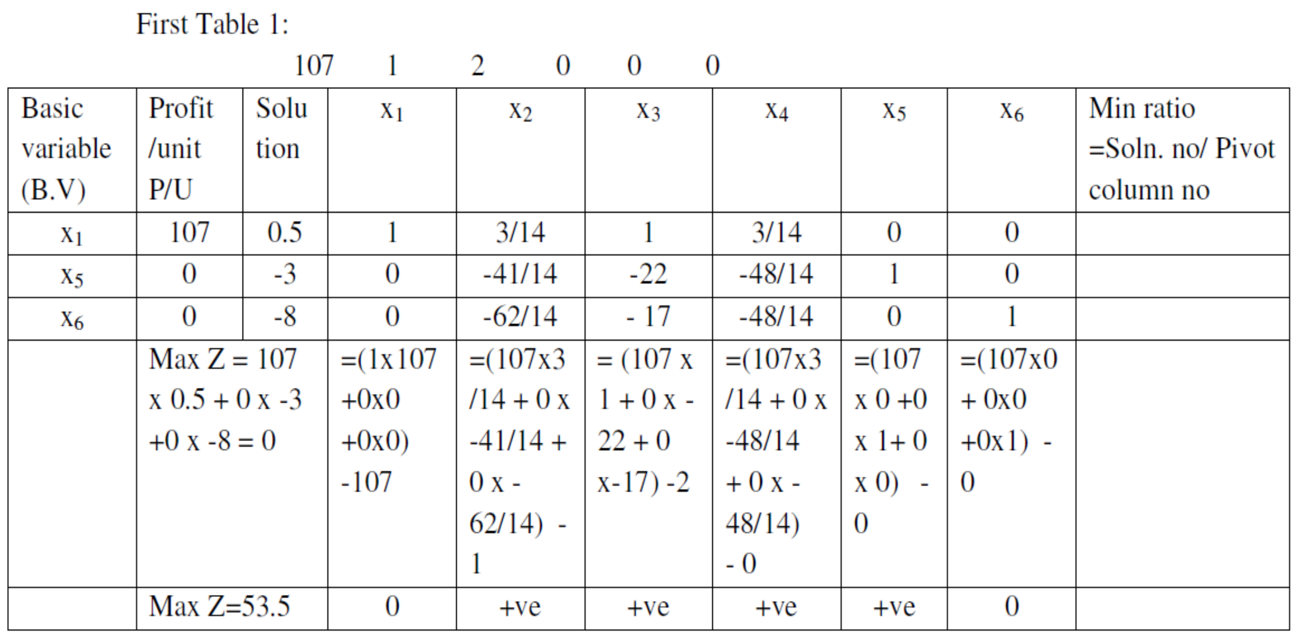
$$\begin{Bmatrix} 14/3 & 1/3 & -2 & 1 & 0 & 0 \\ 16 & 1/2 & -6 & 0 & 1 & 0 \\ 16 & -1 & -1 & 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} = \begin{Bmatrix} 7/3 \\ 5 \\ 0 \end{Bmatrix}$$

$$x_1, x_2, x_3, x_4\ x_5\ x_6 \ >= 0$$

$$\begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix}$$

Now, the canonical form from the above matrix is the corresponding variables are (x_4 x_5 x_6) and their obvious solution is (7/3 5 0)

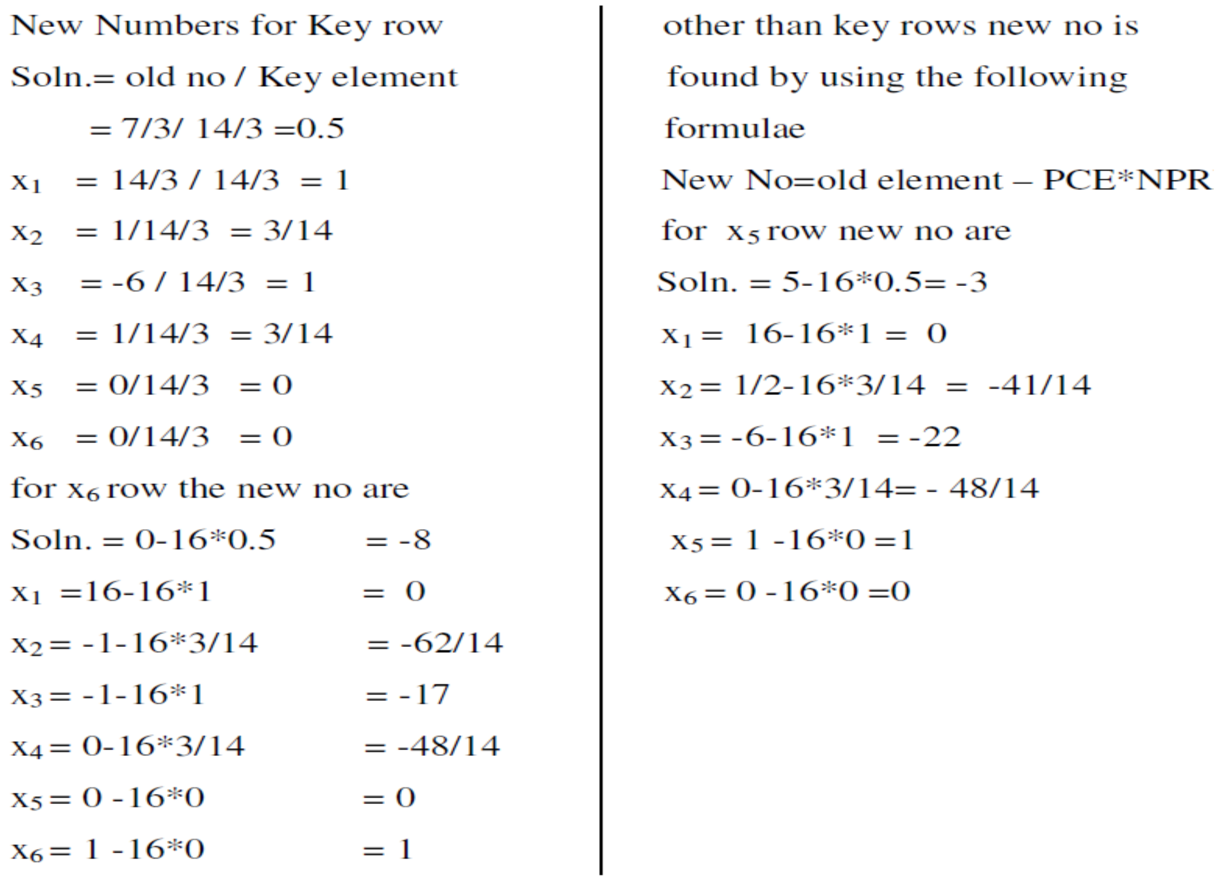
	Starting Table:		107	1	2	0	0	0	
Basic variable (B.V)	Profit /unit P/U	Solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratio =Soln. no/ Pivot column no
x_4	0	7/3	14/3	1	-6	1	0	0	=7/3 / 14/3=0.5
x_5	0	5	16	½	-6	0	1	0	=16/5= 3.2
x_6	0	0	16	-1	-1	0	0	1	=16/0=∞
	Max Z = 0 x 7/3 + 0 x 5 + 0 x 0 = 0		=(0x14/3 +0x16+0x 16) -107	=(0x1+0 x1/2+0x -1) -1	= (0 x-6 +0x- 6+0x-1) -2	=(0x1 +0x0 +0x0) - 0	=(0x0+ 0x1+0x 0)- 0	=(0x0+0 x0+0x1) - 0	
			-107	-1	-2	0	0	0	



First Table 1:

107 1 2 0 0 0

Basic variable (B.V)	Profit /unit P/U	Solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratio =Soln. no/ Pivot column no
x_1	107	0.5	1	3/14	1	3/14	0	0	
x_5	0	-3	0	-41/14	-22	-48/14	1	0	
x_6	0	-8	0	-62/14	- 17	-48/14	0	1	
	Max Z = 107 x 0.5 + 0 x -3 +0 x -8 = 0	=(1x107 +0x0 +0x0) -107	=(107x3 /14 + 0 x -41/14 + 0 x -62/14) - 1	= (107 x 1 + 0 x -22 + 0 x-17) -2	=(107x3 /14 + 0 x -48/14 + 0 x -48/14) - 0	=(107 x 0 +0 x 1+ 0 x 0) - 0	=(107x0 + 0x0 +0x1) - 0		
	Max Z=53.5	0	+ve	+ve	+ve	+ve	0		



New Numbers for Key row

Soln.= old no / Key element

= 7/3/ 14/3 =0.5

x_1 = 14/3 / 14/3 = 1

x_2 = 1/14/3 = 3/14

x_3 = -6 / 14/3 = 1

x_4 = 1/14/3 = 3/14

x_5 = 0/14/3 = 0

x_6 = 0/14/3 = 0

for x_6 row the new no are

Soln. = 0-16*0.5 = -8

x_1 =16-16*1 = 0

x_2 = -1-16*3/14 = -62/14

x_3 = -1-16*1 = -17

x_4 = 0-16*3/14 = -48/14

x_5 = 0 -16*0 = 0

x_6 = 1 -16*0 = 1

other than key rows new no is found by using the following formulae

New No=old element – PCE*NPR

for x_5 row new no are

Soln. = 5-16*0.5= -3

x_1 = 16-16*1 = 0

x_2 = 1/2-16*3/14 = -41/14

x_3 = -6-16*1 = -22

x_4 = 0-16*3/14= - 48/14

x_5 = 1 -16*0 =1

x_6 = 0 -16*0 =0

Since, all the NER is positive then given problem is optimal

Therefore, x1 = 0.5, x2= 0, x3= 0, x4= 0, x5= - 3, x6= - 8

Max Z = 107*0.5 + 1*0 +2*0+ 0*0 +0*-3 +0* -8

= 53.5

Tie Breaking in the SIMPLEX METHOD

Tie for the Entering Basic Variable

Step 1 of each iteration chooses the nonbasic variable having the *negative* coefficient with the *largest absolute value* in the current Eq. (0) as the entering basic variable. Now suppose that two or more nonbasic variables are tied for having the largest negative coefficient (in absolute terms). For example, this would occur in the first iteration for the Wyndor Glass Co. problem if its objective function were changed to $Z = 3x1 + 3x2$, so that the initial Eq. (0) became $Z - 3x1 - 3x2 = 0$. How should this tie be broken?

The answer is that the selection between these contenders may be made *arbitrarily*. The optimal solution will be reached eventually, regardless of the tied variable chosen, and there is no convenient method for predicting in advance which choice will lead there sooner. In this example, the simplex method happens to reach the optimal solution (2, 6) in three iterations with *x*1 as the initial entering basic variable, versus two iterations if *x*2 is chosen.

Tie for the Leaving Basic Variable—Degeneracy

Now suppose that two or more basic variables tie for being the leaving basic variable in step 2 of an iteration. Does it matter which one is chosen? Theoretically it does, and in a very critical way, because of the following sequence of events that could occur. First, all the tied basic variables reach zero simultaneously as the entering basic variable is increased.

Therefore, the one or ones *not* chosen to be the leaving basic variable also will have a value of zero in the new BF solution. (Note that basic variables with a value of *zero* are called **degenerate,** and the same term is applied to the corresponding BF solution.)

Second, if one of these degenerate basic variables retains its value of zero until it is chosen at a subsequent iteration to be a leaving basic variable, the corresponding entering basic variable also must remain zero (since it cannot be increased without making the leaving basic variable negative), so the value of Z must remain unchanged. Third, if Z may remain the same rather than increase at each iteration, the simplex method may then go around in a loop, repeating the same sequence of solutions periodically rather than eventually increasing Z toward an optimal solution. In fact, examples have been artificially constructed so that they do become entrapped in just such a perpetual loop.

Fortunately, although a perpetual loop is theoretically possible, it has rarely been known to occur in practical problems. If a loop were to occur, one could always get out of it by changing the choice of the leaving basic variable. Furthermore, special rules1 have been constructed for breaking ties so that such loops are always avoided. However, these rules frequently are ignored in actual application, and they will not be repeated here. For your purposes, just break this kind of tie arbitrarily and proceed without worrying about the degenerate basic variables that result.

No Leaving Basic Variable—Unbounded *Z*

In step 2 of an iteration, there is one other possible outcome that we have not yet discussed, namely, that *no* variable qualifies to be the leaving basic variable. This outcome would occur if the entering basic variable could be increased *indefinitely* without giving negative values to *any* of the current basic variables. In tabular form, this means that *every* coefficient in the pivot column (excluding row 0) is either negative or zero.

UNIT – 3

Simplex Method – 2

Adapting to other model forms

The only serious problem introduced by the other forms for functional constraints (the = or >= forms, or having a negative right-hand side) lies in identifying an *initial BF solution.* Before, this initial solution was found very conveniently by letting the slack variables be the initial basic variables, so that each one just equals the *nonnegative* right-hand side of its equation. Now, something else must be done. The standard approach that is used for all these cases is the **artificial-variable technique.** This technique constructs a more convenient *artificial problem* by introducing a dummy variable (called an *artificial variable*) into each constraint that needs one. This new variable is introduced just for the purpose of being the initial basic variable for that equation. The usual nonnegativity constraints are placed on these variables, and the objective function also is modified to impose an exorbitant penalty on their having values larger than zero. The iterations of the simplex method then automatically force the artificial variables to disappear (become zero), one at a time, until they are all gone, after which the *real* problem is solved.

To illustrate the artificial-variable technique, first we consider the case where the only nonstandard form in the problem is the presence of one or more equality constraints.

Equality Constraints and Functional Constraints in ≥Form

Any equality constraint

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

actually is equivalent to a pair of inequality constraints:

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \le b_i$$
$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \ge b_i.$$

However, rather than making this substitution and thereby increasing the number of constraints, it is more convenient to use the artificial-variable technique. We shall illustrate this technique with the following example.

Solve the given problem by simplex method

Max Z =107x1 + x2 +2 x3
STC
14x1 + x2 - 6x3 +3x4 = 7
16x1 + 1/2x2 -6x3 <= 5
16x1 - 8x2 - x3 <= 0
x1, x2, x3, x4 >= 0

Solutions:

In the given problem the objective function is MaxZ and it has only three variables.

The I constraint is of standard form already slack variable is introduced as x4 >= 0 and the value should be one but, it is having 3 due this it should be divided by three for enter equation on both sides and II & III constraint are of <= type so we introduce x5 >= 0 x6>= 0 as slack variable.

Then the given problem can be rewritten as

Max Z =107x1 + x2 +2 x3+ 0x4 +0 x5 +0 x6
STC
14/3x1 + 1/3 x2 – 6/3x3 +3/3x4 = 7/3
16x1 + 1/2x2 - 6x3+ x5 = 5
16x1 - 8x2 - x3 + x6 = 0
x1,x2,x3,x4 x5 x6 >= 0

The matrix form
Max Z= (107, 1, 2, 0, 0, 0) (x1, x2, x3, x4 ,x5, x6)
STC

$$\begin{Bmatrix} 14/3 & 1/3 & -2 & 1 & 0 & 0 \\ 16 & 1/2 & -6 & 0 & 1 & 0 \\ 16 & -1 & -1 & 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} = \begin{Bmatrix} 7/3 \\ 5 \\ 0 \end{Bmatrix}$$

$$x_1, x_2, x_3, x_4\ x_5\ x_6 \quad >= 0$$

$$\begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix}$$

Now, the canonical form from the above matrix is the corresponding variables are (x_4 x_5 x_6) and their obvious solution is (7/3 5 0)

	Starting Table:		107	1	2	0	0	0	
Basic variable (B.V)	Profit /unit P/U	Solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratio =Soln. no/ Pivot column no
x_4	0	7/3	14/3	1	-6	1	0	0	=7/3 / 14/3=0.5
x_5	0	5	16	½	-6	0	1	0	=16/5= 3.2
x_6	0	0	16	-1	-1	0	0	1	=16/0=∞
	Max Z = 0 x 7/3 + 0 x 5 + 0 x 0 = 0		=(0x14/3 +0x16+0x 16) -107	=(0x1+0 x1/2+0x -1) -1	= (0 x-6 +0x- 6+0x-1) -2	=(0x1 +0x0 +0x0) - 0	=(0x0+ 0x1+0x 0)- 0	=(0x0+0 x0+0x1) - 0	
			-107	-1	-2	0	0	0	

First Table 1:

107 1 2 0 0 0

Basic variable (B.V)	Profit /unit P/U	Solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratio =Soln. no/ Pivot column no
x_1	107	0.5	1	3/14	1	3/14	0	0	
x_5	0	-3	0	-41/14	-22	-48/14	1	0	
x_6	0	-8	0	-62/14	- 17	-48/14	0	1	
	Max Z = 107 x 0.5 + 0 x -3 +0 x -8 = 0		=(1x107 +0x0 +0x0) -107	=(107x3 /14 + 0 x -41/14 + 0 x -62/14) - 1	= (107 x 1 + 0 x -22 + 0 x-17) -2	=(107x3 /14 + 0 x -48/14 + 0 x -48/14) - 0	=(107 x 0 +0 x 1+ 0 x 0) - 0	=(107x0 + 0x0 +0x1) - 0	
	Max Z=53.5		0	+ve	+ve	+ve	+ve	0	

New Numbers for Key row

Soln.= old no / Key element

= 7/3/ 14/3 =0.5

x_1 = 14/3 / 14/3 = 1

x_2 = 1/14/3 = 3/14

x_3 = -6 / 14/3 = 1

x_4 = 1/14/3 = 3/14

x_5 = 0/14/3 = 0

x_6 = 0/14/3 = 0

for x_6 row the new no are

Soln. = 0-16*0.5 = -8

x_1 =16-16*1 = 0

x_2 = -1-16*3/14 = -62/14

x_3 = -1-16*1 = -17

x_4 = 0-16*3/14 = -48/14

x_5 = 0 -16*0 = 0

x_6 = 1 -16*0 = 1

other than key rows new no is found by using the following formulae

New No=old element – PCE*NPRE

for x_5 row new no are

Soln. = 5-16*0.5= -3

x_1 = 16-16*1 = 0

x_2 = 1/2-16*3/14 = -41/14

x_3 = -6-16*1 = -22

x_4 = 0-16*3/14= - 48/14

x_5 = 1 -16*0 =1

x_6 = 0 -16*0 =0

Since, all the NER is positive then given problem is optimal

Therefore, x1 = 0.5, x2= 0, x3= 0, x4= 0, x5= - 3, x6= - 8

Max Z = 107*0.5 + 1*0 +2*0+ 0*0 +0*-3 +0* -8

= 53.5

BIG M Method or Methods of Penalties

Whenever the objective function is MinZ and when all or some of the constraints are of >= type or = type. We introduce surplus variable and a artificial variable to LHS of the constraint when it is necessary to complete the identity matrix I.

The general practice is to assign the letter M as the cost in a minimization problem, and – M as the profit in the maximization problem with assumption that M is a very large positive number to the artificial variables in the objective function. The method of solving a LPP in which a high penalty cost has been assigned to the artificial variables is known as the method of penalties or BIG m Method.

<u>Procedures</u>

Step1: At any iteration of the usual simplex method can arise any one of the following three cases:

Case a) if there is no vector corresponding to some artificial variable in the solution column in such case, we proceed to step 2.

Case b) if at least one vector corresponding to some artificial variable, in the basis is basic variable column at the zero level i.e., corresponding entry in solution column is zero and the co-efficient of m in each net evaluation Zj - Cj is non negative.

In such case, the current basic feasible solution is a degenerate one.
If this is a case when an optimum solution. The given LPP includes an artificial basic variable and an optimum basic feasible solution does not exist.

Case c) if at least one artificial vector is in the basis Yb but, not at zero level i.e., the corresponding entry in Xb is non zero. Also co-efficient of M in each net evaluation Zj - Cj is non negative.

In the case, the given LPP does not possess an optimum basic feasible solution. Since, M is involved in the objective function. In such case, the given problem has a pseudo optimum basic feasible solution.

Step 2: application of simplex method is continued until either an optimum basic feasible solution is obtained or there is an indication of the existence of an unbounded solution to the given LPP.

Note: while applying simplex method, whenever a vector corresponding to some artificial variable happens to leave the basis, we drop that vector and omit all the entries corresponding to that vector from the simplex table.

Problem 1:

Solve the give LPP by BIG M Method

$\text{Max } Z = 3x1 - x2$

STC

$2x1 + x2 >= 2$

$x1 + 3x2 <= 3$

$8x2 <= 4$

$x1, x2 >= 0$

Since, the given problem is max z and the I constraint is >= type we introduce surplus variable as x3 >= 0, and a artificial variable as x4 >= 0, the II & III constraint are of <= type, so we introduce x5 >= 0 , x6 >= 0 .

The standard of the given LPP as follow as.

Max Z =3x1 - x2 +0x3 -M x4+0 x5+ 0x6

STC

2x1 + x2 - x3+ x4 = 2

x1 + 3x2 + x5 = 3

x2 + x6 = 4

x1,x2,x3, x4 x5,x6 >= 0

The Matrix form

Max Z= (3,-1, 0,-M, 0, 0) (x1,x2, x3, x4 x5,x6)

STC

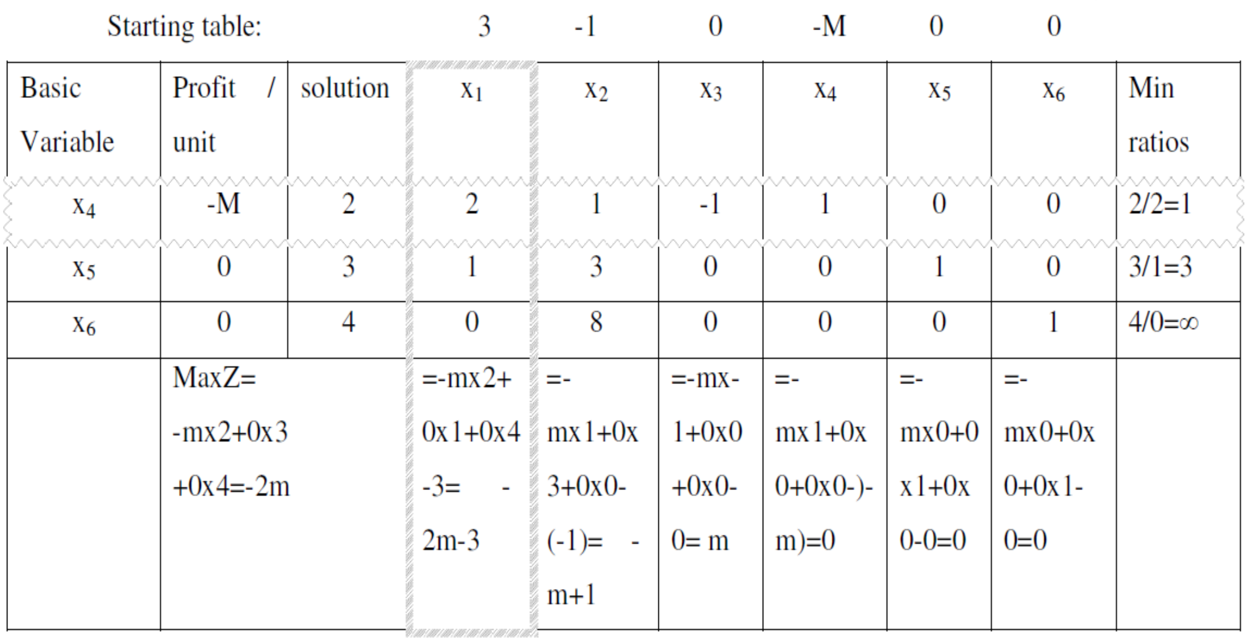
$$\begin{Bmatrix} 2 & 1 & -1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 3 \\ 4 \end{Bmatrix}$$

$x_1, x_2, x_3, x_4\ x_5, x_6 >= 0$

$$\begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix}$$

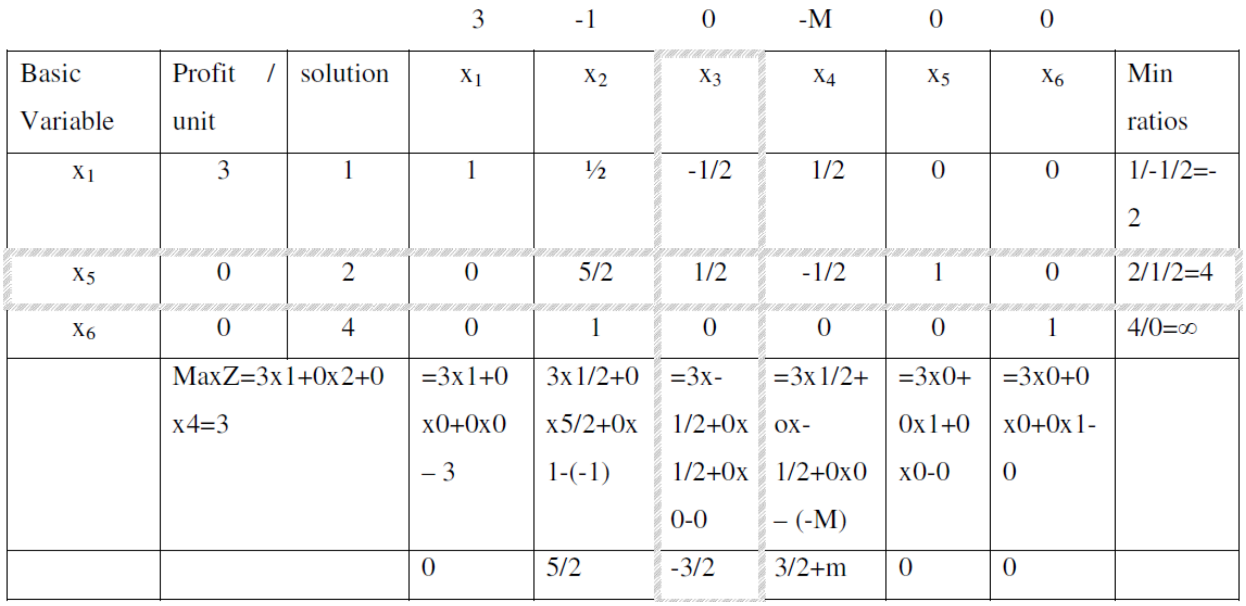
$(x_4\ x_5, x_6)$ canonical system.

The basic variable and their obvious solution



Starting table: 3 -1 0 -M 0 0

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratios
x_4	-M	2	2	1	-1	1	0	0	2/2=1
x_5	0	3	1	3	0	0	1	0	3/1=3
x_6	0	4	0	8	0	0	0	1	4/0=∞
	MaxZ= -mx2+0x3 +0x4=-2m		=-mx2+0x1+0x4 -3= -2m-3	=-mx1+0x3+0x0-(-1)= -m+1	=-mx-1+0x0+0x0-0= m	=-mx1+0x0+0x0-)-m)=0	=-mx0+0x1+0x0-0=0	=-mx0+0x0+0x1-0=0	



3 -1 0 -M 0 0

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratios
x_1	3	1	1	½	-1/2	1/2	0	0	1/-1/2=-2
x_5	0	2	0	5/2	1/2	-1/2	1	0	2/1/2=4
x_6	0	4	0	1	0	0	0	1	4/0=∞
	MaxZ=3x1+0x2+0x4=3		=3x1+0x0+0x0 – 3	3x1/2+0x5/2+0x1-(-1)	=3x-1/2+0x1/2+0x0-0	=3x1/2+ox-1/2+0x0 – (-M)	=3x0+0x1+0x0-0	=3x0+0x0+0x1-0	
			0	5/2	-3/2	3/2+m	0	0	

New Number for Key row is

New no for key row =old element/pivot element

Soln= 2/2=1, x1 =2/2=1, x2 =1/2, x3 = -1/2, x4 =1/2, x5 =0/2=0, x6 =0/2=0

Other than key row new no. = old element - PCE*NPRE

Soln= 3-1*1=2, x1 =1-1*1=0, x2 =3-1*1/2=5/2, x3 =0-1*-1/2=1/2, x4 =0-1*1/2=-1/2, x5

=1-1*0=1, x6 =0-1*0=0

For x6 new no are

Soln= 4-0*1=4, x1 =0-0*1=0, x2 =8-0*1/2=8, x3 =0-0*-1/2=0, x4 =0-0*1/2=0, x5 =0-0*0=0, x6 =1-0*0=1

3 -1 0 -M 0 0

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratios
x_1	3	3	1	3	0	0	1	0	
x_3	0	4	0	5	1	-1	2	0	
x_6	0	4	0	1	0	0	0	1	
	Max Z =3x3+0x4+0x4=9		=3x1+0 x0+0x0-3	=3x3+0 x5+0x1-(-1)	=3x0+ 0x1+0 x0-0	3x0+0x-1+0x0-(-M)	=3x1+ 0x2+0 x0-0	=0x3+0 x0+0x1-0	
			0	10	0	M	3	0	

New Number for Key row is

New no for key row =old element/pivot element (x3)

Soln= 2/1/2=4, x1 =0/1/2=0, x2 =5/2/1/2=5, x3 = 1/2/1/2=1, x4 =-1/2/1/2=-1, x5 =1/1/2=2, x6 =0/1/2=0

Other than key row new no. x1 row = old element – PCE *NPRE

Soln = 1-(-1/2)*4=3, x1 =1+1/2*0=1, x2 =1/2+1/2*5=3, x3 =-1/2-(-1/2)*1=0, x4 =1/2+1/2*-1=0, x5 =0+1/2*2=1, x6 =0-(-1/2)*0=0

For x6 new no are

Soln= 4-0*4=4, x1 =0-0*0=0, x2 =1-0*5=1, x3 =0-0*1=0, x4 =0-0*-1=0, x5 =0-0*0=0, x6 =1-0*0=1

Since all the NER is +ve and at zero level which is optimum.

This problem is of case A, which means no artificial vector appears at the optimal table and therefore, the given problem as attained the optimality.

x1=3,x2 =0, x3 =4, x4 =0, x5 =0x6 =4 substitute this values in the objection function.

Max Z =3x3 - 0 +0x4 –Mx0+0x0+ 0x4 = 9

Minimization

One straightforward way of minimizing *Z* with the simplex method is to exchange the roles of the positive and negative coefficients in row 0 for both the optimality test and step 1 of an iteration. However, rather than changing our instructions for the simplex method for this case, we present the following simple way of converting any minimization problem to an equivalent maximization problem:

i.e., the two formulations yield the same optimal solution(s).

The two formulations are equivalent because the smaller Z is, the larger -Z is, so the solution that gives the smallest value of Z in the entire feasible region must also give the largest value of -Z in this region.

Therefore, in the radiation therapy example, we make the following change in the formulation:

Solve the given LPP

Min $Z = 4x_1 + x_2$

STC

$3x_1 + 2x_2 \quad = \quad 3$

$4x_1 + 3x_2 \quad >= \quad 6$

$x_1 + 2x_2 \quad <= \quad 4$

$x_1, \ x_2 \quad >= 0$

The I constraint is already in standard form, only artificial variable is added as $x_3 >= 0$ and II constraint is of >= type we introduce

$x_4 > = 0$ as surplus variable and an artificial variable as $x_5 > = 0$ and third constraint is of <= type we introduce $x_6 > = 0$ as slack variable.

Then, the given LPP is in standard form

Min $Z = 4x_1 + x_2 + Mx_3 + 0x_4 + Mx_5 + 0x_6$ converting MinZ= -Z

or

Max $Z = -4x_1 - x_2 - Mx_3 - 0x_4 - Mx_5 - 0x_6$

STC

$3x_1 + 2x_2 + x_3 \quad = \quad 3$

$4x_1 + 3x_2 - x_4 + x_5 \quad = \quad 6$

$x_1 + 2x_2 + x_6 \quad = \quad 4$

$x_1, x_2, x_3, x_4, x_5, x_6 \quad > = \quad 0$

The matrix form

Max Z = (-4, -1, -M, 0, -M, 0) $(x_1, x_2, x_3, x_4, x_5, x_6)$

STC

$$\left\{\begin{matrix} 3 & 2 & 1 & 0 & 0 & 0 \\ 4 & 3 & 0 & -1 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{matrix}\right\} \left\{\begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{matrix}\right\} = \left\{\begin{matrix} 3 \\ 6 \\ 4 \end{matrix}\right\}$$

$$x_1, x_2, x_3, x_4, x_5, x_6 >= 0$$

Starting table: - 4 -1 -m 0 -m 0

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratios
X_3	-m	3	3	2	1	0	0	0	3/3=1
x_5	-m	6	4	3	0	-1	1	0	6/4=1.5
x_6	0	4	1	2	0	0	0	1	4/1
	Max z= -9m		-7m+4	-5m+1	0	m	0	0	

Starting table: - 4 -1 -m 0 -m 0

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratios
X_1	-4	1	1	2/3	1/3	0	0	0	3/2
x_5	-m	2	0	1/3	-4/3	-1	1	0	6
x_6	0	3	0	4/3	-1/3	0	0	1	9/4
	Max z= -4-2m		0	.-5-m /3	4m-4 /3	m	0	0	

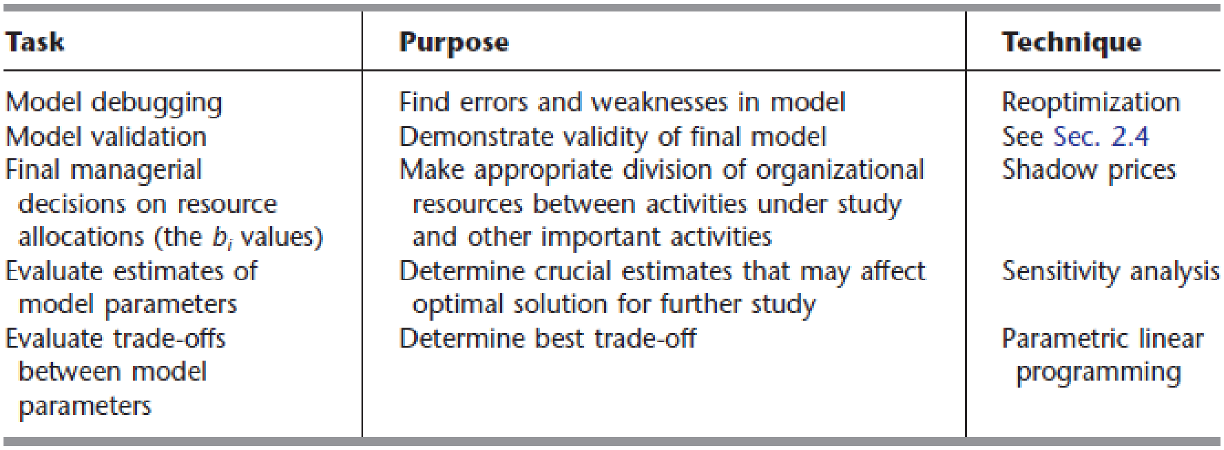
Starting table: - 4 -1 -m 0 -m 0

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratios
X_2	-1	3/2	3/2	1	1/2	0	0	0	
x_5	-m	3/2	-1/2	0	-2	-1	1	0	
x_6	0	1	-2	0	-3	0	0	1	
	Max z= -3/2-3/2m		5+m/2	0	-1/2+2m	m	0	0	

Post optimality analysis

postoptimality analysis—the analysis done *after* an optimal solution is obtained for the initial version of the model—constitutes a very major and very important part of most operations research studies. The fact that postoptimality analysis is very important is particularly true for typical linear programming applications.

In this section, we focus on the role of the simplex method in performing this analysis. Table summarizes the typical steps in postoptimality analysis for linear programming studies. The rightmost column identifies some algorithmic techniques that involve the simplex method. These techniques are introduced briefly here with the technical details deferred to later chapters.



Task	**Purpose**	**Technique**
Model debugging	Find errors and weaknesses in model	Reoptimization
Model validation	Demonstrate validity of final model	See Sec. 2.4
Final managerial decisions on resource allocations (the b_i values)	Make appropriate division of organizational resources between activities under study and other important activities	Shadow prices
Evaluate estimates of model parameters	Determine crucial estimates that may affect optimal solution for further study	Sensitivity analysis
Evaluate trade-offs between model parameters	Determine best trade-off	Parametric linear programming

TABLE: Postoptimality analysis for linear programming

UNIT 4

Simplex Method – 2, Duality Theory

Duality Theory

The linear programming model we develop for a situation is referred to as the **primal** problem. The dual problem can be derived directly from the primal problem.

The standard form has three properties

1. All the constraints are equations (with nonnegative right-hand side)
2. All the variables are nonnegative
3. The sense of optimization may be maximization of minimization.

Comparing the primal and the dual problems, we observe the following relationships.

1. The objective function coefficients of the primal problem have become the right hand side constants of the dual. Similarly, the right-hand side constants of the primal have become the cost coefficients of the dual
2. The inequalities have been reversed in the constraints
3. The objective is changed from maximization in primal to minimization in dual
4. Each column in the primal correspondence to a constraint (row) in the dual. Thus, the number of dual constraints is equal to the number of primal variables.
5. Each constraint (row) in the primal corresponds to a column in the dual. Hence, there is one dual variable for every primal constraints
6. The dual of the dual is the primal problem.
7. if the primal constraints >= the dual constraints will be <= & vice versa

Duality is an extremely important and interesting feature of linear programming. The various useful aspects of this property are

i) If the primal problem contains a large no of rows constraints and smaller no of columns variables computational procedure can be considerably reduced by converting it into dual and then solving it.

ii) It gives additional information as to how the optimal solution changes as a result of the changes in the coefficients and the formulation of its problem.

iii) Calculations of the dual checks the accuracy of the primal solution

iv) This indicates that fairly close relationships exist between LP and theory of games

Note: it is not necessary that only the Max problem be taken as the primal problem we can as well consider the minimization LPP as the primal

Formulation of dual to the primal problem

Problem no 1

Write the dual of the primal problem

Max Z = 3x1 +5x2
STC
2x1 +6x2 <= 50
3x1 +2x2 < = 35
5x1 - 3x2 <= 10
x2 <= 20
x1, x2, x3 >= 0

To write dual for the above primal, since the primal as 4 constraints the dual will have 4 variables as y1 y2 y3 y4 then dual for the primal will be as follows.

Min Z = 50y1 +35y2 +10y3+20y4
STC
2 y1 +3y2 +5y3+0y4 >=3
6y1 +2y2 -3y3+y4 > = 5
y1 ,y2 ,y3,y4 >= 0

it can be observed from the dual problem has less no constraint as compared to the primal problem (in case of primal they are 4 and in case of dual they are 2) which requires less work and effort to solve it

Problem 2

Construct the dual of the given problem

Min Z = 3x1 -2x2 + 4x3

STC

3x1 +5x2+4 x3 >=7

6x1 +x2+ 3x3 > = 4

7x1 -2 x3 -x3 < = 10

x1 -2x2+5 x3 >= 3

4x1 +7x2-2 x3 >= 2

x1, x2, x3 >= 0

Solution:

The given problem is minimization type and all the constraint should be >= type. In the given problem third constraint is <= type so we must convert constraint to >= type by multiply both side of the constraint by -1, we get

-7x1 +2 x3 +x3 > = - 10

Then given the problem can be written restated

Min Z = 3x1 -2x2 + 4x3

STC

3x1 +5x2+4 x3 >=7

6x1 +x2+ 3x3 > = 4

-7x1 +2 x2 +x3 > = -10

x1 -2x2+5 x3 >= 3

4x1 +7x2-2 x3 >= 2

x1, x2, x3 >= 0

Then dual of the given problem is as follows and the dual variables are y1 ,y2 ,y3,y4 y4

Max Z = 7y1 +4y2 -10y3+3y4 +2y5

STC

3 y1 +6y2 -7y3+y4+4y4 >= 3

5y1 +y2 +2y3-2y4 +7y4 > = -2

4y1 +3y2 +y3+5y4 -2y4 > = 4

y1 ,y2 ,y3,y4 ,y5 >= 0

Dual simplex method

Dual simplex method applies to problems which start with dual feasible solns. The objective function may be either in the maximization form or in the minimization form.

After introducing the slack variables, if any right-hand side element is –ve and if the optimality condition is satisfied.

The problem can be solved by the dual simplex method

Procedure for dual simplex method

Step1) obtain an initial basic solution to the LPP and put the solution in the starting dual simplex table

Step2) test the nature of Zj-Cj in the starting simplex table

a)if all Zj-Cj and solution column are non-negative for all I and j, then an optimum basic feasible solution has been obtained.

b) if all Zj-Cj are non negative and at least on basic variable in the solution column is negative go step 3

c) if at least one Zj-Cj is –ve the method is not applicable to the given problem

Step 3) selects the most –ve in solution column

step4) test the nature of

a) if all xij are non negative the given problem does not exist any feasible solution

b) if at least one xij is –ve, compute the ratios Zj-Cj / xij , xij >=0 chose the maximum of theses ratios

Step 5) test the new iterated dual simplex table for optimality

Repeat the procedure until either an optimum feasible solution has been obtained or there is an indication of the non existence of a feasible solution.

Use dual simplex method to solve the LPP

Min Z = x1 +x2

STC

2x1 + x2 >= 2

-x1 - x2 >=1

x1, x2 >= 0

Since, the given problem in min z form it is converted to max z and all the constraint is of >= type it should be converted to <= type by multiplying -1 on the both sides then given problem will be as follows

Max Z = -x1 -x2

STC

-2x1 - x2 <= -2

x1 + x2 <= -1

x1, x2 >= 0

Now, introducing slack variable for the I and II constraint as x3>= 0, x4 >= 0

The standard form as follows

Max Z = -x1 -x2 +0 x3 + 0x4

STC

-2x1 - x2+ x3 = -2

x1 + x2+ x4 = -1

x1, x2 , x3 x4 >= 0

The matrix form

Max Z = (-1,-1,0, 0) (x1, x2 , x3 x4)

STC

$$\begin{bmatrix} -2 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$x_1, x_2, x_3\ x_4 \quad >= 0$

the identity matrix form are $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ the correspond variables $(x_3\ x_4)$ and their solutions are -2 -1 starting table -1 -1 0 0

Basic Variable	Profit / Unit	solution	x_1	x_2	x_3	x_4
x_3	0	-2	-2	-1	1	0
x_4	0	-1	1	1	0	1
	Max Z= 0*-2+0*-1		=0*-2+0*1+1	=0*-1+0*1+1	=0*1+0*0 -0	=0*0+0*1-0
	0		1	1	0	0

$$\text{Ratios} = \text{NER} / X_{1j}$$

$$= \left[1/-2,\ 1/-1,\ 0/1,\ 0/0 \right]$$

The max negative in the ratios are 1/-2

First table -1 -1 0 0

Basic Variable	Profit / Unit	solution	x_1	x_2	x_3	x_4
x_1	-1	1	1	0.5	-1/2	0
x_4	0	-2	0	0.5	1/2	1
	Max Z= -1*1=0*-2		=-1*1+0*0+1	=0.5*-1+0.5*0+1	=-1*-1/2+0*1/2 -0	=-1 * 0 + 0*1-0
	-1		0	0.5	0	0

New no for key row are

Soln = -2/-2 = 1, -2/-2 =1, -1/-2=0.5, - 1/2,0/-2

New no for other than key rows is x4

Soln=-1-1*1=-2, 1-1*1=0, 1-1*0.5=0.5, 0-1*-1/2=1/2, 1-1*0=1

Since, all the values in NER is positive and in solution column one variable is –ve and that is selected as key row and to select key column or pivot column at least on variable in the row should be –ve but, no vector corresponding to that row is –ve and we cannot find the ratios and So we cannot select the key column and the given problem does not given any feasible solution to the LPP.

Use dual simplex table to solve the LPP

Min Z = x1 +2x2 +3 x3

STC

x1 - x2+ x3 >= 4

x1 + x2+ 2x3 <= 8

x2 - x3 >= 2

x1, x2 , x3 >= 0

the given problem is set for the requirement of dual simplex method

Max Z = -x1 -2x2 -3 x3

STC

-x1 + x2- x3 <= - 4

x1 + x2+ 2x3 <= 8

-x2 + x3 <= - 2

x1, x2 , x3 >= 0

since, all the constraint are of <= type we introduce three slack variable for I II and III constraint as x4>= 0, x5 >= 0, x6>= 0 the standard form

Max Z = -x1 -2x2 -3 x3 +0 x4+ 0x5+ 0x6

STC

-x1 + x2- x3 + x4 = - 4

x1 + x2+ 2x3 + x5 = 8

-x2 + x3 + x6 = - 2

x1, x2 , x3 x4 x5 x6 >= 0

The matrix form

Max Z (-1,-2,-3,0,0,0) (x1, x2 , x3 x4 x5 x6)

STC

$$\left\{\begin{matrix} -1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{matrix}\right\} \left\{\begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{matrix}\right\} = \left\{\begin{matrix} -4 \\ 8 \\ -2 \end{matrix}\right\} \quad \left\{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right\}$$

The IBFS are $x_4\ x_5\ x_6$

$x_1, x_2, x_3\ x_4\ x_5\ x_6 >= 0$

and their obvious soln (-4,8,-2)

Starting table -1 -2 -3 0 0 0

Basic Variable	Profit / unit	soln	x_1	x_2	x_3	x_4	x_5	x_6
x_4	0	-4	-1	1	-1	1	0	0
x_5	0	8	1	1	2	0	1	0
x_6	0	-2	0	-1	1	0	0	1
	MaxZ =0*-4+0*8+0*-8		1	2	3	0	0	0

Other than key row new nos are

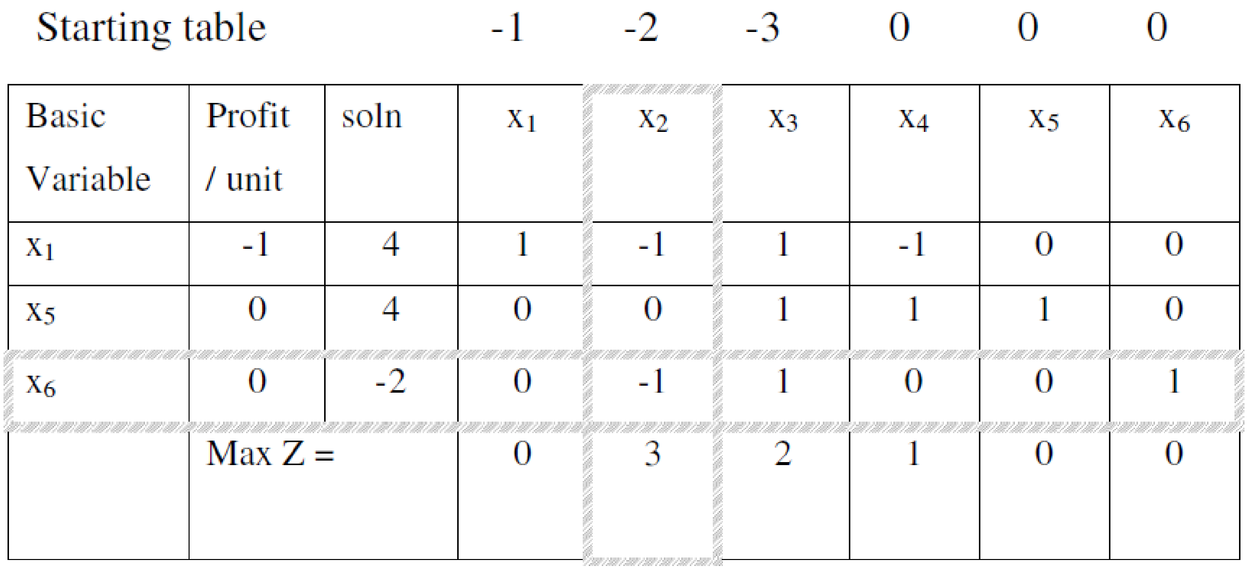
x5

soln = 8-1*4=4, x1 =1-1*1=0, x2 =1-1*-1=0, x3 =2-1*1=1, x4 =0-1*-1=1

x5 = 1-1*0=1, x6 =0-1*0=0

x6

soln= -2-0*4=-2, all other value in the row remains the same because of corresponding value of pivot column element is zero



Starting table -1 -2 -3 0 0 0

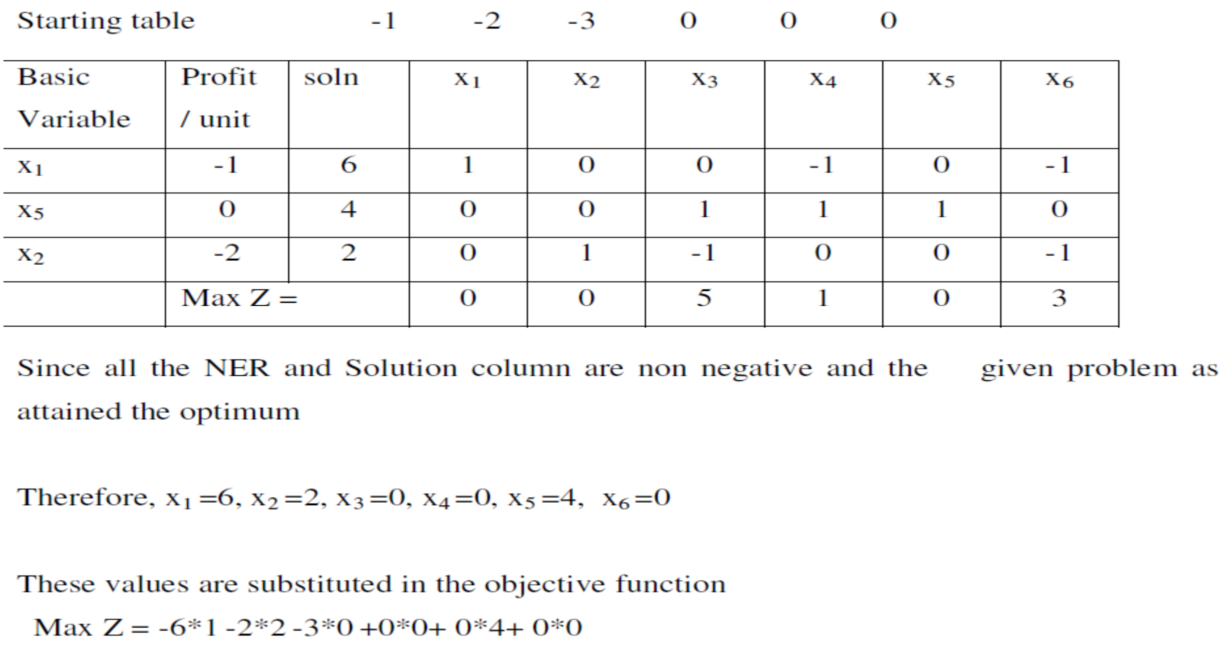
Basic Variable	Profit / unit	soln	x_1	x_2	x_3	x_4	x_5	x_6
x_1	-1	4	1	-1	1	-1	0	0
x_5	0	4	0	0	1	1	1	0
x_6	0	-2	0	-1	1	0	0	1
	Max Z =		0	3	2	1	0	0

For x1 row

soln=4-(-1)*2=6, x1 =1-(-1)*0=1, x2 =-1-(-1)1=0, x3 =1-(-1)-1=0, x4 =-1-(-1)*0=-1, x5 =0-(-1)*0=0, x6 =0-(-1)*-1=-1

For x5 row

soln =4 - 0*2=4, any thing multiplied by 0 is 0 so the value in this row remains same.



Starting table -1 -2 -3 0 0 0

Basic Variable	Profit / unit	soln	x_1	x_2	x_3	x_4	x_5	x_6
x_1	-1	6	1	0	0	-1	0	-1
x_5	0	4	0	0	1	1	1	0
x_2	-2	2	0	1	-1	0	0	-1
	Max Z =		0	0	5	1	0	3

Since all the NER and Solution column are non negative and the given problem as attained the optimum

Therefore, $x_1=6$, $x_2=2$, $x_3=0$, $x_4=0$, $x_5=4$, $x_6=0$

These values are substituted in the objective function

Max Z = -6*1 -2*2 -3*0 +0*0+ 0*4+ 0*0

UNIT 5

Duality Theory and Sensitivity Analysis, Other Algorithms for LP

The Essence of Duality Theory

Given our standard form for the *primal problem* at the left (perhaps after conversion from another form), its *dual problem* has the form shown to the right.

Thus, the dual problem uses exactly the same *parameters* as the primal problem, but in different locations. To highlight the comparison, now look at these same two problems in matrix notation, where c and y _ [*y*1, *y*2, . . . , *ym*] are row vectors but b and x are column vectors.

To illustrate, the primal and dual problems for the Wyndor Glass Co. example are shown in Table in both algebraic and matrix form. The primal-dual table for linear programming also helps to highlight the correspondence between the two problems. It shows all the linear programming parameters (the *aij*, *bi*, and *cj*) and how they are used to construct the two problems. All the headings for the primal problem are horizontal, whereas the headings for the dual problem are read by turning the book sideways. For the primal problem, each *column* (except the Right

Side column) gives the coefficients of a single variable in the respective constraints and then in the objective function, whereas each *row* (except the bottom one) gives the parameters for a single contraint. For the dual problem, each *row* (except the Right Side row) gives the coefficients of a single variable in the respective constraints and then in the objective function, whereas each *column* (except the rightmost one) gives the parameters for a single constraint. In addition, the Right Side column gives the right-hand sides for the primal problem and the objective function coefficients for the dual problem, whereas the bottom row gives the objective function coefficients for the primal problem and the right hand sides for the dual problem.

Consequently, (1) the parameters for a *constraint* in either problem are the coefficients of a *variable* in the other problem and (2) the coefficients for the *objective function* of either problem are the *right sides* for the other problem. Thus, there is a direct correspondence between these entities in the two problems, as summarized in Table. These correspondences are a key to some of the applications of duality theory, including sensitivity analysis.

The role of duality in sensitive analysis

Changes in the Coefficients of a Non basic Variable

Suppose that the changes made in the original model occur in the coefficients of a variable that was nonbasic in the original optimal solution. What is the effect of these changes on this solution? Is it still feasible? Is it still optimal?

Because the variable involved is nonbasic (value of zero), changing its coefficients cannot affect the feasibility of the solution. Therefore, the open question in this case is whether it is still optimal. As Tables and indicate, an equivalent question is whether the complementary basic solution for the dual problem is still feasible after these changes are made. Since these changes affect the dual problem by changing only one constraint, this question can be answered simply by checking whether this complementary basic solution still satisfies this revised constraint. We shall illustrate this case in the corresponding subsection of after developing a relevant example.

Introduction of a New Variable

As indicated in Table 6.6, the decision variables in the model typically represent the levels of the various activities under consideration. In some situations, these activities were selected from a larger group of *possible* activities, where the remaining activities were not included in the original model because they seemed less attractive. Or perhaps these other activities did not come to light until after the original model was formulated and solved.

Either way, the key question is whether any of these previously unconsidered activities are sufficiently worthwhile to warrant initiation. In other words, would adding any of these activities to the model change the original optimal solution?

Adding another activity amounts to introducing a new variable, with the appropriate coefficients in the functional constraints and objective function, into the model. The only resulting change in the dual problem is to add a *new constraint* (see Table 6.3).

After these changes are made, would the original optimal solution, along with the new variable equal to zero (non basic), still be optimal for the primal problem? As for the preceding case, an equivalent question is whether the complementary basic solution for the dual problem is still feasible. And, as before, this question can be answered simply by checking whether this complementary basic solution satisfies one constraint, which in this case is the new constraint for the dual problem.

The essence of sensitivity analysis

The work of the operations research team usually is not even nearly done when the simplex method has been successfully applied to identify an optimal solution for the model.
As we pointed out at the end, one assumption of linear programming is that all the parameters of the model (*aij*, *bi*, and *cj*) are *known constants.* Actually, the parameter values used in the model normally are just *estimates* based on a *prediction of future conditions.* The data obtained to develop these estimates often are rather crude or non- existent, so that the parameters in the original formulation may represent little more than quick rules of thumb provided by harassed line personnel. The data may even represent deliberate overestimates or underestimates to protect the interests of the estimators.

Thus, the successful manager and operations research staff will maintain a healthy skepticism about the original numbers coming out of the computer and will view them in many cases as only a starting point for further analysis of the problem. An “optimal” solution is optimal only with respect to the specific model being used to represent the real problem, and such a solution becomes a reliable guide for action only after it has been verified as performing well for other reasonable representations of the problem. Furthermore, the model parameters (particularly *bi*) sometimes are set as a result of managerial policy decisions (e.g., the amount of certain resources to be made available to the activities), and these decisions should be reviewed after their potential consequences are recognized.

For these reasons it is important to perform sensitivity analysis to investigate the effect on the optimal solution provided by the simplex method if the parameters take on other possible values. Usually there will be some parameters that can be assigned any reasonable value without the optimality of this solution being affected. However, there may also be parameters with likely alternative values that would yield a new optimal solution.

This situation is particularly serious if the original solution would then have a substantially inferior value of the objective function, or perhaps even be infeasible!

Therefore, one main purpose of sensitivity analysis is to identify the sensitive parameters

(i.e., the parameters whose values cannot be changed without changing the optimal solution). For certain parameters that are not categorized as sensitive, it is also very helpful to determine the *range of values* of the parameter over which the optimal solution will remain unchanged. (We call this range of values the *allowable range to stay optimal.*) In some cases, changing a parameter value can affect the *feasibility* of the optimal BF solution.

For such parameters, it is useful to determine the range of values over which the optimal BF solution (with adjusted values for the basic variables) will remain feasible. (We call this range of values the *allowable range to stay feasible.*) In the next section, we will describe the specific procedures for obtaining this kind of information. Such information is invaluable in two ways. First, it identifies the more important parameters, so that special care can be taken to estimate them closely and to select a solution that performs well for most of their likely values. Second, it identifies the parameters that will need to be monitored particularly closely as the study is implemented. If it is discovered that the true value of a parameter lies outside its allowable range, this immediately signals a need to change the solution.

For small problems, it would be straightforward to check the effect of a variety of changes in parameter values simply by reapplying the simplex method each time to see if the optimal solution changes. This is particularly convenient when using a spreadsheet formulation. Once the Solver has been set up to obtain an optimal solution, all you have to do is make any desired change on the spreadsheet and then click on the Solve button again.

However, for larger problems of the size typically encountered in practice, sensitivity analysis would require an exorbitant computational effort if it were necessary to reapply the simplex method from the beginning to investigate each new change in a parameter value. Fortunately, the fundamental insight discussed in Sec. 5.3 virtually eliminates computational effort. The basic idea is that the fundamental insight *immediately* reveals just how any changes in the original model would change the numbers in the final simplex tableau (assuming that the *same* sequence of algebraic operations originally performed by the simplex method were to be *duplicated*). Therefore, after making a few simple calculations to revise this tableau, we can check easily whether the original optimal

BF solution is now non optimal (or infeasible). If so, this solution would be used as the initial basic solution to restart the simplex method (or dual simplex method) to find the new optimal solution, if desired. If the changes in the model are not major, only a very few iterations should be required to reach the new optimal solution from this "advanced" initial basic solution.

Applying sensitivity analysis

Sensitivity analysis often begins with the investigation of changes in the values of *bi*, the amount of resource *i* (*i* _ 1, 2, . . . , *m*) being made available for the activities under consideration.
The reason is that there generally is more flexibility in setting and adjusting these values than there is for the other parameters of the model.

Case 1—Changes in *bi*

Suppose that the only changes in the current model are that one or more of the *bi* parameters (*i* _ 1, 2, . . . , *m*) has been changed. In this case, the *only* resulting changes in the final simplex tableau are in the *right-side* column. Consequently, the tableau still will be in proper form from Gaussian elimination and all the nonbasic variable coefficients in row 0 still will be nonnegative. Therefore, both the *conversion to proper form from Gaussian elimination* and the *optimality test* steps of the general procedure can be skipped. After revising the right-side column of the tableau, the only question will be whether all the basic variable values in this column still are nonnegative (the feasibility test).

Case 2*a*—Changes in the Coefficients of a Nonbasic Variable

Consider a particular variable *xj* (fixed *j*) that is a nonbasic variable in the optimal solution shown by the final simplex tableau. In Case 2*a*, the only change in the current model is that one or more of the coefficients of this variable—*cj*, *a*1*j*, *a*2*j* , . . . , *amj*—have been changed. Thus, letting *c_j* and *a_ij* denote the new values of these parameters, with **A**_*j* (column *j* of matrix **A**_) as the vector containing the *a_ij*, we have

cj □→ *c_j*, **A***j* □→ **A**_*j* for the revised model.

As described at the beginning, duality theory provides a very convenient way of checking these changes. In particular, if the *complementary* basic solution **y*** in the dual problem still satisfies the single dual constraint that has changed, then the original optimal solution in the primal problem *remains optimal* as is. Conversely, if **y*** violates this dual constraint, then this primal solution is *no longer optimal*.

UNIT 6

Transportation and Assignment Problems

Transportation Problems

Introduction:

The objective of the transportation problem is to transport various quantities of a single homogenous commodity, which are initially stored at various origins to various destinations in such a way that the total transportation cost is minimum.

Definitions:

Basic Feasible solution: A feasible solution to a m-origin, n-destination problem is said to be basic if the number of positive allocations are equal to (m+n-1).

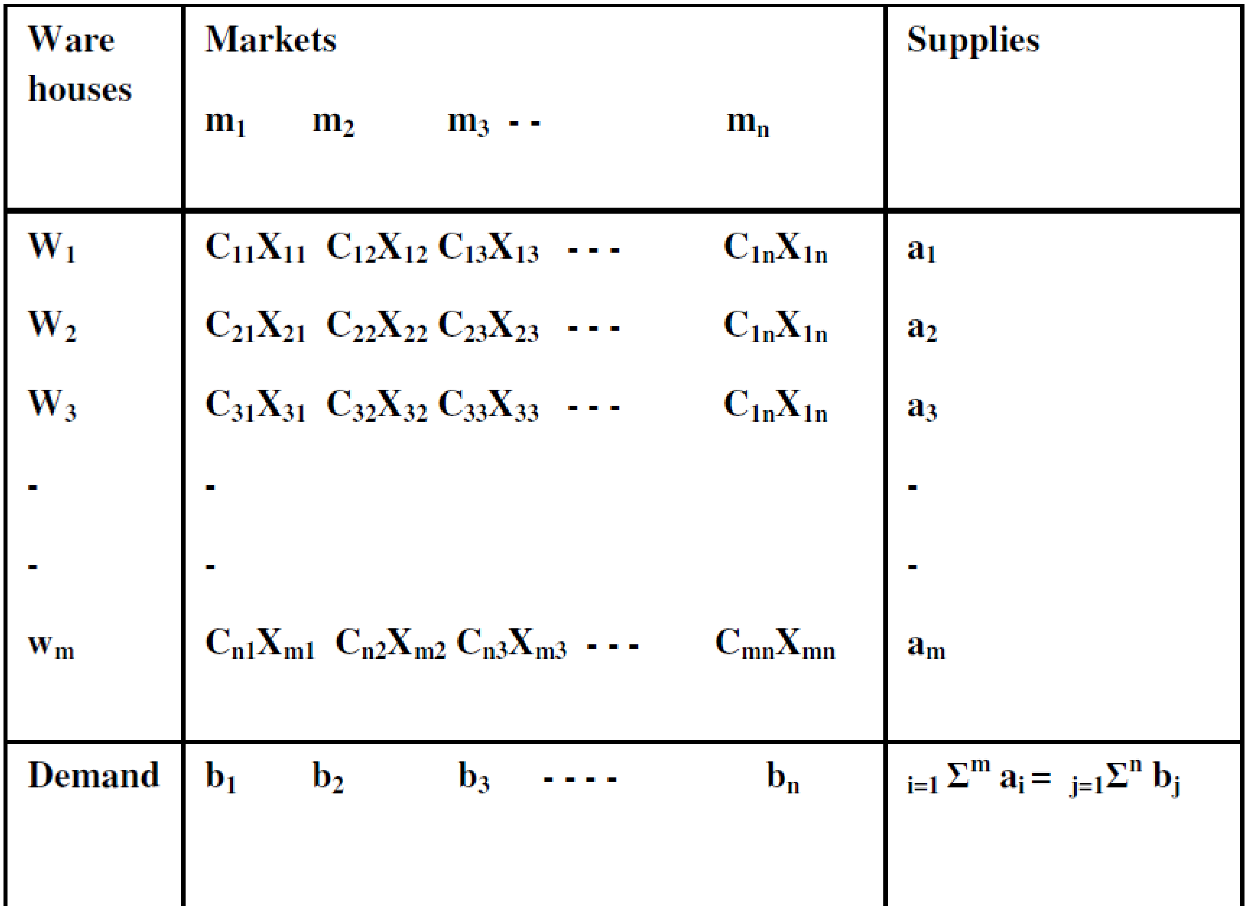
Feasible Solution: A set of positive individual allocations which simultaneously removes deficiencies is called a feasible solution.

Optimal Solution: A feasible solution (not basically basic) is said to be optimal if it minimises the total transportation cost.

Mathematical Formulation of Transportation Problems

- Suppose there are ‘m’ ware houses (w1,w2,w3, _, _, wm), where the commodity is stocked and ‘n’ markets where it is needed.
- Let the supply available in wear houses be a1, a2, a3, _,_,_ am and
- The demands at the markets (m1, m2, m3, _, _, mn) be b1, b2, b3, _, _ , _ bn.
- The unit cost of shipping from ware house i to a market j is Cij (C11,C12,_, _ Cn),
- Let X11, X12,X13,_, _, Xmn be the distances from warehouse to the markets

- we want to find an optimum shipping schedule which minimises the total cost of transportation from all warehouses to all the markets



Ware houses	Markets					Supplies
	m_1	m_2	m_3	- -	m_n	
W_1	$C_{11}X_{11}$	$C_{12}X_{12}$	$C_{13}X_{13}$	- - -	$C_{1n}X_{1n}$	a_1
W_2	$C_{21}X_{21}$	$C_{22}X_{22}$	$C_{23}X_{23}$	- - -	$C_{1n}X_{1n}$	a_2
W_3	$C_{31}X_{31}$	$C_{32}X_{32}$	$C_{33}X_{33}$	- - -	$C_{1n}X_{1n}$	a_3
-	-					-
-	-					-
w_m	$C_{n1}X_{m1}$	$C_{n2}X_{m2}$	$C_{n3}X_{m3}$	- - -	$C_{mn}X_{mn}$	a_m
Demand	b_1	b_2	b_3	- - - -	b_n	$_{i=1}\Sigma^{m} a_i = {}_{j=1}\Sigma^{n} b_j$

The total minimum transportation cost is

Z = i=1 _m j=1_n Xij * Cij i.e. Z = X11C11 + X12C12 + _ _ _ + XmnCmn

Types of Transportation Problems

1. Minimisation Balanced Transportation Problems
2. Minimisation Unbalanced Transportation Problems
3. Maximisation Balanced Transportation Problems
4. Maximisation unbalanced Transportation Problems
5. All the above models with degeneracy.

Ware

houses

Markets

m1 m2 m3 - - mn

Supplies

W1

W2

W3

-

-

wm

C11X11 C12X12 C13X13 - - - C1nX1n

C21X21 C22X22 C23X23 - - - C1nX1n

C31X31 C32X32 C33X33 - - - C1nX1n

-

-

Cn1Xm1 Cn2Xm2 Cn3Xm3 - - - CmnXmn

a1

a2

a3

-

-

am

Demand b1 b2 b3 - - - - bn i=1 _m ai = j=1_n bj

Methods of solving Transportation Problems

1. North- West Corner Rule method
2. Row-minima Method
3. Column minima method
4. Matrix Minima Method or least cost method
5. Vogel's Approximation method (VAM)

Methods for checking Optimality

1. Modified Distribution Method, UV or MODI method

PROBLEMS:

1. Solve the following transportation problem by North-West corner rule, Row Minima, Column Minima, Matrix Minima and VAM Method:

Factories	W1	W2	W3	W4	Supply
F1	6	4	1	5	14
F2	8	9	2	7	16
F3	4	3	6	2	05
Demand	6	10	15	4	35

Solution:

This is a balanced transportation problem, since supply is equal to demand

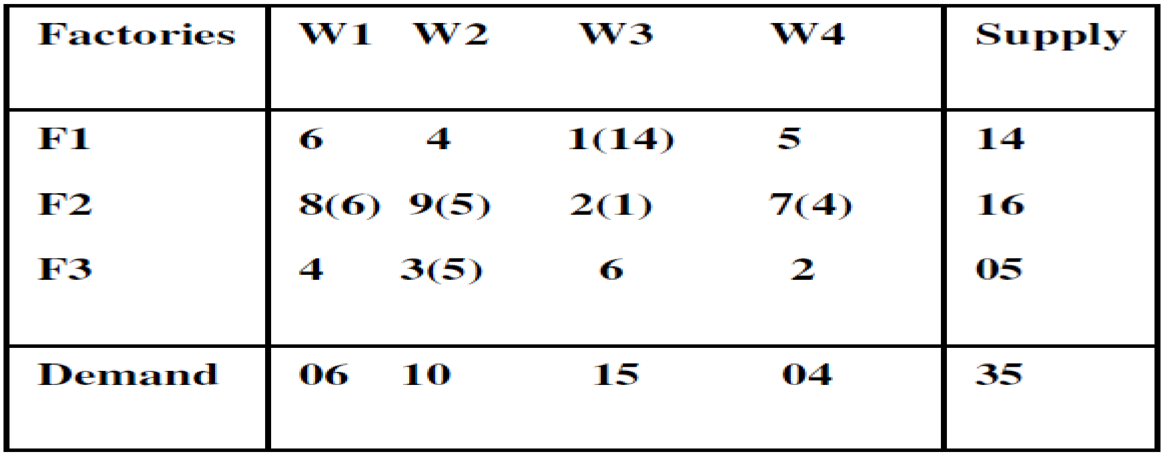
North-West corner rule Method:

Factories	W1	W2	W3	W4	Supply
F1	6(6)	4(8)	1	5	14
F2	8	9(2)	2(14)	7	16
F3	4	3	6(1)	2(4)	05
Demand	06	10	15	04	35

The Total feasible transportation cost

= 6(6) + 4(8) + 9(2) + 2(14) + 6(1) + 2(4) = Rs. 128/-

Row Minima Method:



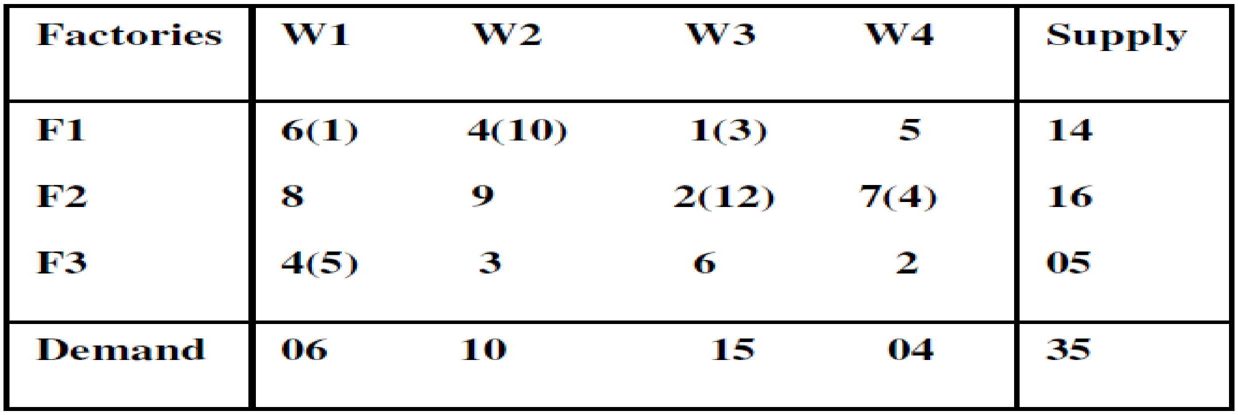
Factories	W1	W2	W3	W4	Supply
F1	6	4	1(14)	5	14
F2	8(6)	9(5)	2(1)	7(4)	16
F3	4	3(5)	6	2	05
Demand	06	10	15	04	35

The Total feasible transportation cost

= 1(14)+8(6)+9(5)+2(1)+7(4)+3(5)

= Rs.152/-

Column-Minima Method



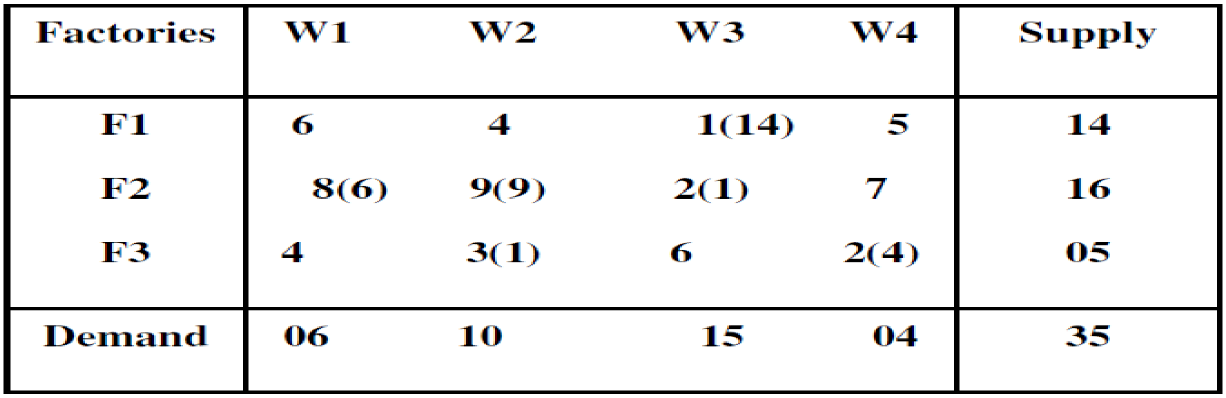
Factories	W1	W2	W3	W4	Supply
F1	6(1)	4(10)	1(3)	5	14
F2	8	9	2(12)	7(4)	16
F3	4(5)	3	6	2	05
Demand	06	10	15	04	35

The Total feasible transportation cost

= 6(1)+4(10)+1(3)+2(12)+7(4)+4(5)

= Rs. 121/-

Matrix-Minima Method or Least Cost method



Factories	W1	W2	W3	W4	Supply
F1	6	4	1(14)	5	14
F2	8(6)	9(9)	2(1)	7	16
F3	4	3(1)	6	2(4)	05
Demand	06	10	15	04	35

The Total feasible transportation cost

= 1(14)+8(6)+9(9)+2(1)+3(1)+2(4)

= 156/-

VAM- Vogel's approximation method:

Step-I: Against each row and column of the matrix, denote the difference between the two least cost in that particular row and column.

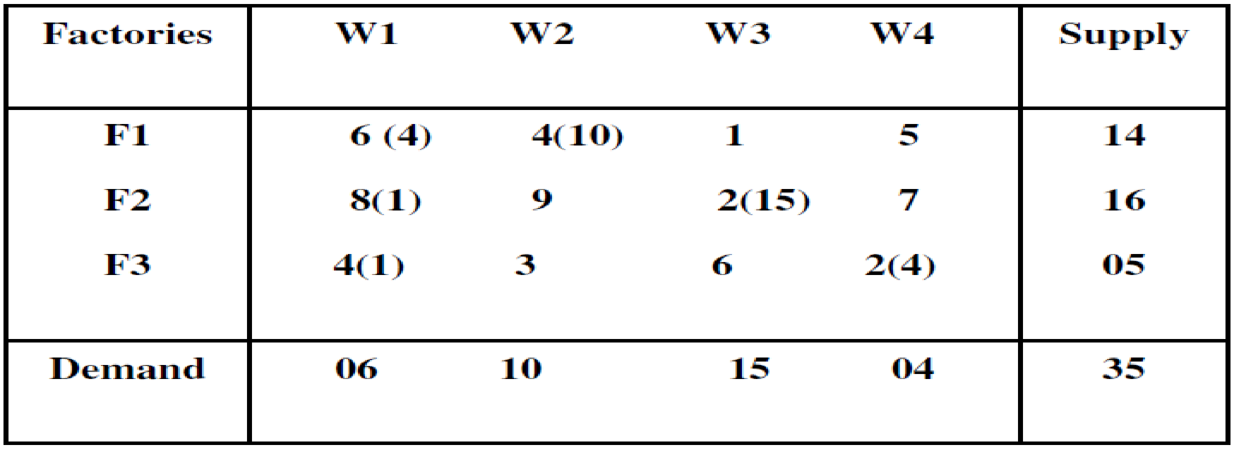
Step-II: Select the maximum value noted as per step-I, in this row or column select the cell which has the least cost

Step-III: Allocate the maximum possible quantity

Step-IV: After fulfilling the requirements of that particular row or column, Ignore that particular row or column and recalculate the difference by the two lowest cost for each of the remaining rows or columns, Again select the maximum of these differences and allocate the maximum possible quantity in the cell with the lowest cost in that particular / corresponding row or column.

Step-V: Repeat the procedure till the initial allocation is completed

VAM- Vogel's approximation method



Factories	W1	W2	W3	W4	Supply
F1	6 (4)	4(10)	1	5	14
F2	8(1)	9	2(15)	7	16
F3	4(1)	3	6	2(4)	05
Demand	06	10	15	04	35

The Total feasible transportation cost

= 6(4) + 4(10) + 8(1) + 2(15) + 4 (1) + 2(4)

= 114/-

II- Check for degeneracy:

If (m+n-1) is not equal to the number of allocated cells, then it is called degeneracy in transportation problems,

Where m= number of rows, n= number of columns.

This will occur if the source and destination is satisfied simultaneously.

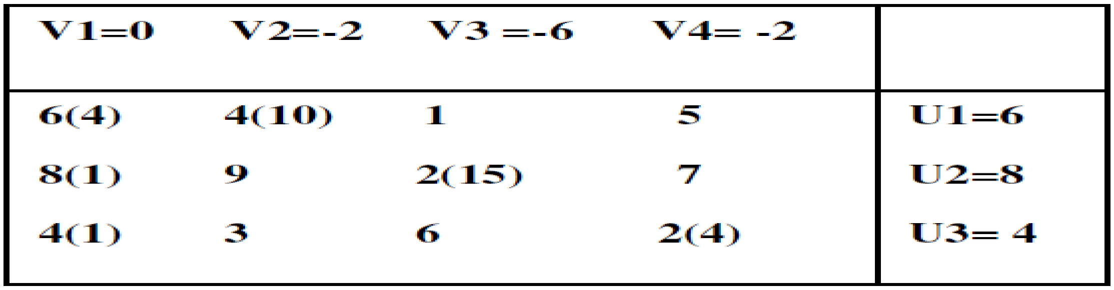
The degeneracy can be avoided by introducing a dummy allocation cell. To equate the number of allocated cells equal to (m+n-1)

For the above problem (m+n-1) = (3+4-1) =6 = Number of allocations = 6

Hence there is no degeneracy.

III- Checking optimality using MODI method:

- For allocated cells Cij – (Ui +Vj) = 0
- For unallocated cells Cij – (Uij) +Vj) _ 0



V1=0	V2=-2	V3 =-6	V4= -2	
6(4)	4(10)	1	5	U1=6
8(1)	9	2(15)	7	U2=8
4(1)	3	6	2(4)	U3= 4

The Total feasible transportation cost

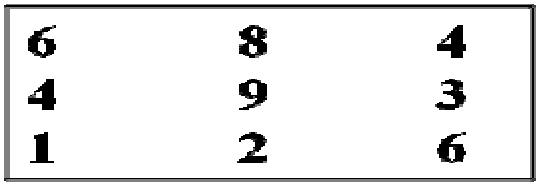
= 6(4)+4(10)+8(1)+2(15)+4(1)+2(4)

= 114/-

Problem.2

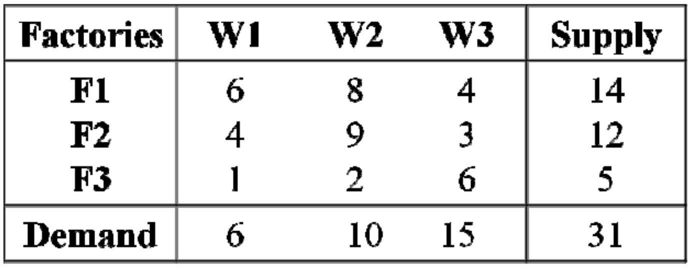
There are 3 Parties who supply the following quantity of coal P1= 14t, P2=12t, P3= 5t.

There are 3 consumers who require the coal as follows C1=6t, C2=10t, C3=15t. The cost matrix in Rs. Per ton is as follows. Find the schedule of transportation policy which minimises the cost:



6	8	4
4	9	3
1	2	6

Solution:



Factories	W1	W2	W3	Supply
F1	6	8	4	14
F2	4	9	3	12
F3	1	2	6	5
Demand	6	10	15	31

Factories	W1	W2	W3	Supply
F1	6	8(5)	4(9)	14
F2	4(6)	9	3(6)	12
F3	1	2(5)	6	05
Demand	6	10	15	31

Therefore the total feasible transportation cost

= 8(5)+ 4(9) + 4(6) +3 (6) + 2(5)

= Rs. 128/-

II. Check for Degeneracy:

(m+n-1) = (3+3-1) = 5 = Number of allocations

Hence there is no degeneracy

III- Checking optimality using MODI method:

- **For allocated cells C_{ij} – (U_i +V_j) = 0**
- **For unallocated cells C_{ij} – (U_{ij}) +V_j) _ 0**

V1=5	V2=8	V3=4	
6	8(5)	4(9)	U1= 0
4(6)	9	3(6)	U2=-1
1	2(5)	6	U3=-6

Since all the marginal costs for the unallocated cells are positive, it gives an optimal solution and the total minimum transportation cost = Rs. 128/-

ASSIGNMENT PROBLEM

The assignment problem is a special case of transportation problem in which the objective is to assign ‘m’ jobs or workers to ‘n’ machines such that the cost incurred is minimized.

The element Cij represents the cost of assigning worker I to job (I,j= 1,2,---n). There is no loss in generality in assuming that the number of workers always equals the number of jobs because we can always add fictitious (untrue or fabricated) workers or fictitious jobs to effect this result.

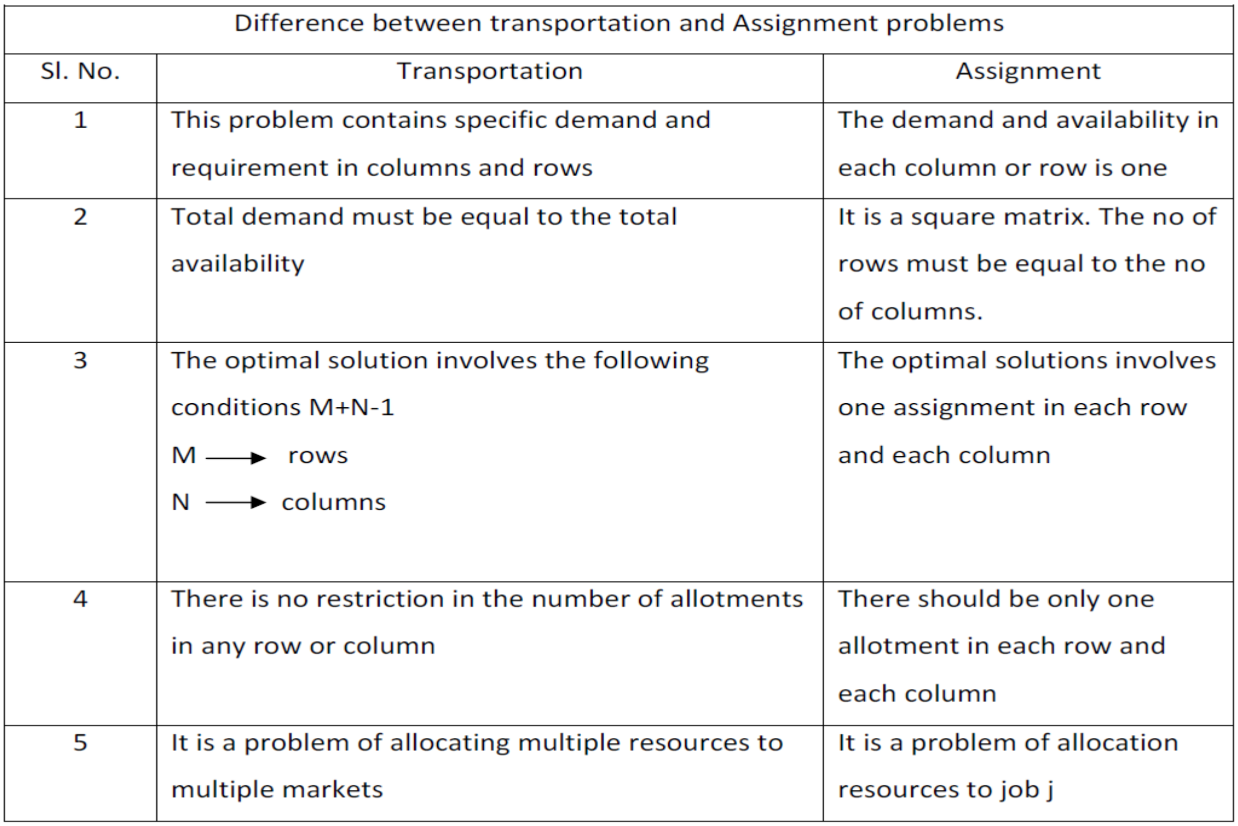
The assignment model is actually a special case of the transportation model in which the workers represent the sources and the jobs represent the destinations.

The supply amount at each source and the demand amount at each destination exactly equal 1.

The cost of transporting workers I to job j is Cij .

The assignment model can be solved directly as a regular transportation model.

The fact that all the supply and demand amounts equal 1 has led to the development of a simple solution algorithm called the Hungarian method.



Difference between transportation and Assignment problems		
Sl. No.	Transportation	Assignment
1	This problem contains specific demand and requirement in columns and rows	The demand and availability in each column or row is one
2	Total demand must be equal to the total availability	It is a square matrix. The no of rows must be equal to the no of columns.
3	The optimal solution involves the following conditions M+N-1 M ⟶ rows N ⟶ columns	The optimal solutions involves one assignment in each row and each column
4	There is no restriction in the number of allotments in any row or column	There should be only one allotment in each row and each column
5	It is a problem of allocating multiple resources to multiple markets	It is a problem of allocation resources to job j

Assignment Algorithm (Hungarian Method)

Step I:- Create Zero elements in the cost matrix by subtract the smallest element in each row column for the corresponding row and column.

Step II:- Drop the least number of horizontal and vertical lines so as to cover all zeros if the no of here lines are 'N'

i) If N = n (n=order of the square matrix) then an optimum assignment has been obtained

ii) If N<n proceeds to step III

Step III :- determine the smallest cost cell from among the uncrossed cells subtract. This cost from all the uncrossed cells and add the same to all those cells laying in the intersection of horizontal and vertical lines.

Step IV:- repeat steps II and III until N=n.

Step V:- examine the rows (column) successively until a row (column) with are zero is found enclose the zero in a square (0) and cancel out (0) any other zeros laying in the column (row) of the Matrix. Continue in this way until all the rim requirements are satisfied i.e N=n.

Step VI:- repeat step 5 successively one of the following arises.

i) No unmarked zero is left

ii) If more then one unmarked zeros in one column or row.

In case i) the algorithm stops

ii)Encircle one of the unmarked zeros arbitrary and mark a cross in the cells of remaining zeroes in it's row and column. Repeat the process until no unmarked zero is left in the cost matrix.

Step VII) we now have exactly one encircled zero in each row and each column of the cost matrix. The assignment schedule corresponding to there zeros is the optimum (maximal) assignment.

Note: the above procedure for assignment is Hungarian assignment method

Problem 1.

Three jobs A B C are to be assigned to three machines x Y Z. The processing costs are as given in the matrix shown below. Find the allocation which will minimize the overall processing cost.

		Machines		
		X	Y	Z
Jobs	A	19	28	31
	B	11	17	16
	C	12	15	13

Solution:

Step 1: create zero in each row or column by subtracting by selecting least number in each row and column

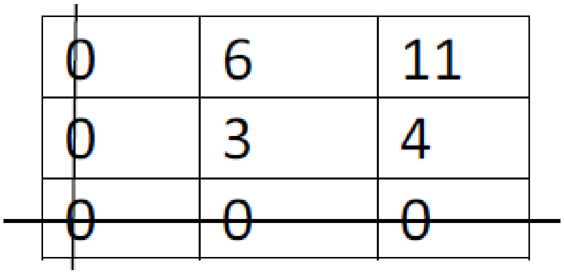
Row Minimization

0	9	12
0	6	5
0	3	1

Column Minimization

0	6	11
0	3	4
0	0	0

Now draw Horizontal and vertical lines



0	6	11
0	3	4
0	0	0

Here, no of horizontal lines is one and vertical line is one

The order of matrix is 3 x 3, therefore, N≠ n

Now, in the uncrossed cell the least cost is selected and subtracted for the remaining uncrossed# cell by the least value and for the intersection of the horizontal line and vertical line the least value should be added and the resulting matrix.

0	3	8
0	0	1
3	0	0

The above matrix has two horizontal line and one vertical line which satisfies our condition N= n

{0}	3	8
~~0~~	{0}	~~1~~
3	~~0~~	{0}

The assignment are A $\longrightarrow$ X = 19

B $\longrightarrow$ Y = 17

C $\longrightarrow$ Z = 13

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Problem 2

Solve of the assignment problem

	I	II	III	IV	V
1	11	17	8	16	20
2	9	7	12	6	15
3	13	16	15	12	16
4	21	24	17	28	26
5	14	10	12	11	15

Solns:

Row Minimization

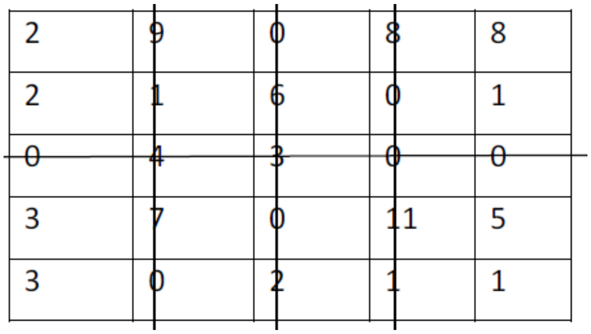
3	9	0	8	12
3	1	6	0	9
1	4	3	0	4
4	7	0	11	9
4	0	2	1	5

Column Minimization

2	9	0	8	8
2	1	6	0	1
0	4	3	0	0
3	7	0	11	5
3	0	2	1	1

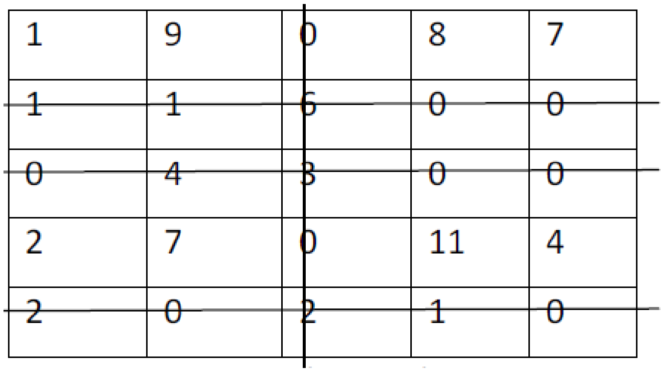
N≠n

5≠4



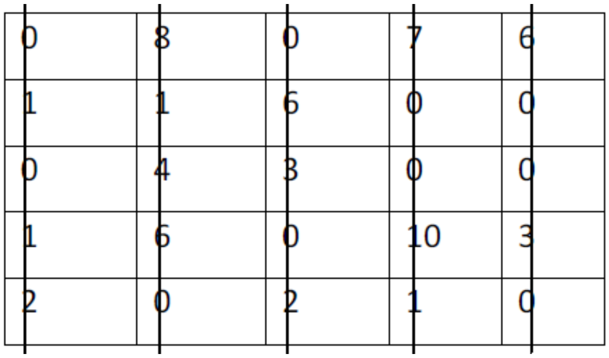
2	9	0	8	8
2	1	6	0	1
0	4	3	0	0
3	7	0	11	5
3	0	2	1	1

The least value in the uncrossed cell is 1, it is subtracted for the uncrossed cell and added for intersection of the vertical line and horizontal.



1	9	0	8	7
1	1	6	0	0
0	4	3	0	0
2	7	0	11	4
2	0	2	1	0

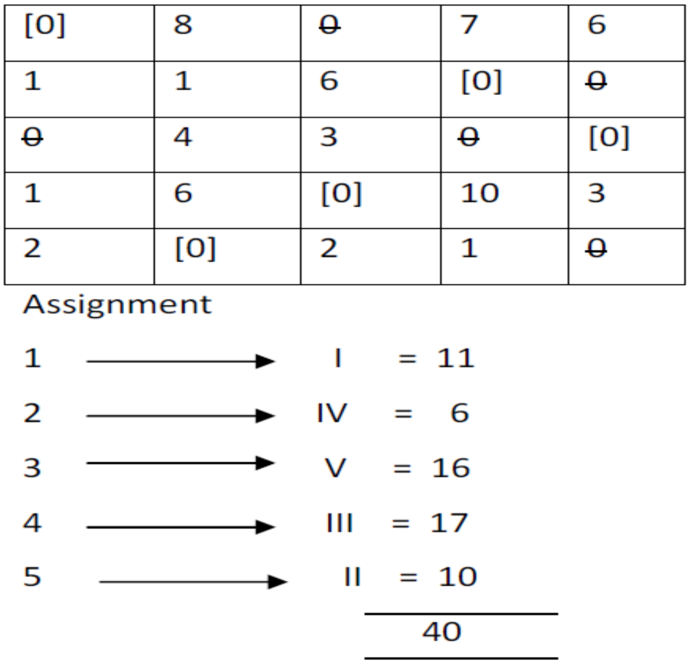
Again N ≠ n least value is one again



0	8	0	7	6
1	1	6	0	0
0	4	3	0	0
1	6	0	10	3
2	0	2	1	0

Here, it satisfies our condition N=n

Now, the assignment for the optimum table



[0]	8	0	7	6
1	1	6	[0]	0
0	4	3	0	[0]
1	6	[0]	10	3
2	[0]	2	1	0

Assignment

1 → I = 11

2 → IV = 6

3 → V = 16

4 → III = 17

5 → II = 10

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UNIT – 7

Game Theory, Decision Analysis

INTRODUCTION

Competition is a 'Key factor' of modern life. We say that a competitive situation exists if two or more individuals are taking decisions in situation that involves conflicting interests and in which the outcome is controlled by the decisions of all parties concerned. We assume that in a ompetitive situation each participant acts in a rational manner and tries to resolve the conflicts of interests in his favour. It is in this context that game theory has developed.

Professor John von Newmann and Oscar Morgensten published their book entitled "The Theory of Games and Economic Behaviour" where in they provided a new approach to many problems involving conflict situations. This approach is now widely used in Economics, Business

Administration, Sociology, Psychology and Political Science as well as in Military Training. In games like chess, draught, pocker etc. which are played as per certain rules victory of one side and the defeat of the other is dependent upon the decisions based in skillful evaluation of the alternatives of the opponent and also upon the selection of the right alternative.

Game Theory is a body of knowledge which is concerned with the study of decision making in situation where two or more rational opponents are involved under condition of competition and conflicting interests. It deals with human processes in which an individua1 decision making unit is not in complete control of the other decision making units. Here e unit may be an individual group, organisation, society or a country.

Game Theory is a type of decision theory which is based on reasoning in which the choice of action is determined after considering the possible alternatives available to the opponents playing the same game. The aim is to choose the best course of action, because every player has got an alternative course of action.

Significance of Game Theory

We know that competition is a major factor in modern life. There are so many competition like business competition, elections competitions, sport competition etc. In all these competition persons have conflicting interest, and everybody tries to maximise his gains and minimise loss.

Game theory is a type of decision theory which is based on the choice of action. A choice of action is determined after considering the possible alternatives available to the opponent. It involves the players i.e. decision makers who have different goals or objectives.

The game theory determine the rules of rational behaviour of these players in which outcomes are dependent on the actions of the interdependent players.

In a game there are number of possible outcomes, with different values to the decision makers. They might have some control but do not have the complete control over others example players in a chess game, labour union striking against the management, companies, striving for larger share of market etc are the situations where theory of games is applicable because each situation can be viewed as games. So, game theory is important weapon in the hands of management. Game theory is a scientific approach rational decision making.

Essential features of Game Theory

A competitive situation is called a game if it has the following features:

i. **Finite Number of Competitors**. There are finite number of competitors, called players.

The players need not be individuals, they can be groups, corporations, political parties, institutions or even nations.

ii. **Finite Number of Action.** A list of finite number of possible courses of action is available to each player. The list need not be the same for each player.

iii. **Knowledge of Alternatives.** Each player has the knowledge of alternatives available to his opponent.

iv. **Choice.** Each player makes a choice, i.e., the game is played. The choices are assumed to be made simultaneously, so that no player knows his opponents' choice until he has decided his own course of action.

v. **Outcome or Gain**. The play is associated with an outcome known as gain. Here the loss is considered negative gain.

vi. **Choice of Opponent.** The possible gain or loss of each player depends upon not only the choice made by him but also the choice made by his opponent.

Two Persons Zero-sum Game. Two person zero-sum game is the situation which involves two persons or players and gains made by one person is equals to the loss incurred by the other. For example there are two companies Coca-cola & Pepsi and are struggling for a larger share in the market. Now any share of the market gained by the Coca-cola company must be the lost share of Pepsi, and therefore, the sums of the gains and losses equals zero. In other words if gain of Coca-cola is 40% so the lost share of the Pepsi will be 40%. The sum of game and losses is zero

i.e. (+40%)+(-40%) = 0

n-persons game. A game involving n persons is called a n persons game. In this two persons game are most common. When there are more than two players in a game, obviously the complexity of the situation is increased.

Pay offs. Outcomes of a game due to adopting the different courses of actions by the competing players in the form of gains or losses for each of the players is known as pay offs.

Payoff Matrix. In a game, the gains and losses, resulting from different moves and counter moves, when represented in the form of a matrix are known as payoff matrix or gain matrix.

This matrix shows how much payment is to be made or received at the end of the game in case a particular strategy is adopted by a player.

Payoff matrix shows the gains and losses of one of the two players, who is indicated on the left hand side of the pay off matrix. Negative entries in the matrix indicate losses. This is generally prepared for the maximising player. However the same matrix can be interpreted for the other player also, as in a zero sum game, the gains of one player represent the losses of the other player, and *vice versa*. Thus, the payoff matrix of Mr A is the negative payoff matrix for Mr B.

The other player is known as the minimising player. He is indicated on the top of the table.

The payoff matrix can be understood by the following example.

$$\begin{array}{ccc|cc|} & & & \multicolumn{2}{c}{Y} \\ & & & Y_1 & Y_2 \\ \hline X & & X_1 & 6 & 9 \\ & & X_2 & 12 & 15 \\ \hline \end{array}$$

In our example we have two players X and Y. X has two strategies Xl and X2. Similarly Y also has two strategies Y1 and Y2.:

X's	:	X_1	X_2	X_1	X_2
Y's,	:	Y_1	Y_1	Y_2	Y_2
Payoff	:	6	12	9	15

The above arrangement of the matrix is known as payoff matrix. It should always be written from the X's point of view *i.e.* from the player who is on the left hand side.

Decision of a Game. In game theory best strategy for each player is determined on the basis of some criteria. Since both the players are expected to be rational in their approach, this is known as the criteria of optimality. Each player lists the possible outcomes from his objective and then selects the best strategy out of these outcomes from his point of view or as per his objective.
The decision criteria in game theory is known as the criteria of optimality, *i.e.*, maximin for the maximising player and minimax for the minimising player.

Limitation of Game Theory

1. **Infinite number of strategy.** In a game theory we assume that there is finite number of possible courses of action available to each player. But in practice a player may have infinite number of strategies or courses of action.

2. **Knowledge about strategy.** Game theory assumes that each player as the knowledge of strategies available to his opponent. But some times knowledge about strategy about the opponent is not available to players. This leads to the wrong conclusions.

Zero outcomes. We have assumed that gain of one person is the loss of another person. But in practice gain of one person may not be equal to the loss of another person i.e. opponent.

Risk and uncertainty. Game theory does not takes into consideration the concept of probability. So game theory usually ignores the presence of risk and uncertainty.

Finite number of competitors. There are finite number of competitors as has been assumed in the game theory. But in real practice there can be more than the expected number of players.

Certainity of Pay off. Game theory assume that payoff is always known in advance. But sometimes it is impossible to know the pay off in advance. The decision situation infact becomes multidimensional with large number of variables.

Rules of Game. Every game is played according to the set of rules i.e specific rules which governs the behaviour of the players. *As* there we have set of rules of playing Chess, Badminton, Hockey etc.

Strategy. It is the pre-determined rule by which each player decides his course of action from his list available to him. How one course of action is selected out of various courses available to him is known as strategy of the game.

Types of Strategy. Generally two types of strategy are employed

(i) Pure Strategy (ii) Mixed Strategy

(i.) Pure Strategy. It is the predetermined course of action to be employed by the player. The players knew it in advance. It is usually represented by a number with which the course of action is associated.

(ii.) Mixed Strategy. In mixed strategy the player decides his course of action in accordance with some fixed probability distribution. Probability are associated with each course of action and the selection is done as per these probabilities.

In mixed strategy the opponent cannot be sure of the course of action to be taken on any particular occasion.

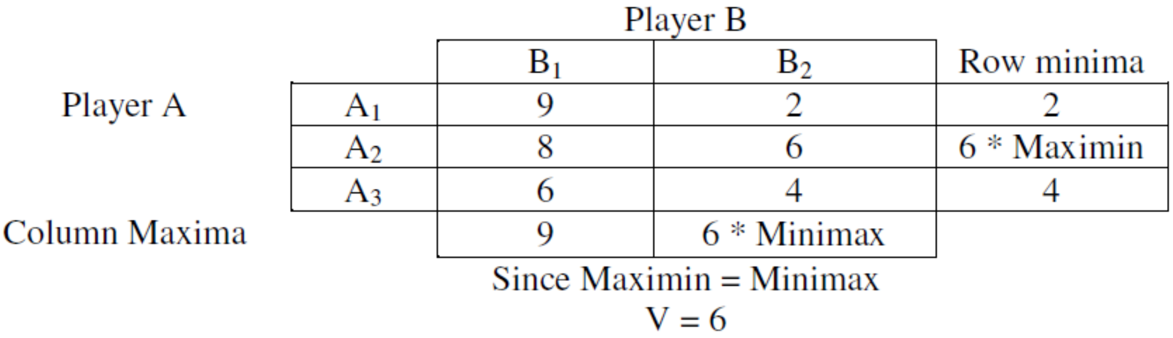
Decision of a Game. In Game theory, best strategy for each player is determined on the basis of some rule. Since both the players are expected to be rational in their approach this is known as the criteria of optimality. Each player lists the possible outcomes from his action and selects the best action to achieve his objectives. This criteria of optimality is expressed as Maximin for the maximising player and Minimax for the minimising player.

THE MAXIMIN-MINIMAX PRINCIPLE

(i) **Maximin Criteria:** The maximising player lists his minimum gains from each strategy and selects the strategy which gives the maximum out of these minimum gains.

(ii) **Minimax Criteria A:** The minimising player lists his maximum loss from each strategy and selects the strategy which gives him the minimum loss out of these maximum losses.

For Example Consider a two person zero sum game involving the set of pure strategy for Maximising player A say Al A2 & As and for player B, B1 & B2, with the following payoff



Player B

		B_1	B_2	Row minima
Player A	A_1	9	2	2
	A_2	8	6	6 * Maximin
	A_3	6	4	4
Column Maxima		9	6 * Minimax	

Since Maximin = Minimax
V = 6

Suppose that player A starts the game knowing fully well that whatever strategy he adopts B will select that particular counter strategy which will minimise the payoff to A. If A selects the strategy Al then B will select B2 so that A may get minimum gain. Similarly if A chooses A2 than B will adopt the strategy of B2. Naturally A would like to maximise his maximin gain which is just the largest of row minima. Which is called 'maximin strategy'. Similarly B will minimize his minimum loss which is called 'minimax strategy'. We observe that in the above example, the maximum if row minima and minimum of column maxima are equal. In symbols.

Maxi [Min.] = Mini [Max]

The strategies followed by both the p1ayersarecalled ‘optimum strategy’.

Value of Game. In game theory, the concept value of game is considered as very important. The value of game is the maximum guaranted gain to the maximising player if both the players use their best strategy. It refers to the average payoff per play of the game over a period of time.

Consider the following the games.

$$\text{Player X} \quad \overset{\text{Player Y}}{\begin{bmatrix} 3 & 4 \\ -6 & -2 \end{bmatrix}} \qquad \text{Player X} \quad \overset{\text{Player Y}}{\begin{bmatrix} -7 & 2 \\ -3 & -1 \end{bmatrix}}$$

$$\text{(with positive value)} \qquad\qquad \text{(with negative value)}$$

In the first game player X wins 3 points and the value of the value is three with positive sign and in the second game the player Y wins 3 points and the value of the game is -ve which indicates that Y is the Winner. The value is denoted by '*v*.

Saddle Point. The Saddle point in a pay off matrix is one which is the smallest value in its row and the largest value in its column. The saddle point is also known as equilibrium point in the theory of games. An element of a matrix that is simultaneously minimum of the row in which it occurs and the maximum of the column in which it occurs is a saddle point of the matrix game.
In a game having a saddle point optimum strategy for a player X is always to play the row containing saddle point and for a player Y to play the column that contains saddle point. The following steps are required to find out Saddle point;
(i) Select the minimum value of each row & put a circle around it.
(ii) Select the maximum value of each column and put square around it.
(iii) The value with both circle & square is the saddle point.

In case there are more than one Saddle point there exist as many optimum points or solutions of the game. There may or may not be the saddle point in the game. When there is no saddle point we have to use algebraic methods for working out the solutions concerning the game problem.
Saddle point can be found out in different ways.

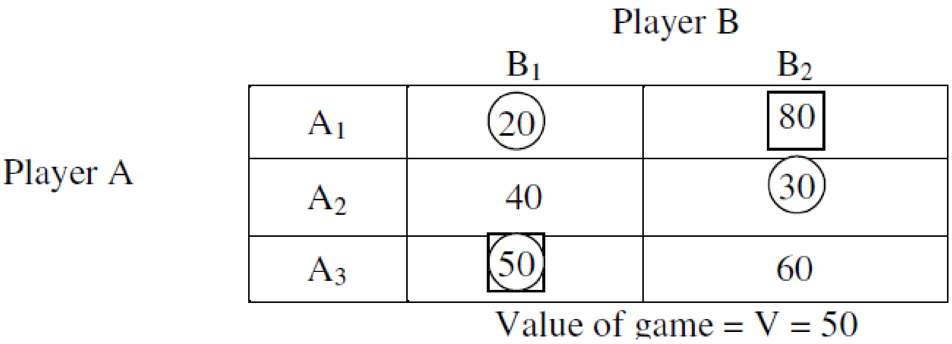
	Player B
Player A	$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & 20 & 80 \\ A_2 & 40 & 30 \\ A_3 & 50 & 60 \end{array}$

1st Method

Step 1. Putting circle around Row minima.

Step 2. Putting square around Column Maxima.

Step 3. Saddle point is a point where circle & square are both combined.



Player B

Player A		B_1	B_2
	A_1	20	80
	A_2	40	30
	A_3	50	60

Value of game = V = 50

IInd Method

(i) Putting mark* on each Row minima.

(ii) Putting mark* on each Column Maxima.

(iii) Saddle point where both* and (different star) appears.

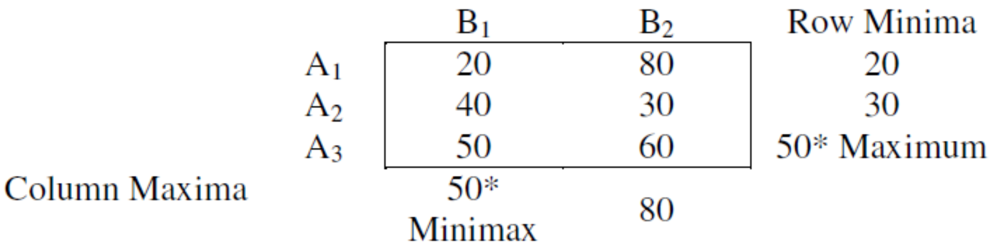
Player A		B_1	B_2
	A_1	20*	80*
	A_2	40	30*
	A_3	50**	60

Value of game (υ) = 50

IIIrd Method

(i) Creating column for minimum value of each row

(ii) Creating Row for Maximum value of each column.



	B_1	B_2	Row Minima
A_1	20	80	20
A_2	40	30	30
A_3	50	60	50* Maximum
Column Maxima	50* Minimax	80	

The same value in Row minima & Column maxima is the value of game. The optimal strategy for A is A3 and for B is B1 (Students can apply any method they like.)

Points to remember

(i) Saddle point mayor may not exist in a given game.

(ii) There may be more than one saddle point then there will be more than one solution.

(Such situation is rare in the rea1life).

(iii) The value of game may be +ve or -ve.

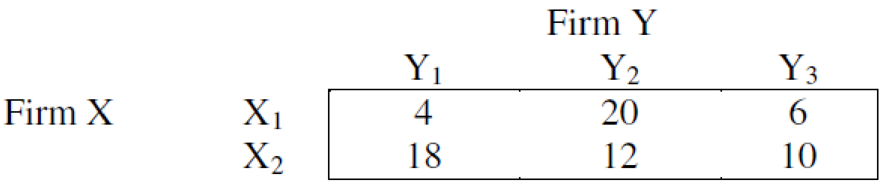
(iv) The value of game may be zero which means 'fair game'.

Types of Problems

(1) GAMES WITH PURE STRATEGIES OR TWO PERSON ZERO SUM GAME WITH SADDLE POINT.OR TWO PERSON ZERO SUM WITH PURE STRATEG'Y.

In case of pure strategy, the maximising player arrives at his optimal strategy on the basis of maximin criterion. The game is solved when maximin value equals minimax value.

Example 2. Solve the following game



		Firm Y		
		Y_1	Y_2	Y_3
Firm X	X_1	4	20	6
	X_2	18	12	10

Solution X is maximising player & Y is minimising player. If Firm X chooses Xl then Firm Y will choose Y1 as a counter strategy resulting in payoff equal to 4 to X.

On the other hand if X chooses X2 then firm Y will choose Y3 as counter strategy these giving payoff 10 to X. From the firm Y point of view. Strategy Y3 is better than Y2. If Y choose Y1, X will choose X2 and Y will lose 18 points. On the other hand if Y choose Y3, firm X will choose X2 and Y will lose 10 points. So preferred strategy for X is X2 and for Y is Y3.

The above problem can be simplified with the help of maximin and minimax criterion as follows

Firm Y

		Y_1	Y_2	Y_3
Firm X	X_1	$\textcircled{4}$	$\boxed{20}$	6
	X_2	$\boxed{18}$	12	$\boxed{\textcircled{10}}$

○ = Row Minima
□ = Column Maxima

The saddle point exists and the value of Game *(v)* is 10 and the pure strategy for X is X2 and for Y is Y3

All game problems, where saddle point does not exist are taken as mixed strategy problems. Where row minima is not equal to column maxima, then different methods are used to solve the different types of problems. Both players will use different strategies with certain probabilities to optimise. For the solution of games with mixed strategies, any of the following methods can be applied.

1. ODDS METHOD
(2x2 game without saddle point)
2. Dominance Method.
3. Sub Games Method. – For (mx2) or (2xn) Matrices
4. Equal Gains Method.
5. Linear Programming Method-Graphic solution
6. Algebraic method.
7. Linear programming - Simplex method

8. Itirative method

These methods are explained one by one with examples, in detail.

1. ODDS Method - For 2 x 2 Game

Use of odds method is possible only in case of games with 2 x 2 matrix. Here it should be ensured that the sum of column odds and row odds is equal.

METHOD OF FINDING OUT ODDS

Step 1. Find out the difference in the value of in cell (1, 1) and the value in the cell (1,2) of the first row and place it in front of second row.

Step 2. Find out the difference in the value of cell (2, 1) and (2, 2) of the second row and place it in front of first row.

Step 3. Find out the differences in the value of cell (1, 1) and (2, 1) of the first column and place it below the second column.

Step 4. Similarly find the difference between the value of the cell (1, 2) and the value in cell (2, 2) of the second column and place it below the first column.

The above odds or differences are taken as positive (ignoring the negative sign)

$$\begin{array}{cccccc}
\text{Strategy} & \rightarrow & & Y & & \\
 & \downarrow & Y_1 & Y_2 & & \text{ODDS} \\
X & X_1 & a_1 & a_2 & \rightarrow & (b_1 - b_2) \\
 & X_2 & b_1 & b_2 & \rightarrow & (a_1 - a_2) \\
 & & \downarrow & \downarrow & & \\
\text{ODDS} & & (a_2 - b_2) & (a_1 - b_1) & &
\end{array}$$

The value of game is determined with the help of following equation.

Value of the game (υ) $= \dfrac{a_1(b_1-b_2)+b_1(a_1-a_2)}{(b_1-b_2)+(a_1-a_2)}$

Probabilities for X_1 $= \dfrac{b_1-b_2}{(b_1-b_2)+(a_1-a_2)}$, $X_2 = \dfrac{a_1-a_2}{(b_1-b_2)+(a_1-a_2)}$

Probabilities for Y_1 $= \dfrac{a_2-b_2}{(a_2-b_2)+(a_1-b_1)}$, $Y_2 = \dfrac{a_1-b_1}{(a_2-b_2)+(a_1-b_1)}$

Example: Solve the following game by odds method.

Strategy		Player B	
		B_1	B_2
Player A	A_1	1	5
	A_2	4	2

Solutions:

		Player B	
		B_1	B_2
Player A	A_1	①	$\boxed{5}$
	A_2	④	②

Since the game does not have saddle point, the players will use mixed strategy. We apply odds methods to solve the game.

	B_1	B_2	odds
A_1	1	5	4 – 2 = 2
A_2	4	2	1 – 5 = 4
Odds	5 – 2 = 3	1 – 4 = 3	

$$\text{Value of the game } (\upsilon) = \frac{(1\times 2)+(4\times 4)}{2+4} = 3, \qquad \upsilon = 3$$

Probabilities of Selecting Strategies

		I	II
	A	1/3	2/3
Players	B	1/2	1/2

2. Dominance Method. Dominance method is also applicable to pure strategy and mixed strategy problem. In pure strategy the solution is obtained by itself while in mixed strategy it can be used for simplifying the problem.

Principle of Dominance. The Principle of Dominance states that if the strategy of a player dominates over the other strategy in all condition then the later strategy is ignored because it will not affect the solution in any way. For the gainer point of view if a strategy gives more gain than another strategy, then first strategy dominates over the other and the second strategy can be ignored altogether. Similarly from loser point of view, if a strategy involves lesser loss than other in all condition then second can be ignored. So determination of superior or inferior strategy is based upon the objective of the player. Since each player is to select his best strategy, the inferior strategies can be eliminated. In other words, ineffective rows & column can be deleted from the game matrix and only effective rows & columns of the matrix are retained in the reduced matrix. For deleting the ineffective rows & columns the following general rules are to be followed.

1. If all the elements of a row (say ith row) ofa pay off matrix are less than or equal to (£) the corresponding each element of the other row (say jth row) then the player A will never choose the ith strategy OR ith row is dominated by jth row. Then delete ith row.

$$E_{ij} - [R_{ith}] \le E_{ij}\begin{pmatrix} R_{ith} \\ Row \end{pmatrix} \quad \text{Delete } R_{ith} \text{ rows.}$$

2. If all the elements of a column (say jth column are greater than or equal to the corresponding elements of any other column (say jth column) then ith column is dominated by jth column.

Then delete ith column.

$$E_{ij}(c_i) \geq E_{ij}(cj)$$
$$\text{delete} = c_i \text{ th}$$

3. A pure strategy of a player may also be dominated if it is inferior to some convex combination of two or more pure strategies. As a particular case, if all the elements of a column are greater than or equal to the average of two or more other columns then this column is dominated by the group of columns. Similarly if all the elements of row are less than or equal to the average of two or more rows then this row is dominated by other group of row.

4. By eliminating some of the dominated rows a columns and if the game is reduced to 2 x 2 form it can be easily solved by odds method.

Example: Solve the game.

	B		
A	5	20	-10
	10	6	2
	20	15	18

		B		
		I	II	III
A	I	5	$\boxed{20}$	-10
	II	10	6	2
	III	$\boxed{20}$	15	$\boxed{18}$

Since there is no saddle point, so we apply dominance method. Here Row II dominates Row II s we will delete Row II.

		B		
		I	II	III
A	I	5	20	-10
	III	20	15	18

Since column III dominates column I, we delete column I we get:

		B		
		II	III	odds
A	I	20	-10	3
	III	15	18	30
	Odds	28	5	33

$$\text{Value of the game} \quad = \frac{20(3)+15(30)}{3+30} = \frac{510}{33} = \frac{170}{11}$$

UNIT – 8

Metaheuristics

TABU SEARCH

Tabu search algorithm was proposed by Glover. In 1986, he pointed out the controlled randomization in SA to escape from local optima and proposed a deterministic algorithm. In a parallel work, a similar approach named “steepest ascent/mildest descent” has been proposed by Hansen. In the 1990s, the tabu search algorithm became very popular in solving optimization problems in an approximate manner.

Nowadays, it is one of the most widespread S-metaheuristics. The use of memory, which stores information related to the search process, represents the particular feature of tabu search.
TS behaves like a steepest LS algorithm, but it accepts nonimproving solutions to escape from local optima when all neighbors are nonimproving solutions. Usually, the whole neighborhood is explored in a deterministic manner, whereas in SA a random neighbor is selected. As in local search, when a better neighbor is found, it replaces the current solution. When a local optima is reached, the search carries on by selecting a candidate worse than the current solution. The best solution in the neighborhood is selected as the new current solution even if it is not improving the current solution.

Tabu search may be viewed as a dynamic transformation of the neighborhood. This policy may generate cycles; that is, previous visited solutions could be selected again.
To avoid cycles, TS discards the neighbors that have been previously visited. It memorizes the recent search trajectory. Tabu search manages a memory of the solutions or moves recently applied, which is called the tabu list. This tabu list constitutes the short-term memory. At each iteration of TS, the short-term memory is updated.

Storing all visited solutions is time and space consuming. Indeed, we have to check at each iteration if a generated solution does not belong to the list of all visited solutions.

The tabu list usually contains a constant number of tabu moves. Usually, the attributes of the moves are stored in the tabu list. By introducing the concept of solution features or move features in the tabu list, one may lose some information about the search memory. We can reject solutions that have not yet been generated. If a move is “good,” but it is tabu, do we still reject it? The tabu list may be too restrictive; a nongenerated solution may be forbidden.

Yet for some conditions, called aspiration criteria, tabu solutions may be accepted. The admissible neighbor solutions are those that are nontabu or hold the aspiration criteria

In addition to the common design issues for S-metaheuristics such as the definition of the neighborhood and the generation of the initial solution, the main design issues that are specific to a simple TS are

- Tabu list: The goal of using the short-term memory is to prevent the search from revisiting previously visited solutions. As mentioned, storing the list of all visited solutions is not practical for efficiency issues.

- Aspiration criterion: A commonly used aspiration criteria consists in selecting a tabu move if it generates a solution that is better than the best found solution.

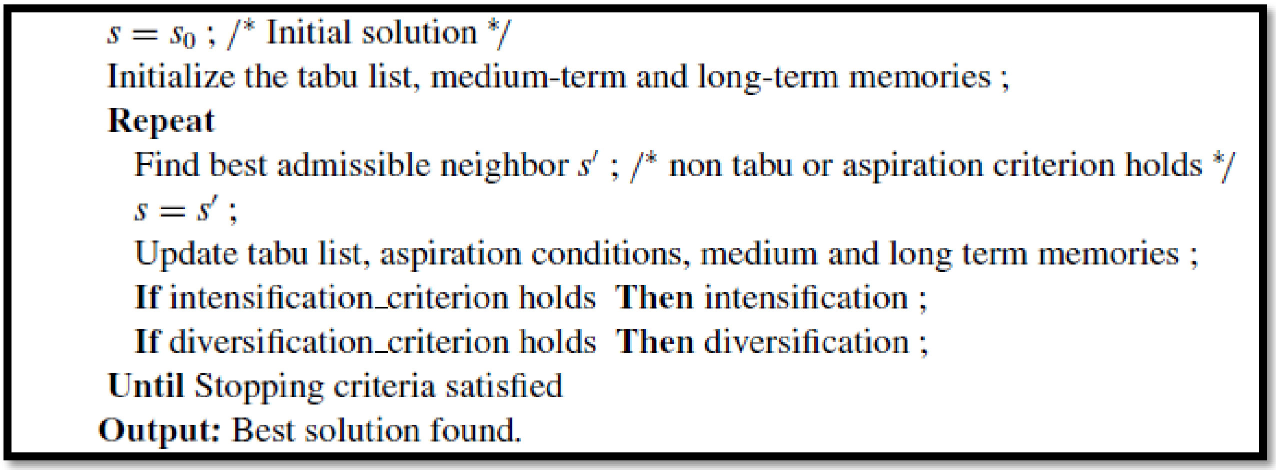
Another aspiration criterion may be a tabu move that yields a better solution among the set of solutions possessing a given attribute.

Some advanced mechanisms are commonly introduced in tabu search to deal with the intensification and the diversification of the search:

- Intensification (medium-term memory): The medium-term memory stores the elite (e.g., best) solutions found during the search. The idea is to give priority to attributes of the set of elite solutions, usually in weighted probability manner.

The search is biased by these attributes.

- Diversification (long-term memory): The long-term memory stores information on the visited solutions along the search. It explores the unvisited areas of the solution space. For instance, it will discourage the attributes of elite solutions in the generated solutions to diversify the search to other areas of the search space.



```
$s = s_0$ ; /* Initial solution */
Initialize the tabu list, medium-term and long-term memories ;
Repeat
  Find best admissible neighbor $s'$ ; /* non tabu or aspiration criterion holds */
  $s = s'$ ;
  Update tabu list, aspiration conditions, medium and long term memories ;
  If intensification_criterion holds Then intensification ;
  If diversification_criterion holds Then diversification ;
Until Stopping criteria satisfied
Output: Best solution found.
```

Tabu Search Algorithm

Simulated Annealing:

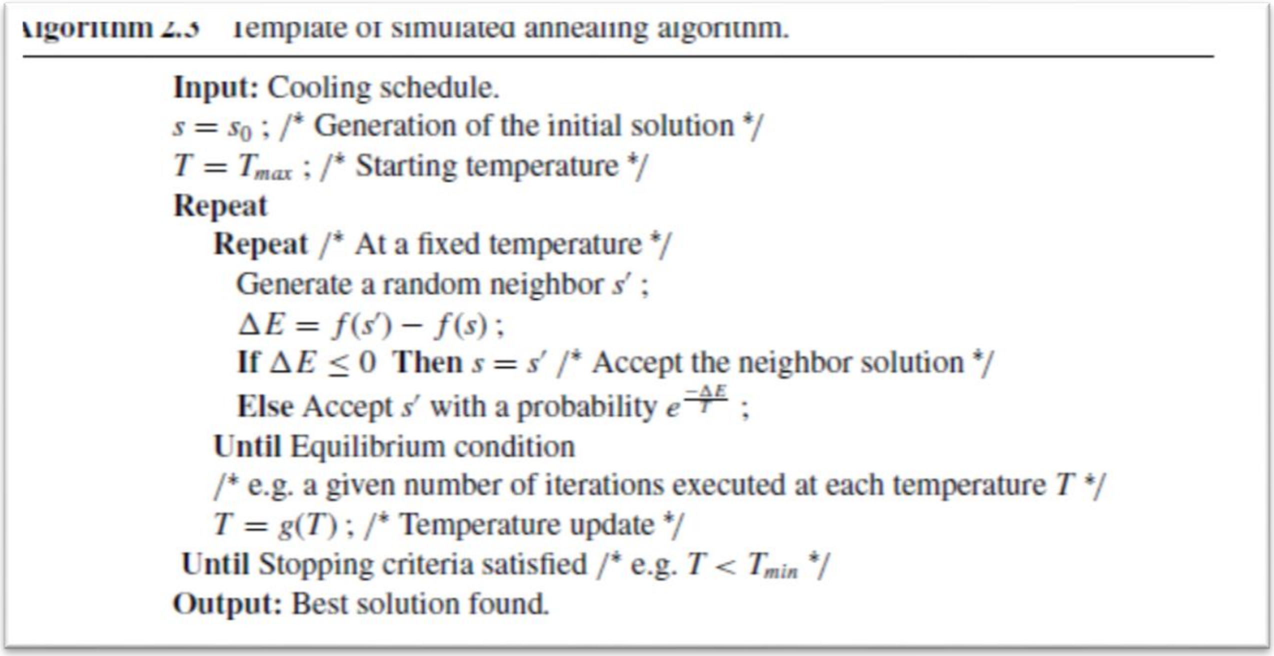
Simulated annealing applied to optimization problems emerges from the work of S. Kirkpatrick et al. and V. Cerny. In these pioneering works, SA has been applied to graph partitioning and VLSI design. In the 1980s, SA had a major impact on the field of heuristic search for its simplicity and efficiency in solving combinatorial optimization problems. Then, it has been extended to deal with continuous optimization problems.

SA is based on the principles of statistical mechanics whereby the annealing process requires heating and then slowly cooling a substance to obtain a strong crystalline structure. The strength of the structure depends on the rate of cooling metals.

If the initial temperature is not sufficiently high or a fast cooling is applied, imperfections (metastable states) are obtained. In this case, the cooling solid will not attain thermal equilibrium at each temperature. Strong crystals are grown from careful and slow cooling. The SA algorithm simulates the energy changes in a system subjected to a cooling process until it converges to an equilibrium state (steady frozen state).

This scheme was developed in 1953 by Metropolis. Table illustrates the analogy between the physical system and the optimization problem. The objective function of the problem is analogous to the energy state of the system. A solution of the optimization problem corresponds to a system state.

The decision variables associated with a solution of the problem are analogous to the molecular positions. The global optimum corresponds to the ground state of the system. Finding a local minimum implies that a metastable state has been reached. SA is a stochastic algorithm that enables under some conditions the degradation of a solution. The objective is to escape from local optima and so to delay the convergence. SA is a memory less algorithm in the sense that the algorithm does not use any information gathered during the search. From an initial solution, SA proceeds in several iterations. At each iteration, a random neighbor is generated. Moves that improve the cost function are always accepted. Otherwise, the neighbor is selected with a given probability that depends on the current temperature and the amount of degradation _E of the objective function. _E represents the difference in the objective value (energy) between the current solution and the generated neighboring solution. As the algorithm progresses, the probability that such moves are accepted decreases.



lgorithm 2.3 Template of simulated annealing algorithm.

```
Input: Cooling schedule.
$s = s_0$ ; /* Generation of the initial solution */
$T = T_{max}$ ; /* Starting temperature */
Repeat
   Repeat /* At a fixed temperature */
     Generate a random neighbor $s'$ ;
     $\Delta E = f(s') - f(s)$ ;
     If $\Delta E \leq 0$ Then $s = s'$ /* Accept the neighbor solution */
     Else Accept $s'$ with a probability $e^{\frac{-\Delta E}{T}}$ ;
   Until Equilibrium condition
   /* e.g. a given number of iterations executed at each temperature $T$ */
   $T = g(T)$ ; /* Temperature update */
Until Stopping criteria satisfied /* e.g. $T < T_{min}$ */
Output: Best solution found.
```

Simulated Annealing Algorithm

Genetic Algorithms

Genetic algorithms have been developed by J. Holland in the 1970s (University of Michigan, USA) to understand the adaptive processes of natural systems. Then, they have been applied to optimization and machine learning in the 1980s.

GAs are a very popular class of EAs. Traditionally, GAs are associated with the use of a binary representation but nowadays one can find GAs that use other types of representations. A GA usually applies a crossover operator to two solutions that plays a major role, plus a mutation operator that randomly modifies the individual contents to promote diversity (Tables 3.4 and 3.5). GAs use a probabilistic selection that is originally the proportional selection. The replacement (survivor selection) is generational, that is, the parents are replaced systematically by the offsprings. The crossover operator is based on the *n*-point or uniform crossover while the mutation is bit flipping. A fixed probability pm (resp. pc) is applied to the mutation (resp. crossover) operator.

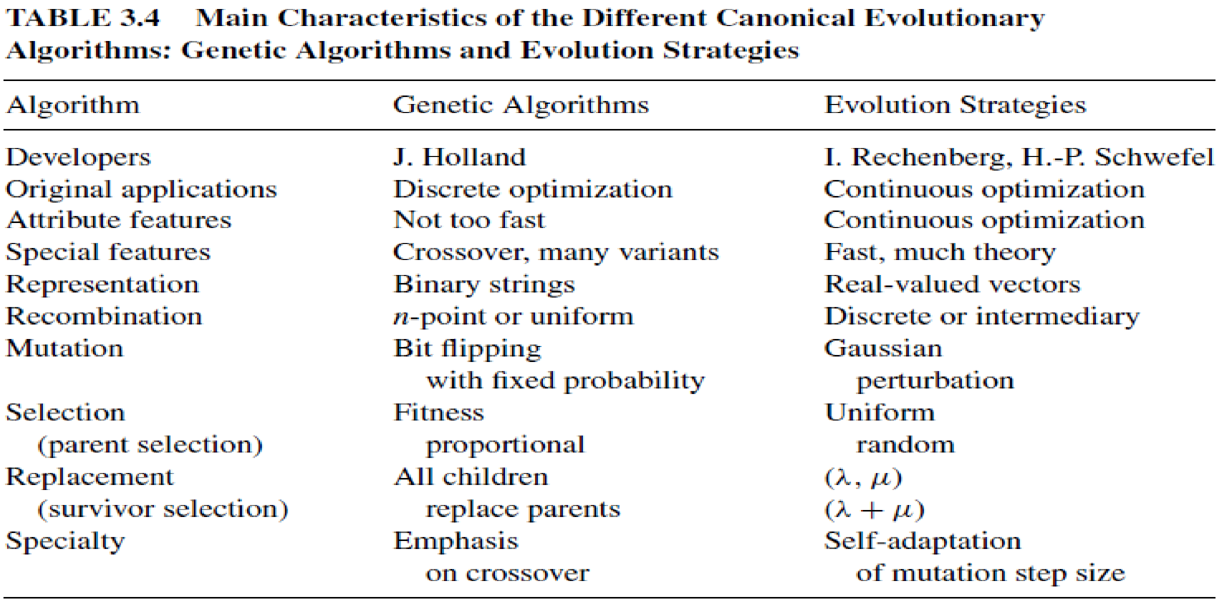


TABLE 3.4 Main Characteristics of the Different Canonical Evolutionary Algorithms: Genetic Algorithms and Evolution Strategies

Algorithm	Genetic Algorithms	Evolution Strategies
Developers	J. Holland	I. Rechenberg, H.-P. Schwefel
Original applications	Discrete optimization	Continuous optimization
Attribute features	Not too fast	Continuous optimization
Special features	Crossover, many variants	Fast, much theory
Representation	Binary strings	Real-valued vectors
Recombination	*n*-point or uniform	Discrete or intermediary
Mutation	Bit flipping with fixed probability	Gaussian perturbation
Selection (parent selection)	Fitness proportional	Uniform random
Replacement (survivor selection)	All children replace parents	(λ, μ) $(\lambda + \mu)$
Specialty	Emphasis on crossover	Self-adaptation of mutation step size

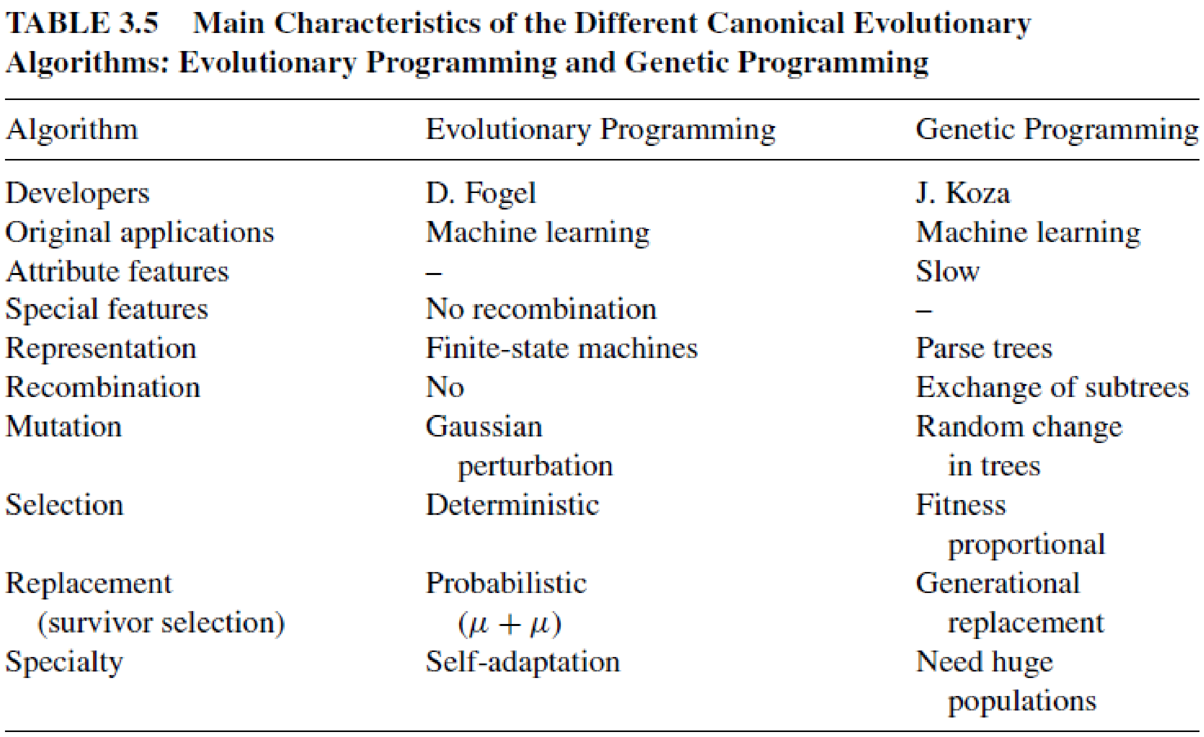


TABLE 3.5 **Main Characteristics of the Different Canonical Evolutionary Algorithms: Evolutionary Programming and Genetic Programming**

Algorithm	Evolutionary Programming	Genetic Programming
Developers	D. Fogel	J. Koza
Original applications	Machine learning	Machine learning
Attribute features	–	Slow
Special features	No recombination	–
Representation	Finite-state machines	Parse trees
Recombination	No	Exchange of subtrees
Mutation	Gaussian perturbation	Random change in trees
Selection	Deterministic	Fitness proportional
Replacement (survivor selection)	Probabilistic $(\mu + \mu)$	Generational replacement
Specialty	Self-adaptation	Need huge populations