A FILL IN THE BLANKS

1. Let $\vec{A}, \vec{B}, \vec{C}$ be vectors of length 3, 4, 5 respectively. Let $\vec{A}$ be perpendicular to $\vec{B}+\vec{C}$, $\vec{B}$ to $\vec{C}+\vec{A}$ and $\vec{C}$ to $\vec{A}+\vec{B}$. Then the length of vector $\vec{A}+\vec{B}+\vec{C}$ is ... (IIT 1981; 2M)

2. A, B, C and D, are four points in a plane with position vectors $\hat{a}, \hat{b}, \hat{c}$ and $\hat{d}$ respectively such that
$$(\hat{a}-\hat{d})\cdot(\hat{b}-\hat{c})=(\hat{b}-\hat{d})\cdot(\hat{c}-\hat{a})=0$$
The point D, then, is the... of the triangle ABC. (IIT 1984; 2M)

3. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix}=0$ and the vectors $\vec{A}=(1, a, a^2)$, $\vec{B}=(1, b, b^2)$, $\vec{C}(1, c, c^2)$ are non-coplanar, then the product $abc=$ ... (IIT 1985; 2M)

4. If $\vec{A}, \vec{B}, \vec{C}$ are three non-coplanar vectors, then $\dfrac{\vec{A}\cdot(\vec{B}\times\vec{C})}{(\vec{C}\times\vec{A})\cdot\vec{B}}+\dfrac{\vec{B}\cdot(\vec{A}\times\vec{C})}{\vec{C}\cdot(\vec{A}\times\vec{B})}=\ldots$ (IIT 1985; 2M)

5. If $\vec{A}=(1,1,1)$, $\vec{C}=(0,1,-1)$ are given vectors, then a vector $\vec{B}$ satisfying the equations $\vec{A}\times\vec{B}=\vec{C}$ and $\vec{A}\cdot\vec{B}=3$ is ...... (IIT 1985, 91; 2M)

6. If the vectors $a\hat{i}+\hat{j}+\hat{k}$, $\hat{i}+b\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}+c\hat{k}$ $(a\neq b\neq c\neq 1)$ are coplanar, then the value of
$$\frac{1}{(1-a)}+\frac{1}{(1-b)}+\frac{1}{(1-c)}=\ldots$$
(IIT 1987; 2M)

7. Let $\vec{b}=4\hat{i}+3\hat{j}$ and $\vec{c}$ be two vectors perpendicular to each other in the xy-plane. All vectors in the same plane having projections 1 and 2 along $\vec{b}$ and $\vec{c}$, respectively, are given by... (IIT 1987; 2M)

8. The components of a vector $\vec{a}$ along and perpendicular to a non-zero vector $\vec{b}$ are ...and...respectively. (IIT 1988; 2M)

9. A unit vector coplanar with $\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$ and perpendicular to $\hat{i}+\hat{j}+\hat{k}$ is ... (IIT 1992; 2M)

10. A non-zero vector $\vec{a}$ is parallel to the line of intersection of the plane determined by the vectors $\hat{i}, \hat{i}+\hat{j}$ and the plane determined by the vectors $\hat{i}-\hat{j}, \hat{i}+\hat{k}$. The angle between $\vec{a}$ and the vector $\hat{i}-2\hat{j}+2\hat{k}$ is... (IIT 1996; 2M)

11. If $\vec{b}$ and $\vec{c}$ are any two non-collinear unit vectors and $\vec{a}$ is any vector, then
$$(\vec{a}\cdot\vec{b})\vec{b}+(\vec{a}\cdot\vec{c})\vec{c}+\frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{|\vec{b}\times\vec{c}|}(\vec{b}\times\vec{c})=\ldots\ldots$$
(IIT 1996; 2M)

12. Let $\vec{OA}=\vec{a}$, $\vec{OB}=10\vec{a}+2\vec{b}$, and $\vec{OC}=\vec{b}$, where O, A and C are non-collinear points. Let p denote the area of the quadrilateral $OABC$, and let q denote the area of the parallelogram with OA and OC as adjacent sides. If $p=kq$, then $k=\ldots$ (IIT 1997; 2M)

13. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors having magnitudes 1, 1 and 2 respectively. If $\vec{a} \times (\vec{a} \times \vec{c}) + \vec{b} = 0$, then the acute angle between $\vec{a}$ and $\vec{c}$ is ... **(IIT 1997; 2M)**

B TRUE / FALSE

1. Let $\vec{A}$, $\vec{B}$ and $\vec{C}$ be unit vectors. Suppose that $\vec{A}.\vec{B} = \vec{A}.\vec{C} = 0$, and that the angle between $\vec{B}$ and $\vec{C}$ is $\pi/6$. Then $\vec{A} = \pm 2(\vec{B} \times \vec{C})$. **(IIT 1981; 2M)**

2. If $\vec{X}\cdot\vec{A} = 0$, $\vec{X}\cdot\vec{B} = 0$, $\vec{X}\cdot\vec{C} = 0$ for some non-zero vector $\vec{X}$, then $[\vec{A}\ \vec{B}\ \vec{C}] = 0$ **(IIT 1983; 1M)**

3. The points with position vectors $\vec{a} + \vec{b}$, $\vec{a} - \vec{b}$ and $\vec{a} + k\vec{b}$ are collinear for all real values of k. **(IIT 1984; 1M)**

4. For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, $(\vec{a} - \vec{b})\{(\vec{b} - \vec{c}) \times (\vec{c} - \vec{a})\} = 2\vec{a}.(\vec{b} \times \vec{c})$. **(IIT 1989; 1M)**

C OBJECTIVE QUESTIONS

➡ ***Only one option is correct :***

1. The scalar $\vec{A}\cdot\{(\vec{B} + \vec{C}) \times (\vec{A} + \vec{B} + \vec{C})\}$ equals : **(IIT 1981; 2M)**

   (a) 0 (b) $[\vec{A}\ \vec{B}\ \vec{C}] + [\vec{B}\ \vec{C}\ \vec{A}]$

   (c) $[\vec{A}\ \vec{B}\ \vec{C}]$ (d) none of these

2. For non-zero vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $|(\vec{a} \times \vec{b}).\vec{c}| = |\vec{a}||\vec{b}||\vec{c}|$ holds, if and only if : **(IIT 1982; 2M)**

   (a) $\vec{a}.\vec{b} = 0$, $\vec{b}.\vec{c} = 0$

   (b) $\vec{b}.\vec{c} = 0$, $\vec{c}.\vec{a} = 0$

   (c) $\vec{c}.\vec{a} = 0$, $\vec{a}.\vec{b} = 0$

   (d) $\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$

3. The volume of the parallelopiped whose sides are given by $\vec{OA} = 2\hat{i} - 3\hat{j}$, $\vec{OB} = \hat{i} + \hat{j} - \hat{k}$, $\vec{OC} = 3\hat{i} - \hat{k}$, is : **(IIT 1983; 1M)**

   (a) $\frac{4}{13}$ (b) 4

   (c) $\frac{2}{7}$ (d) none of these

4. The points with position vectors $60\hat{i} + 3\hat{j}$, $40\hat{i} - 8\hat{j}$, $a\hat{i} - 52\hat{j}$ are collinear, if : **(IIT 1983; 1M)**

   (a) $a = -40$ (b) $a = 40$
   (c) $a = 20$ (d) none of these

5. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$, then

   $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$ is equal to : **(IIT 1986; 2M)**

   (a) 0
   (b) 1
   (c) $\frac{1}{4}(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$
   (d) $\frac{3}{4}(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)(c_1^2 + c_2^2 + c_3^2)$
   (e) none of these

6. A vector $\vec{a}$ has components $2p$ and 1 with respect to a rectangular cartesian system. This system is rotated through a certain angle about the origin in the counter clockwise sense. If, with respect to the new system, $\vec{a}$ has components $p+1$ and 1, then : **(IIT 1986; 2M)**

   (a) $p = 0$ (b) $p = 1$ or $p = -\frac{1}{3}$

   (c) $p = -1$ or $p = \frac{1}{3}$ (d) $p = 1$ or $p = -1$

   (e) none of these

7. The number of vectors of unit length perpendicular to vectors $\vec{a} = (1, 1, 0)$ and $\vec{b} = (0, 1, 1)$ is : **(IIT 1987; 2M)**

   (a) one (b) two
   (c) three (d) infinite
   (e) none of these

8. Let a, b, c be distinct non-negative numbers. If the vectors $a\hat{i} + a\hat{j} + c\hat{k}$, $\hat{i} + \hat{k}$ and $c\hat{i} + c\hat{j} + b\hat{k}$ lie in a plane, then c is : **(IIT 1993; 1M)**

(a) the Arithmetic Mean of a and b
(b) the Geometric Mean of a and b
(c) the Harmonic Mean of a and b
(d) equal to zero

9. Let $\vec a=\hat i-\hat j$, $\vec b=\hat j-\hat k$, $\vec c=\hat k-\hat i$. If $\vec d$ is a unit vector such that $\vec a\cdot\vec d=0=[\vec b\ \vec c\ \vec d]$, then $\vec d$ equals : **(IIT 1995S)**

(a) $\pm\dfrac{\hat i+\hat j-2\hat k}{\sqrt6}$ (b) $\pm\dfrac{\hat i+\hat j-\hat k}{\sqrt3}$

(c) $\pm\dfrac{\hat i+\hat j-\hat k}{\sqrt3}$ (d) $\pm\hat k$

10. If $\vec a,\vec b,\vec c$ are non coplanar unit vectors such that $\vec a\times(\vec b\times\vec c)=\dfrac{(\vec b+\vec c)}{\sqrt2}$, then the angle between $\vec a$ and $\vec b$ is : **(IIT 1995S)**

(a) $\dfrac{3\pi}{4}$ (b) $\dfrac{\pi}{4}$

(c) $\dfrac{\pi}{2}$ (d) π

11. Let $\vec u,\vec v$ and $\vec w$ be vectors such that $\vec u+\vec v+\vec w=0$. If $|\vec u|=3$, $|\vec v|=4$ and $|\vec w|=5$, then $\vec u\cdot\vec v+\vec v\cdot\vec w+\vec w\cdot\vec u$ is : **(IIT 1995S)**

(a) 47 (b) -25
(c) 0 (d) 25

12. If $\vec a,\vec b$ and $\vec c$ are three non coplanar vectors, then $(\vec a+\vec b+\vec c)\cdot[(\vec a+\vec b)\times(\vec a+\vec c)]$ equals : **(IIT 1995S)**

(a) 0 (b) $[\vec a\ \vec b\ \vec c]$

(c) $2\cdot[\vec a\ \vec b\ \vec c]$ (d) $-[\vec a\ \vec b\ \vec c]$

13. If $\vec p,\vec q,\vec r$ be three mutually perpendicular vectors of the same magnitude. If a vector $\vec x$ satisfies the equation

$$\vec p\times\{(\vec x-\vec q)\times\vec p\}+\vec q\times\{(\vec x-\vec r)\times\vec q\}+\vec r\times\{(\vec x-\vec p)\times\vec r\}=\vec 0,$$

then $\vec x$ is given by : **(IIT 1997C; 2M)**

(a) $\frac12(\vec p+\vec q-2\vec r)$ (b) $\frac12(\vec p+\vec q+\vec r)$

(c) $\frac13(\vec p+\vec q+\vec r)$ (d) $\frac13(2\vec p+\vec q-\vec r)$

14. If $\vec a=\hat i+\hat j+\hat k$, $\vec b=4\hat i+3\hat j+4\hat k$ and $\vec c=\hat i+\alpha\hat j+\beta\hat k$ are linearly dependent vectors and $|\vec c|=\sqrt3$, then : **(IIT 1998; 2M)**

(a) $\alpha=1,\beta=-1$ (b) $\alpha=1,\beta=\pm1$
(c) $\alpha=-1,\beta=\pm1$ (d) $\alpha=\pm1,\beta=1$

15. For three vectors $\vec u,\vec v,\vec w$ which of the following expressions is not equal to any of the remaining three ? **(IIT 1998; 2M)**

(a) $\vec u\cdot(\vec v\times\vec w)$ (b) $(\vec v\times\vec w)\cdot\vec u$

(c) $\vec v\cdot(\vec u\times\vec w)$ (d) $(\vec u\times\vec v)\cdot\vec w$

16. Let $\vec a=2\hat i+\hat j-2\hat k$ and $\vec b=\hat i+\hat j$. If $\vec c$ is a vector such that $\vec a\cdot\vec c=|\vec c|$, $|\vec c-\vec a|=2\sqrt2$ and the angle between $(\vec a\times\vec b)$ and $\vec c$ is 30°, then $|(\vec a\times\vec b)\times\vec c|$ is equal to : **(IIT 1999; 2M)**

(a) $\dfrac23$ (b) $\dfrac32$
(c) 2 (d) 3

17. Let $\vec a=2\hat i+\hat j+\hat k$, $\vec b=\hat i+2\hat j-\hat k$ and a unit vector $\vec c$ be coplanar. If $\vec c$ is perpendicular to $\vec a$, then $\vec c$ is equal to : **(IIT 1999; 2M)**

(a) $\dfrac{1}{\sqrt2}(-\hat j+\hat k)$ (b) $\dfrac{1}{\sqrt3}(-\hat i-\hat j-\hat k)$

(c) $\dfrac{1}{\sqrt5}(\hat i-2\hat j)$ (d) $\dfrac{1}{\sqrt5}(\hat i-\hat j-\hat k)$

18. If the vectors $\vec a,\vec b$ and $\vec c$ from the sides BC, CA and AB respectively of a triangle ABC, then : **(IIT 2000)**

(a) $\vec a\cdot\vec b+\vec b\cdot\vec c+\vec c\cdot\vec a=0$

(b) $\vec a\times\vec b=\vec b\times\vec c=\vec c\times\vec a$

(c) $\vec a\cdot\vec b=\vec b\cdot\vec c=\vec c\cdot\vec a$

(d) $\vec a\times\vec b+\vec b\times\vec c+\vec c\times\vec a=0$

19. Let the vectors $\vec a,\vec b,\vec c$ and $\vec d$ be such that $(\vec a\times\vec b)\times(\vec c\times\vec d)=0$. Let P_1 and P_2 be planes determined by the pairs of vectors $\vec a,\vec b$ and $\vec c,\vec d$ respectively, then the angle between P_1 and P_2 is : **(IIT 2000)**

(a) 0 (b) $\pi/4$
(c) $\pi/3$ (d) $\pi/2$

20. If $\vec a,\vec b$ and $\vec c$ are unit coplanar vectors, then the scalar triple product $[2\vec a-\vec b, 2\vec b-\vec c, 2\vec c-\vec a]$ is equal to : **(IIT 2000)**

(a) 0 (b) 1
(c) $-\sqrt3$ (d) $\sqrt3$

21. If $\vec a, \vec b$ and $\vec c$ are unit vectors, then $|\vec a-\vec b|^2+|\vec b-\vec c|^2+|\vec c-\vec a|^2$ does not exceed : **(IIT 2001)**

(a) 4 (b) 9 (c) 8 (d) 6

22. Let $\vec a=\hat i-\hat k$, $\vec b=x\hat i+\hat j+(1-x)\hat k$ and $\vec c=y\hat i+x\hat j+(1+x-y)\hat k$. Then $[\vec a\ \vec b\ \vec c]$ depends on : **(IIT 2001)**

(a) only x (b) only y
(c) neither x nor y (d) both x and y

23. If $\vec a$ and $\vec b$ are two unit vectors such that $\vec a+2\vec b$ and $5\vec a-4\vec b$ are perpendicular to each other then the angle between $\vec a$ and $\vec b$ is : **(IIT 2002)**

(a) 45° (b) 60°
(c) $\cos^{-1}\left(\frac{1}{3}\right)$ (d) $\cos^{-1}\left(\frac{2}{7}\right)$

24. Let $\vec V=2\hat i+\hat j-\hat k$ and $\vec W=\hat i+3\hat k$. If $\vec U$ is a unit vector, then the maximum value of the scalar triple product $[\vec U\ \vec V\ \vec W]$ is : **(IIT 2002)**

(a) 1 (b) $\sqrt{10}+\sqrt{6}$
(c) $\sqrt{59}$ (d) $\sqrt{60}$

25. The value of 'a' so that the volume of parallelopiped formed by $\hat i+a\hat j+\hat k$, $\hat j+a\hat k$ and $a\hat i+\hat k$ because minimum is : **(IIT 2003)**

(a) -3 (b) 3
(c) $1/\sqrt3$ (d) $\sqrt3$

26. If $\vec a=(\hat i+\hat j+\hat k)$, $\vec a\cdot\vec b=1$ and $\vec a\times\vec b=\hat j-\hat k$, then $\vec b$ is : **(IIT 2003)**

(a) $\hat i-\hat j+\hat k$ (b) $2\hat j-\hat k$
(c) $\hat i$ (d) $2\hat i$

27. The unit vector which is orthogonal to the vector $3\hat i+2\hat j+6\hat k$ and is coplanar with the vectors $2\hat i+\hat j+\hat k$ and $\hat i-\hat j+\hat k$ is : **(IIT 2004)**

(a) $\frac{2\hat i-6\hat j+\hat k}{\sqrt{41}}$ (b) $\frac{2\hat i-3\hat j}{\sqrt{13}}$
(c) $\frac{3\hat j-\hat k}{\sqrt{10}}$ (d) $\frac{4\hat i+3\hat j-3\hat k}{\sqrt{34}}$

28. If $\vec a, \vec b, \vec c$ are three non-zero, non-coplanar vectors and $\vec b_1=\vec b-\frac{\vec b\cdot\vec a}{|\vec a|^2}\vec a$, $\vec b_2=\vec b+\frac{\vec b\cdot\vec a}{|\vec a|^2}\vec a$,

$$\vec c_1=\vec c-\frac{\vec c\cdot\vec a}{|\vec a|^2}\vec a-\frac{\vec c\cdot\vec b}{|\vec b|^2}\vec b,$$

$$\vec c_2=\vec c-\frac{\vec c\cdot\vec a}{|\vec a|^2}\vec a-\frac{\vec c\cdot\vec b_1}{|\vec b_1|^2}\vec b_1,$$

$$\vec c_3=\vec c-\frac{\vec c\cdot\vec a}{|\vec a|^2}\vec a-\frac{\vec c\cdot\vec b_2}{|\vec b_2|^2}\vec b_2,\quad \vec c_4=\vec a-\frac{\vec c\cdot\vec a}{|\vec a|^2}\vec a,$$

Then which of the following is a set of mutually orthogonal vectors? **(IIT 2005)**

(a) $\{\vec a, \vec b_1, \vec c_1\}$ (b) $\{\vec a, \vec b_1, \vec c_2\}$
(c) $\{\vec a, \vec b_2, \vec c_3\}$ (d) $\{\vec a, \vec b_2, \vec c_4\}$

29. Let, $\vec a=\hat i+2\hat j+\hat k$, $\vec b=\hat i-\hat j+\hat k$, $\vec c=\hat i+\hat j-\hat k$. A vector coplanar to $\vec a$ and $\vec b$ has a projection along $\vec c$ of magnitude $\frac{1}{\sqrt3}$, then the vector is : **(IIT 2006)**

(a) $4\hat i-\hat j+4\hat k$ (b) $4\hat i+\hat j-4\hat k$
(c) $2\hat i+\hat j+\hat k$ (d) none of these

30. The number of distinct real values of λ, for which the vectors $-\lambda^2\hat i+\hat j+\hat k$, $\hat i-\lambda^2\hat j+\hat k$ and $\hat i+\hat j-\lambda^2\hat k$ are coplanar, is : **(IIT 2007)**

(a) zero (b) one
(c) two (d) three

31. Let $\vec a, \vec b, \vec c$ be unit vectors such that $\vec a+\vec b-\vec c=\vec 0$. Which one of the following is correct ? **(IIT 2007)**

(a) $\vec a\times\vec b-\vec b\times\vec c=\vec c\times\vec a=\vec 0$

(b) $\vec a\times\vec b=\vec b\times\vec c=\vec c\times\vec a\neq\vec 0$

(c) $\vec a\times\vec b=\vec b\times\vec c-\vec a\times\vec c-\vec 0$

(d) $\vec a\times\vec b$, $\vec b\times\vec c$, $\vec c\times\vec a$ are mutually perpendicular

D OBJECTIVE QUESTIONS

➡ ***More than one options are correct :***

1. Let $\vec a=2\hat i-\hat j+\hat k$, $\vec b=\hat i+2\hat j-\hat k$ and $\vec c=\hat i+\hat j-2\hat k$ be three vectors. A vector in the plane of $\vec b$ and $\vec c$, whose projection on $\vec a$ is of magnitude $\sqrt{2/3}$, is : **(IIT 1993; 2M)**

(a) $2\hat i+3\hat j-3\hat k$ (b) $2\hat i+3\hat j+3\hat k$
(c) $-2\hat i-\hat j+5\hat k$ (d) $2\hat i+\hat j+5\hat k$

2. Which of the following expressions are meaningfull question ? (IIT 1998; 2M)

(a) $\vec{u}\cdot(\vec{v}\times\vec{w})$ (b) $(\vec{u}\cdot\vec{v})\cdot\vec{w}$

(c) $(\vec{u}\cdot\vec{v})\vec{w}$ (d) $\vec{u}\times(\vec{v}\cdot\vec{w})$

3. Let $\vec{a}$ and $\vec{b}$ be two non-collinear unit vectors. If $\vec{u}=\vec{a}-(\vec{a}\cdot\vec{b})\vec{b}$ and $\vec{v}=\vec{a}\times\vec{b}$, then $|\vec{v}|$ is : (IIT 1999; 3M)

(a) $|\vec{u}|$ (b) $|\vec{u}|+|\vec{u}\cdot\vec{a}|$

(c) $|\vec{u}|+|\vec{u}\cdot\vec{b}|$ (d) $|\vec{u}|-\vec{u}\cdot(\vec{a}+\vec{b})$

4. Let $\vec{A}$ be vector parallel to line of intersection of planes P_1 and P_2 through origin. P_1 is parallel to the vectors $2\hat{j}+3\hat{k}$ and $4\hat{j}-3\hat{k}$ and P_2 is parallel to $\hat{j}-\hat{k}$ and $3\hat{i}+3\hat{j}$, then the angle between vector $\vec{A}$ and $2\hat{i}+\hat{j}-2\hat{k}$ is : (IIT 2006)

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{6}$ (d) $\frac{3\pi}{4}$

E SUBJECTIVE QUESTIONS

1. $A_1, A_2, \ldots\ldots, A_n$ are the vertices of a regular plane polygon with n sides and O is its centre. Show that (IIT 1982; 2M)

$$\sum_{i=1}^{n-1}(\vec{OA}_i\times\vec{OA}_{i+1})=(1-n)(\vec{OA}_2\times\vec{OA}_1)$$

2. Find all values of λ such that $x, y, z \neq (0,0,0)$ and

$$(\vec{i}-\vec{j}+3\vec{k})x+(3\vec{i}-3\vec{j}+\vec{k})y+(-4\vec{i}+5\vec{j})z=\lambda(\vec{i}x+\vec{j}y+\vec{k}z)$$

where $\vec{i}, \vec{j}, \vec{k}$ are unit vectors along the co-ordinate axes. (IIT 1982; 2M)

3. (a) If C be a given non-zero scalar and $\vec{A}$ and $\vec{B}$ be given non-zero vectors such that $\vec{A}\perp\vec{B}$, find the vector $\vec{X}$ which satisfies the equations $\vec{A}\cdot\vec{X}=c$ and $\vec{A}\times\vec{X}=\vec{B}$. (IIT 1983)

(b) $\vec{A}$ vector A has components A_1, A_2, A_3 in a right-handed rectangular cartesian coordinate system oxyz. The coordinate system is rotated about the x-axis through an angle $\frac{\pi}{2}$. Find the components of A in the new coordinate system, in terms of A_1, A_2, A_3. (IIT 1983; 2M)

4. The position vectors of the points A, B, C and D are $3\hat{i}-2\hat{j}-\hat{k}, 2\hat{i}+3\hat{j}-4\hat{k}, -\hat{i}+\hat{j}+2\hat{k}$ and $4\hat{i}+5\hat{j}+\lambda\hat{k}$, respectively. If the points A, B, C and D lie on a plane, find the value of λ. (IIT 1986; $2\frac{1}{2}$ M)

5. If A, B, C, D are any four points in space, prove that :

$$|\vec{AB}\times\vec{CD}+\vec{BC}\times\vec{AD}+\vec{CA}\times\vec{BD}| = 4\,(\text{area of triangle } ABC).$$

(IIT 1987; 2M)

6. Let $OACB$ be a parallelogram with O at the origin and OC a diagonal. Let D be the midpoint of OA. Using vector methods prove that BD and CO intersect in the same ratio. Determine this ratio. (IIT 1988; 3M)

7. If vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar, show that

$$\begin{vmatrix}\vec{a} & \vec{b} & \vec{c}\\ \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c}\end{vmatrix}=\vec{0}$$

(IIT 1989; 2M)

8. In a triangle OAB, E is the midpoint of BO and D is a point on AB such that $AD:DB=2:1$. If OD and AE intersect at P, determine the ratio $OP:PD$ using vector methods. (IIT 1989; 4M)

9. Let $\vec{A}=2\hat{i}+\hat{k}$, $\vec{B}=\hat{i}+\hat{j}+\hat{k}$, and $\vec{C}=4\hat{i}-3\hat{j}+7\hat{k}$. Determine a vector $\vec{R}$ satisfying $\vec{R}\times\vec{B}=\vec{C}\times\vec{B}$ and $\vec{R}\cdot\vec{A}=0$. (IIT 1990; 3M)

10. Determine the value of 'c' so that for all real x, the vector $cx\hat{i}-6\hat{j}-3\hat{k}$ and $x\hat{i}+2\hat{j}+2cx\hat{k}$ make an obtuse angle with each other. (IIT 1991; 4M)

11. In a triangle ABC, D and E are points on BC and AC respectively, such that $BD=2DC$ and $AE=3EC$. Let P be the point of intersection of AD and BE. Find BP/PE using vector methods. (IIT 1993; 5M)

12. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be non-coplanar unit vectors, equally inclined to one another at an angle θ. If $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}=p\vec{a}+q\vec{b}+r\vec{c}$, find scalars p, q and r in terms of θ. (IIT 1997C; 5M)

13. Prove, by vector methods or otherwise, that the point of intersection of the diagonals of a trapezium lies on the line passing through the mid-points of the parallel sides. (You may assume that the trapezium is not a parallelogram). (IIT 1998; 8M)

14. If the vectors $\vec{b}, \vec{c}, \vec{d}$, are not coplanar, then prove that the vector $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})+(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b})+(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$ is parallel to $\vec{a}$. (IIT 1994; 4M)

15. The position vectors of the vertices A, B and C of a tetrahedron $ABCD$ are $\hat{i}+\hat{j}+\hat{k}, \hat{i}$ and $3\hat{i}$, respectively. The altitude from vertex D to the opposite face ABC meets the median line through A of the triangle ABC at a point E. If the length of the side AD is 4 and the volume of the tetrahedron is $\frac{2\sqrt{2}}{2}$, find the position vector of the point E for all its possible positions. **(IIT 1996; 5M)**

16. If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are vectors such that $|\vec{B}| = |\vec{C}|$. Prove that $[(\vec{A}+\vec{B})\times(\vec{A}+\vec{C})]\times(\vec{B}\times\vec{C}).(\vec{B}-\vec{C})=0$ **(IIT 1997; 5M)**

17. For any two vectors $\vec{u}$ and $\vec{v}$ prove that

(a) $|\vec{u}.\vec{v}|^2 + |\vec{u}\times\vec{v}|^2 = |\vec{u}|^2|\vec{v}|^2$ and

(b) $(1+|\vec{u}|^2)(1+|\vec{v}|^2)$

$= |1-\vec{u}.\vec{v}|^2 + |\vec{u}+\vec{v}+(\vec{u}\times\vec{v})|^2$ **(IIT 1998; 8M)**

18. Let $\vec{u}$ and $\vec{v}$ be unit vectors. If $\vec{w}$ is a vector such that $\vec{w}+(\vec{w}\times\vec{u})=\vec{v}$, then prove that $|(\vec{u}\times\vec{v}).\vec{w}| \le \frac{1}{2}$ and that the equality holds if and only if $\vec{u}$ is perpendicular to $\vec{v}$. **(IIT 1999; 10M)**

19. Let ABC and PQR be any two triangles in the same plane. Assume that the perpendiculars from the points A, B, C to the sides QR, RP, PQ respectively are concurrent. Using vector methods or otherwise, prove that the perpendiculars from P, Q, R to BC, CA, AB respectively are also concurrent. **(IIT 2000)**

20. Show, by vector methods, that the angular bisectors of a triangle are concurrent and find an expression for the position vector of the point of concurrency in terms of the position vectors of the vertices. **(IIT 2001; 5M)**

21. Find 3-dimensional vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ satisfying

$\vec{v}_1.\vec{v}_1 = 4, \vec{v}_1.\vec{v}_2 = -2, \vec{v}_1.\vec{v}_3 = 6,$

$\vec{v}_2.\vec{v}_2 = 2, \vec{v}_2.\vec{v}_3 = -5, \vec{v}_3.\vec{v}_3 = 29$ **(IIT 2001; 5M)**

22. Let $\vec{A}(t) = f_1(t)\hat{i} + f_2(t)\hat{j}$ and $\vec{B}(t) = g_1(t)\hat{i} + g_2(t)\hat{j}, t \in [0,1]$, where f_1, f_2, g_1, g_2 are continuous functions. If $\vec{A}(t)$ and $\vec{B}(t)$ are non-zero vectors.

for all t and $\vec{A}(0) = 2\hat{i}+3\hat{j}$, $\vec{A}(1) = 6\hat{i}+2\hat{j}$,

$\vec{B}(0) = 3\hat{i}+2\hat{j}$ and $\vec{B}(1) = 2\hat{i}+6\hat{j}$

Then show that $\vec{A}(t)$ and $\vec{B}(t)$ are parallel for some t. **(IIT 2001; 5M)**

23. Let V be the volume of the parallelopiped formed by the vectors

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$.

If a_r, b_r, c_r, where $r = 1, 2, 3$ are non-negative real numbers and $\sum_{r=1}^{3}(a_r + b_r + c_r) = 3L$. Show that $V \le L^3$. **(IIT 2002; 5M)**

24. If $\vec{u}, \vec{v}, \vec{w}$ are three non-coplanar unit vectors and α, β, γ are the angles between $\vec{u}$ and $\vec{v}$, $\vec{v}$ and $\vec{w}$, $\vec{w}$ and $\vec{u}$ respectively and $\vec{x}, \vec{y}, \vec{z}$ are unit vectors along the bisectors of the angles α, b, g respectively. Prove that

$[\vec{x}\times\vec{y} \quad \vec{y}\times\vec{z} \quad \vec{z}\times\vec{x}]$

$= \frac{1}{16}[\vec{u} \quad \vec{v} \quad \vec{w}]^2 \sec^2\frac{\alpha}{2}\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}$. **(IIT 2003; 4M)**

25. If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are four distinct vectors satisfying the conditions $\vec{a}\times\vec{b} = \vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c} = \vec{b}\times\vec{d}$, then prove that $\vec{a}.\vec{b} + \vec{c}.\vec{d} \ne \vec{a}.\vec{c} + \vec{b}.\vec{d}$. **(IIT 2004; 2M)**

26. Incident ray is along the unit vector $\hat{v}$ and the reflected ray is along the unit vector $\hat{w}$. The normal is along unit vector $\hat{a}$ outwards. Express $\hat{w}$ in terms of $\hat{a}$ and $\hat{v}$. **(IIT 2005)**

G ASSERTION AND REASON

This question contains STATEMENT-I (Assertion) and STATEMENT-II (Reason).

1. Let the vectors $\vec{PQ}, \vec{QR}, \vec{RS}, \vec{ST}, \vec{TU}$ and $\vec{UP}$ represent the sides of a regular hexagon. **(IIT 2007)**

Statement-1 : $\vec{PQ}\times(\vec{RS}+\vec{ST}) \ne \vec{0}$.

Because

Statement-II : $\vec{PQ}\times\vec{RS} = \vec{0}$ and $\vec{PQ}\times\vec{ST} \ne \vec{0}$

(a) Statement-I is True, Statement-II is True; Statement-II is a correct explanation for Statement-I

(b) Statement-I is True, Statement-II is True; Statement-II is NOT a correct explanation for Statement-I

(c) Statement-I is True, Statement-II is False

(d) Statement-I is False, Statement-II is True

ANSWERS

A Fill in the Blanks

1. $5\sqrt{2}$ 2. orthocentre 3. 1 4. 0 5. $\left(\frac{5}{3}, \frac{2}{3}, \frac{2}{3}\right)$ 6. 1

7. $(2\hat{i} - \hat{j})$ or $\frac{1}{5}(-2\hat{i} + 11\hat{j})$ 8. $\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$ and $\vec{a} - \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$ 9. $\pm\frac{1}{\sqrt{2}}(-\hat{j} + \hat{k})$ 10. $\frac{\pi}{4}$ or $\frac{3\pi}{4}$ 11. $\vec{a}$ 12. 6 13. $\frac{\pi}{6}$

B True / False

1. True 2. False 3. False 4. True

C Objective Questions (Only one option)

1. (a)	2. (d)	3. (b)	4. (a)	5. (c)	6. (b)	7. (b)
8. (b)	9. (a)	10. (a)	11. (b)	12. (d)	13. (b)	14. (d)
15. (c)	16. (b)	17. (a)	18. (b)	19. (a)	20. (a)	21. (b)
22. (c)	23. (b)	24. (c)	25. (c)	26. (c)	27. (c)	28. (b)
29. (a)	30. (c)	31. (b)				

D Objective Questions (More than one option)

1. (a, c) 2. (a, c) 3. (b, c) 4. (b, d)

E Subjective Questions

2. 0, –1 3. (a) $\vec{X} = \left(\frac{c}{|\vec{A}|^2}\right)\vec{A} - \left(\frac{1}{|\vec{A}|^2}\right)(\vec{A} \times \vec{B})$ (b) $(A_2\hat{i} - A_1\hat{j} + A_3\hat{k})$

4. $\frac{146}{17}$ 6. 2 : 1 8. 3 : 2 9. $-\hat{i} - 8\hat{j} + 2\hat{k}$ 10. $\left(-\frac{4}{3}, 0\right)$ 11. $\frac{8}{3}$

12. $p = \frac{1}{\sqrt{1 + 2\cos 2\theta}}$, $q = \frac{-\cos\theta}{\sqrt{1 + \cos\theta}}$ and $r = \frac{1}{\sqrt{1 + \cos\theta}}$ 15. $-\hat{i} + 3\hat{j} - 3\hat{k}$ and $3\hat{i} - \hat{j} - \hat{k}$

21. $\vec{v}_1 = 2\hat{i}$, $\vec{v}_2 = -\hat{i} + \hat{j}$ and $\vec{v}_3 = 3\hat{i} \pm 2\hat{j} \pm 4\hat{k}$ 26. $\hat{w} = \hat{v} - 2(\hat{a}\cdot\hat{v})\hat{a}$

G Assertion and Reason

1. (c)

SOLUTIONS

A FILL IN THE BLANKS

1. $|\vec{A}| = 3, |\vec{B}| = 4, |\vec{C}| = 5$

$$\vec{A}\cdot(\vec{B} + \vec{C}) = \vec{B}\cdot(\vec{C} + \vec{A}) = \vec{C}\cdot(\vec{A} + \vec{B}) = 0 \quad \dots (1)$$

$$\therefore |\vec{A} + \vec{B} + \vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2 + |\vec{C}|^2 + 2(\vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{C} + \vec{C}\cdot\vec{A})$$

$$= 9 + 16 + 25 + 0$$

{using (1) $\vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{C} + \vec{C}\cdot\vec{A} = 0$}

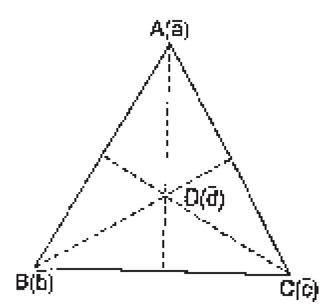
$$\therefore \quad |\vec{A} + \vec{B} + \vec{C}|^2 = 50$$

or $$|\vec{A} + \vec{B} + \vec{C}| = 5\sqrt{2}$$

2. As, $(\hat{a} - \hat{d}).(\hat{b} - \hat{c}) = (\hat{b} - \hat{d}).(\hat{c} - \hat{a}) = 0$

$\Rightarrow$ $AD \perp BC$ and $BD \perp CA$

which clearly represents from figure that D is orthocentre of ΔABC



3. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix}=0$

$$\Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}=0$$

$$\Rightarrow (1+abc)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}=0$$

i.e., either $(1+abc)=0$ or $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}=0$

But. $(1, a, a^2)$, $(1, b, b^2)$, $(1, c, c^2)$ are non-coplanar.

$$\Rightarrow \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0$$

$$\therefore \quad abc=-1$$

4. $$\frac{\vec{A}\cdot(\vec{B}\times\vec{C})}{(\vec{C}\times\vec{A})\cdot\vec{B}}+\frac{\vec{B}\cdot(\vec{A}\times\vec{C})}{\vec{C}\cdot(\vec{A}\times\vec{B})}$$

$$\Rightarrow \frac{[\vec{A}\ \vec{B}\ \vec{C}]}{[\vec{C}\ \vec{A}\ \vec{B}]}+\frac{[\vec{B}\ \vec{A}\ \vec{C}]}{[\vec{C}\ \vec{A}\ \vec{B}]}$$

$$\Rightarrow \frac{[\vec{A}\ \vec{B}\ \vec{C}]+[\vec{B}\ \vec{A}\ \vec{C}]}{[\vec{C}\ \vec{A}\ \vec{B}]}=\frac{[\vec{A}\ \vec{B}\ \vec{C}]-[\vec{A}\ \vec{B}\ \vec{C}]}{[\vec{C}\ \vec{A}\ \vec{B}]}=0$$

5. Let $\vec{B}=x\hat{i}+y\hat{j}+z\hat{k}$

Also $\quad \vec{A}=\hat{i}+\hat{j}+\hat{k}$

$\quad \vec{C}=\hat{j}-\hat{k}$

from $\vec{A}\times\vec{B}=\vec{C}$, we get

$$z-y=0,\ x-z=1,\ y-x=-1$$

also $\quad \vec{A}\cdot\vec{B}=3$

$$\Rightarrow \quad x+y+z=3$$

Solving above equations

$$x=\frac{5}{3},\ y=z=\frac{2}{3}$$

$$\therefore \quad \vec{B}\left(\frac{5}{3},\frac{2}{3},\frac{2}{3}\right)$$

6. Since, vectors are coplanar

$$\Rightarrow \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}=0,$$

applying $R_2 \to R_2-R_1$, $R_3 \to R_3-R_1$

$$\Rightarrow \begin{vmatrix} a & 1 & 1 \\ 1-a & b-1 & 0 \\ 1-a & 0 & c-1 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} a/(1-a) & 1/(1-b) & 1/(1-c) \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix}=0$$

$$\Rightarrow \frac{a}{1-a}(1)-\frac{1}{1-b}(-1)+\frac{1}{1-c}(1)=0$$

$$\Rightarrow \frac{a}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=0$$

or $$\frac{a-1+1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=0$$

or $$-1+\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=0$$

$$\Rightarrow \frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=1$$

7. $\vec{b}=4\hat{i}+3\hat{j}$ and $\vec{c}=3\lambda\hat{i}-4\lambda\hat{j}$

for all values of λ

we chose $\vec{c}$ such that $\vec{b}\cdot\vec{c}=0$ as $\vec{b}$ and $\vec{c}$ are perpendicular.

Let the required vertex be $\vec{\alpha}=p\hat{i}+q\hat{j}$

projection of $\vec{\alpha}$ on $\vec{b}$ is $\dfrac{\vec{\alpha}\cdot\vec{b}}{|\vec{b}|}$

$$\therefore \quad 1=\frac{4p+3q}{5} \quad \text{or} \quad 4p+3q=5 \qquad \ldots(1)$$

also projection of $\vec{\alpha}$ on $\vec{c}$ is $\dfrac{\vec{\alpha}\cdot\vec{c}}{|\vec{c}|}$

$$\Rightarrow \quad 2=\frac{3\lambda p-4\lambda q}{\pm 5\lambda}$$

$$\Rightarrow \quad 3p-4q=10 \text{ or } 3p-4q=-10$$

modulus of vector $\vec{c}=\begin{cases} 5\lambda, & \text{if } \lambda>0 \\ -5\lambda, & \text{if } \lambda<0 \end{cases}$

solving above equations $p=2,\ q=-1$ and

$$\therefore \quad \alpha=2\hat{i}-\hat{j} \text{ or } \frac{1}{5}(-2\hat{i}+11\hat{j})$$

8. Vector component of $\vec{a}$ along and perpendicular to $\vec{b}$ are

$$\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b} \quad \text{and} \quad \vec{a}-\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

9. Any vector coplanar with $\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$ is given by

$$\vec{a}=x(\hat{i}+\hat{j}+2\hat{k})+y(\hat{i}+2\hat{j}+\hat{k})$$
$$=(x+y)\hat{i}+(x+2y)\hat{j}+(2x+y)\hat{k}$$

This vector is perpendicular to $\hat{i}+\hat{j}+\hat{k}$ if

$$(x+y)\cdot 1+(x+2y)1+(2x+y)1=0$$
$$\Rightarrow \quad 4x+4y=0 \quad\Rightarrow\quad y=-x$$
$$\Rightarrow \quad \vec{a}=-x\hat{j}+x\hat{k}=x(-\hat{j}+\hat{k})$$

Therefore, $|\vec{a}|=\sqrt{2}|x|$, so that the required unit vector is

$$\hat{a}=\pm\frac{1}{\sqrt{2}}(-\hat{j}+\hat{k})$$

10. Equation of the plane containing $\hat{i}$ and $\hat{i}+\hat{j}$ is

$$[(\vec{r}-\hat{i})\ \hat{i}\ (\hat{i}+\hat{j})]=0$$
$$\Rightarrow \quad (\vec{r}-\hat{i})\cdot[\hat{i}\times(\hat{i}+\hat{j})]=0$$
$$\Rightarrow \quad \{(x\hat{i}+y\hat{j}+z\hat{k})-\hat{i}\}\cdot[\hat{i}\times\hat{i}+\hat{i}\times\hat{j}]=0$$
$$\Rightarrow \quad \{(x-1)\hat{i}+y\hat{j}+z\hat{k}\}[\hat{k}]=0$$
$$\Rightarrow \quad (x-1)\hat{i}\cdot\hat{k}+y\hat{j}\cdot\hat{k}+z\hat{k}\cdot\hat{k}=0$$
$$\Rightarrow \quad z=0 \qquad \ldots(1)$$

Equation of the plane containing $\hat{i}-\hat{j}$ and $\hat{i}+\hat{k}$ is

$$[\vec{r}-(\hat{i}-\hat{j})\ \ (\hat{i}-\hat{j})\ (\hat{i}+\hat{k})]=0$$
$$\Rightarrow \quad (\vec{r}-\hat{i}+\hat{j})\cdot[(\hat{i}-\hat{j})\times(\hat{i}+\hat{k})]=0$$
$$\Rightarrow \{(x\hat{i}-y\hat{j}+z\hat{k})-(\hat{i}-\hat{j})\}\cdot[\hat{i}\times\hat{i}+\hat{i}\times\hat{k}-\hat{j}\times\hat{i}-\hat{j}\times\hat{k}]=0$$
$$\Rightarrow \{(x-1)\hat{i}+(y+1)\hat{j}+z\hat{k}\}\cdot[-\hat{j}+\hat{k}-\hat{i}]=0$$
$$\Rightarrow \quad -(x-1)-(y+1)+z=0 \qquad \ldots(2)$$

Let $\quad \vec{a}=a_1\hat{i}+a_2\hat{j}-a_3\hat{k}$

Since $\vec{a}$ is parallel to (1) and (2), we obtain

$$a_3=0 \quad \text{and} \quad a_1+a_2-a_3=0$$
$$\Rightarrow \quad a_1=-a_2,\ a_3=0$$

Thus a vector in the direction $\vec{a}$ is $\vec{p}=\hat{i}-\hat{j}$

If θ is the angle between $\vec{a}$ and $\hat{i}-2\hat{j}+2\hat{k}$, then

$$\cos\theta=\pm\frac{(1)(1)+(-1)(-2)}{\sqrt{1+1}\sqrt{1+4+4}}=\pm\frac{3}{(\sqrt{2})\cdot 3}$$

$$\cos\theta=\mp\frac{1}{\sqrt{2}} \quad\Rightarrow\quad \theta=\frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

11. There is **printing mistake** in IIT paper. The question should read as follows:

$$(\vec{a}\cdot\vec{b})\vec{b}+(\vec{a}\cdot\vec{c})\vec{c}+\frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{|\vec{b}\times\vec{c}|^2}(\vec{b}\times\vec{c})=\ \ldots\text{(given)}$$

Let $\hat{i}$ be a unit vector in the direction of $\vec{b}$, $\hat{j}$ in the direction of $\vec{c}$. Note that $\vec{c}=\hat{j}$

and $\quad (\vec{b}\times\vec{c})=|\vec{b}||\vec{c}|\sin\alpha\ \hat{k}=\sin\alpha\hat{k}$

Where $\hat{k}$ is a unit vector perpendicular to $\vec{b}$ and $\vec{c}$

$$\Rightarrow \quad |\vec{b}\times\vec{c}|=\sin\alpha$$

$$\Rightarrow \quad \hat{k}=\frac{\vec{b}\times\vec{c}}{|\vec{b}\times\vec{c}|}$$

Any vector $\vec{a}$ can be written as linear combination of $\hat{i}$, $\hat{j}$ and $\hat{k}$.

Let $\quad \vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

Now $\quad \vec{a}\cdot\vec{b}=\vec{a}\cdot\hat{i}=\hat{i}\cdot(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})=a_1$

and $\quad \vec{a}\cdot\vec{c}=\vec{a}\cdot\hat{j}=\hat{j}\cdot(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})=a_2$

and $\quad \vec{a}\cdot\frac{\vec{b}\times\vec{c}}{|\vec{b}\times\vec{c}|}=\vec{a}\cdot\hat{k}=a_3$

Therefore, $(\vec{a}\cdot\vec{b})\vec{b}+(\vec{a}\cdot\vec{c})\vec{c}+\frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{|\vec{b}\times\vec{c}|^2}(\vec{b}\times\vec{c})$

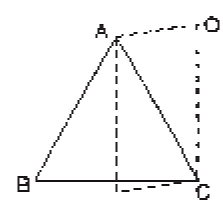
$$=a_1\vec{b}+a_2\vec{c}+a_3\frac{(\vec{b}\times\vec{c})}{|\vec{b}\times\vec{c}|}$$

$$=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}=\vec{a}$$

12. q = area of parallelogram with $\overrightarrow{OA}$ and $\overrightarrow{OC}$ as adjacent sides

$$=|\overrightarrow{OA}\times\overrightarrow{OC}|=|\vec{a}\times\vec{b}|$$

and p = area of quadrilateral $OABC$



$$=\frac{1}{2}|\overrightarrow{OA}\times\overrightarrow{OB}|+\frac{1}{2}|\overrightarrow{OB}\times\overrightarrow{OC}|$$

$$=\frac{1}{2}|\vec{a}\times(10\vec{a}+2\vec{b})|+\frac{1}{2}|(10\vec{a}+2\vec{b})\times\vec{b}|$$

$= |\vec{a} \times \vec{b}| + 5|\vec{a} \times \vec{b}| = 6|\vec{a} \times \vec{b}|$

$\therefore \quad q = 6p \Rightarrow k = 6$

13. $\vec{a} \times (\vec{a} \times \vec{c}) + \vec{b} = 0$

$\Rightarrow \quad (\vec{a} \cdot \vec{c})\vec{a} - (\vec{a} \cdot \vec{a})\vec{c} + \vec{b} = 0$

$\Rightarrow \quad (2\cos\theta)\vec{a} - \vec{c} + \vec{b} = 0$

$\Rightarrow \quad (2\cos\theta\, \vec{a} - \vec{c})^2 = (\vec{b})^2$

$\Rightarrow \quad 4\cos^2\theta \cdot |\vec{a}|^2 + |\vec{c}|^2 - 2 \cdot 2\cos\theta\, \vec{a} \cdot \vec{c} = |\vec{b}|^2$

$\Rightarrow \quad 4\cos^2\theta - 4 - 8\cos^2\theta = 1$

$\Rightarrow \quad 4\cos^2\theta = 3 \quad \Rightarrow \quad \cos\theta = \pm\sqrt{3}/2$

for θ to be acute, $\cos\theta = \sqrt{3}/2 \Rightarrow \theta = \pi/6$

B TRUE / FALSE

1. $\vec{A} \cdot \vec{B} = \vec{A} \cdot \vec{C} = 0$

$\Rightarrow \quad \vec{A} \perp$ to both $\vec{B}$ and $\vec{C}$

or $\quad \vec{A} = \lambda(\vec{B} \times \vec{C})$

$|\vec{A}| = |\lambda||\vec{B} \times \vec{C}|$, where $\vec{A}, \vec{B}, \vec{C}$ are unit vectors

$\Rightarrow \quad |\lambda| = \dfrac{1}{1 \cdot \sin 60^\circ}$

$\Rightarrow \quad |\lambda| = 2$ or $\lambda = \pm 2$.

$\therefore \quad \vec{A} = \pm 2(\vec{B} \times \vec{C})$

2. Since $\vec{X} \cdot \vec{A} = \vec{X} \cdot \vec{B} = \vec{X} \cdot \vec{C} = 0$

$\Rightarrow \vec{X}$ is perpendicular to $\vec{A}, \vec{B}, \vec{C} \neq [\vec{A}\, \vec{B}\, \vec{C}] = 0$

3. If $\vec{A} + \vec{B}, \vec{A} - \vec{B}, \vec{A} + k\vec{B}$ are collinear

$\Rightarrow \quad (a - b) - (a + b) = (a + kb) - (a - b)$

$\Rightarrow \quad 2b = (k + 1)b$

$\therefore \quad k + 1 = -2$

or $\quad k = -3.$

4. $(\vec{a} - \vec{b}) \cdot \{(\vec{b} - \vec{c}) \times (\vec{c} - \vec{a})\}$

$\Rightarrow \quad (\vec{a} - \vec{b}) \cdot \{\vec{b} \times \vec{c} - \vec{b} \times \vec{a} + \vec{c} \times \vec{a}\}$

$\Rightarrow \quad \vec{a} \cdot (\vec{b} \times \vec{c}) - \vec{b} \cdot (\vec{c} \times \vec{a})$

$\Rightarrow \quad [\vec{a}\; \vec{b}\; \vec{c}] + [\vec{a}\; \vec{b}\; \vec{c}]$

$\Rightarrow \quad 2[\vec{a}\; \vec{b}\; \vec{c}] = 2\vec{a} \cdot (\vec{b} \times \vec{c})$

Thus, (true)

C OBJECTIVE (ONLY ONE OPTION)

1. $\vec{A} \cdot \{(\vec{B} + \vec{C}) \times (\vec{A} + \vec{B} + \vec{C})\}$

It is scalar triple product of three vectors namely,

$[\vec{A}.\; \vec{B} + \vec{C}.\; \vec{A} + \vec{B} + \vec{C}]$

$= \vec{A} \cdot \{\vec{B} \times \vec{A} + \vec{B} \times \vec{B} + \vec{B} \times \vec{C} + \vec{C} \times \vec{A} + \vec{C} \times \vec{B} + \vec{C} \times \vec{C}\}$

$= \vec{A} \cdot (\vec{B} \times \vec{A}) + \vec{A} \cdot (\vec{B} \times \vec{C}) + \vec{A} \cdot (\vec{C} \times \vec{B})$

$= [\vec{A}\; \vec{B}\; \vec{C}] - [\vec{A}\; \vec{B}\; \vec{C}]$

$= 0$

2. $|(\vec{a} \times \vec{b}) \cdot \vec{c}| = |\vec{a}||\vec{b}||\vec{c}|$

$\Rightarrow \quad ||\vec{a}||\vec{b}|\sin\theta\, \hat{n} \cdot \vec{c}| = |\vec{a}||\vec{b}||\vec{c}|$

$\Rightarrow \quad |\vec{a}||\vec{b}||\vec{c}||\sin\theta \cdot \cos\alpha| = |\vec{a}||\vec{b}||\vec{c}|$

$\Rightarrow \quad |\sin\theta| \cdot |\cos\alpha| = 1$

$\Rightarrow \quad \theta = \dfrac{\pi}{2}$ and $\alpha = 0$

$\therefore \quad \vec{a} \perp \vec{b}$ and $\vec{c} \,||\, \hat{n}$

i.e., $\vec{a} \perp \vec{b}$ and $\vec{c}$ perpendicular to both $\vec{a}$ and $\vec{b}$.

3. The volume of parallelopiped

$= [\vec{a}\; \vec{b}\; \vec{c}]$

$= \begin{vmatrix} 2 & -3 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix} = 2(-1) + 3(-1 - 3)$

$= -2 + 6 = 4$

4. Three points A, B, C are collinear if $\vec{AB} = -20\hat{i} - 11\hat{j}$ and

$\vec{AC} = (a - 60)\hat{i} - 55\hat{j}$

Since, $\quad \vec{AB} \,||\, \vec{AC}$

$\Rightarrow \quad \dfrac{a - 60}{-20} = \dfrac{-55}{-11} \Rightarrow a = -40$

5. As $\quad (\vec{a} \times \vec{b}) = |\vec{a}||\vec{b}|\sin\dfrac{\pi}{6} \cdot \hat{n}$

$(\vec{a}\times\vec{b})\cdot\vec{c}=\frac{1}{2}|\vec{a}||\vec{b}|\hat{n}\cdot\vec{c}$

$[\vec{a}\ \vec{b}\ \vec{c}]=\frac{1}{2}|\vec{a}||\vec{b}|1.1\cos 0^\circ$

$\because$ $\hat{n}$ is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c}$ is also a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$

$$\therefore \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 = [\vec{a}\ \vec{b}\ \vec{c}]^2 = \frac{1}{4}|\vec{a}|^2|\vec{b}|^2$$

$$=\frac{1}{4}(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$$

Hence, (c) is correct answer.

6. Here, $\vec{a}=(2p)\hat{i}+\hat{j}$, when a system is rotated, the new component of $\vec{a}$ are $(p+1)$ and 1.

i.e., $\vec{b}=(p+1)\hat{i}+\hat{j}$

$\Rightarrow$ $|\vec{a}|^2=|\vec{b}|^2$

or $4p^2+1=(p+1)^2+1$

$\Rightarrow$ $4p^2=p^2+2p+1$

$\Rightarrow$ $3p^2-2p-1=0$

$\Rightarrow$ $3p^2-3p+p-1=0$

$\Rightarrow$ $(3p+1)(p-1)=0 \Rightarrow p=1,-1/3$

7. A vector perpendicular to $\vec{a}$ and $\vec{b}$ is,

$$\pm\frac{(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$$

Hence, (b) is correct answer.

8. Three vectors are coplanar if

$$\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} 0 & a & c \\ 1 & 0 & 1 \\ 0 & c & b \end{vmatrix}=0\ (C_1\to C_1-C_2)$$

$\Rightarrow$ $-1(ab-c^2)=0 \Rightarrow ab=c^2$

9. Let, $\vec{d}=x\hat{i}+y\hat{j}+z\hat{k}$

where $x^2+y^2+z^2=1$ ...(1)

($\vec{d}$ being unit vector)

$\therefore$ $\vec{a}\cdot\vec{d}=0$

$\Rightarrow$ $x-y=0 \Rightarrow x=y$ ...(2)

also, $[\vec{b}\ \vec{c}\ \vec{d}]=0$

$$\Rightarrow \begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ x & y & z \end{vmatrix}=0$$

$\Rightarrow$ $x+y+z=0.$ (using (2))

$\Rightarrow$ $2x+z=0$ ...(3)

from (1), (2) and (3)

$x^2+x^2+4x^2=1$

$\Rightarrow$ $x=\pm\frac{1}{\sqrt{6}}$

$\therefore$ $\vec{d}=\pm\frac{1}{\sqrt{6}}(\hat{i}+\hat{j}-2\hat{k})$

Hence, (a) is the correct answer.

10. Since, $\vec{a}\times(\vec{b}\times\vec{c})=\frac{\vec{b}+\vec{c}}{\sqrt{2}}$

$\Rightarrow$ $(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}=\frac{1}{\sqrt{2}}\vec{b}+\frac{1}{\sqrt{2}}\vec{c}$

$\Rightarrow$ $\vec{a}\cdot\vec{c}=-\frac{1}{\sqrt{2}} \Rightarrow |\vec{a}||\vec{b}|\cos\theta=-\frac{1}{\sqrt{2}}$

$\therefore$ $\cos\theta=-\frac{1}{\sqrt{2}}$ or $\theta=\frac{3\pi}{4}$

Hence (a) is the correct answer.

11. $\because$ $\vec{u}+\vec{v}+\vec{w}=0$

$\Rightarrow$ $|\vec{u}+\vec{v}+\vec{w}|^2=0$

or $(\vec{u}+\vec{v}+\vec{w})\cdot(\vec{u}+\vec{v}+\vec{w})=0$

$\Rightarrow$ $|\vec{u}|^2+|\vec{v}|^2+|\vec{w}|^2+2(\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{w}+\vec{w}\cdot\vec{u})=0$

$\Rightarrow$ $9+16+25+2(\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{w}+\vec{w}\cdot\vec{u})=0$

$\Rightarrow$ $\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{w}+\vec{w}\cdot\vec{u}=-25$

Hence (b) is correct answer

12. $(\vec{a}+\vec{b}+\vec{c})\cdot[(\vec{a}+\vec{b})\times(\vec{a}+\vec{c})]$

$=(\vec{a}+\vec{b}+\vec{c})\cdot[\vec{a}\times\vec{a}+\vec{a}\times\vec{c}+\vec{b}\times\vec{a}+\vec{b}\times\vec{c}]$

$=\{\vec{a}\cdot(\vec{a}\times\vec{c})-\vec{a}\cdot(\vec{b}\times\vec{a})+\vec{a}\cdot(\vec{b}\times\vec{c})\}$

$+\{\vec{b}\cdot(\vec{a}\times\vec{c})+\vec{b}\cdot(\vec{b}\times\vec{a})+\vec{b}\cdot(\vec{b}\times\vec{c})\}$

$+\{\vec{c}\cdot(\vec{a}\times\vec{c})+\vec{c}\cdot(\vec{b}\times\vec{a})+\vec{c}\cdot(\vec{b}\times\vec{c})\}$

$-[\vec{a}\ \vec{b}\ \vec{c}]-[\vec{a}\ \vec{b}\ \vec{c}]-[\vec{a}\ \vec{b}\ \vec{c}]$

$=-[\vec{a}\ \vec{b}\ \vec{c}]$

Hence, (d) is correct answer.

13. As $\vec{p}, \vec{q}, \vec{r}$ are mutually perpendicular vectors of same magnitude, so let us consider

$$\left.\begin{array}{l}|\vec{p}|=|\vec{q}|=|\vec{r}|=\lambda \text{ and} \\ \vec{p}\cdot\vec{q}=\vec{q}\cdot\vec{r}=\vec{r}\cdot\vec{p}=0\end{array}\right\} \quad \ldots(1)$$

where, $\vec{p}\times\{(\vec{x}-\vec{q})\times\vec{p}\}+\vec{q}\times\{(\vec{x}-\vec{r})\times\vec{q}\}$

$+\vec{r}\times\{(\vec{x}-\vec{p})\times\vec{r}\}=\vec{0}$

$\Rightarrow (\vec{p}\cdot\vec{p})(\vec{x}-\vec{q})-(\vec{p}\cdot(\vec{x}-\vec{q}))\vec{p}$

$+(\vec{q}\cdot\vec{q})(\vec{x}-\vec{r})-(\vec{q}\cdot(\vec{x}-\vec{r}))\vec{q}$

$+(\vec{r}\cdot\vec{r})(\vec{x}-\vec{p})-(\vec{r}\cdot(\vec{x}-\vec{p}))\vec{r}$

$=0$

$\Rightarrow \vec{x}\{(\vec{p}\cdot\vec{p})+(\vec{q}\cdot\vec{q})+(\vec{r}\cdot\vec{r})\}-(\vec{p}\cdot\vec{p})\vec{q}$

$-(\vec{q}\cdot\vec{q})\vec{r}+(\vec{r}\cdot\vec{r})\vec{p}=(\vec{x}\cdot\vec{p})\vec{p}+(\vec{x}\cdot\vec{q})\vec{q}+(\vec{x}\cdot\vec{r})\vec{r}$

$\Rightarrow 3\vec{x}|\lambda|^2-(\vec{p}+\vec{q}-\vec{r})|\lambda|^2=(\vec{x}\cdot\vec{p})\vec{p}$

$+(\vec{x}\cdot\vec{q})\vec{q}+(\vec{x}\cdot\vec{r})\vec{r}$

Taking dot of (1) with $\vec{p}$, we get

$$3(\vec{x}\cdot\vec{p})|\lambda|^2-|\lambda|^4=(\vec{x}\cdot\vec{p})\lambda|^2$$

$$\Rightarrow \quad \vec{x}\cdot\vec{p}=\frac{1}{2}|\lambda|^2$$

Similarly, taking dot of (1) with $\vec{q}$ and $\vec{r}$ we get

$$\vec{x}\cdot\vec{q}=\frac{H^2}{2}=\vec{x}\cdot\vec{r}$$

∴ Equation (1) becomes

$$3\vec{x}|\lambda|^2-(\vec{p}+\vec{q}+\vec{r})|\lambda|^2=\frac{|\lambda|^2}{2}(\vec{p}+\vec{q}+\vec{r})$$

$$\Rightarrow \quad 3\vec{x}=\frac{1}{2}(\vec{p}+\vec{q}+\vec{r})+(\vec{p}+\vec{q}+\vec{r})$$

$$\Rightarrow \quad \vec{x}=\frac{1}{2}(\vec{p}+\vec{q}+\vec{r})$$

Hence, (b) is the correct answer.

14. It is given that $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent

$$\Rightarrow \quad [\vec{a}\ .\ \vec{b}\ .\ \vec{c}]=0$$

$$\Rightarrow \quad \begin{vmatrix}1 & 1 & 1\\ 4 & 3 & 4\\ 1 & \alpha & \beta\end{vmatrix}=0$$

$$\Rightarrow \quad \begin{vmatrix}1 & 0 & 0\\ 4 & -1 & 0\\ 1 & \alpha-1 & \beta-1\end{vmatrix}=0$$

Apply $C_2 \to C_2-C_1, C_3 \to C_3-C_1$

Now, expanding along R_1,

$$\Rightarrow \quad -(\beta-1)=0 \Rightarrow \beta=1$$

Also $\quad |\vec{c}|=\sqrt{3}$ (given)

where $\quad c=\hat{i}+\alpha\hat{j}+\beta\hat{k}$ (given)

$\Rightarrow \quad 1+\alpha^2+\beta^2=3$

$\Rightarrow \quad 1+\alpha^2+1=3 \quad \Rightarrow \quad \alpha^2=1 \quad \Rightarrow \alpha=\pm 1$

15. $[\vec{u}\ \vec{v}\ \vec{w}]=[\vec{v}\ \vec{w}\ \vec{u}]=[\vec{w}\ \vec{u}\ \vec{v}]=-[\vec{v}\ \vec{u}\ \vec{w}]$

Therefore (c) is the answer.

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16. **Imp. Note :** In this Question vector $\vec{c}$ is not given, therefore, we cannot apply the formulae of $\vec{a}\times\vec{b}\times\vec{c}$ (vector triple product).

Now, $|(\vec{a}\times\vec{b})\times\vec{c}|=|\vec{a}\times\vec{b}||\vec{c}|\sin 30°$

So, we need now $|\vec{a}\times\vec{b}|$ and $|\vec{c}|$

again $\quad |\vec{a}\times\vec{b}|=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & -2\\ 1 & 1 & 0\end{vmatrix}=2\hat{i}-2\hat{j}-\hat{k}$

$$\Rightarrow \quad |\vec{a}\times\vec{b}|=\sqrt{2^2+(-2)^2+1}$$

$$=\sqrt{4+4+1}=\sqrt{9}=3$$

Next, $\quad |\vec{c}-\vec{a}|=2\sqrt{2}$

$\Rightarrow \quad |\vec{c}-\vec{a}|^2=8$

$\Rightarrow \quad (\vec{c}-\vec{a}).(\vec{c}-\vec{a})=8$

$\Rightarrow \quad \vec{c}.\vec{c}-\vec{c}.\vec{a}-\vec{a}.\vec{c}+\vec{a}.\vec{a}=8$

$\Rightarrow \quad |\vec{c}|^2+|\vec{a}|^2-2\vec{a}.\vec{c}=8$

$\Rightarrow \quad |\vec{c}|^2+9-2|\vec{c}|=8$

$(\because \vec{a}=2\hat{i}-\hat{j}+2\hat{k}$ (given), $\vec{a}.\vec{c}=|\vec{c}|$ (given))

$\Rightarrow \quad |\vec{c}|^2-2|\vec{c}|+1=0$

$\Rightarrow \quad (|\vec{c}|-1)^2=0$

$\Rightarrow \qquad |\vec{c}| = 1$

Now, putting in

$$|(\vec{a} \times \vec{b}) \times \vec{c}| = |\vec{a} \times \vec{b}||\vec{c}| \sin 30^\circ$$

$$= (3)(1) \cdot \left(\frac{1}{2}\right) = \frac{3}{2}.$$

17. It is given that $\vec{c}$ is coplanar with $\vec{a}$ and $\vec{b}$, we take

$$\vec{c} = p\vec{a} + q\vec{b} \qquad \ldots(1)$$

where p, q are scalars.

again $\vec{c} \perp \vec{a}$, (given)

$$\Rightarrow \qquad \vec{c} \cdot \vec{a} = 0$$

taking dot product of $\vec{a}$ in (1)

$$\Rightarrow \qquad \vec{c} \cdot \vec{a} = p\,\vec{a} \cdot \vec{a} + q\,\vec{b} \cdot \vec{a}$$

$$\left[\begin{array}{l} \because \vec{a} = 2\hat{i} + \hat{j} + \hat{k} \\ \Rightarrow |\vec{a}| = \sqrt{2^2 + 1 + 1} = \sqrt{6}. \\ \vec{a} \cdot \vec{b} = (2\hat{i} + \hat{j} + \hat{k}) \cdot (\hat{i} + 2\hat{j} - \hat{k}) \\ = 2 + 2 - 1 = 3 \end{array}\right]$$

$$\Rightarrow \qquad 0 = p|\vec{a}|^2 + q|\vec{b} \cdot \vec{a}|$$

$$\Rightarrow \qquad 0 = p \cdot 6 + q \cdot 3$$

$$\Rightarrow \qquad q = -2p$$

putting in (1)

$$\Rightarrow \qquad \vec{c} = p\vec{a} + \vec{b}(-2p)$$

$$\Rightarrow \qquad \vec{c} = p\vec{a} - 2p\vec{b}$$

$$\Rightarrow \qquad \vec{c} = p(\vec{a} - 2\vec{b})$$

$$\Rightarrow \qquad \vec{c} = p\{(2\hat{i} + \hat{j} + \hat{k}) - 2(\hat{i} + 2\hat{j} - \hat{k})\}$$

$$\Rightarrow \qquad \vec{c} = p\{-3\hat{j} + 3\hat{k}\}$$

again $|\vec{c}| = 1$ (given) $\Rightarrow |\vec{c}| = p\sqrt{(-3)^2 + 3^2}$

$$\Rightarrow \qquad |\vec{c}|^2 = p^2(\sqrt{18})^2$$

$$\Rightarrow \qquad |\vec{c}|^2 = p^2 \cdot 18$$

$$\Rightarrow \quad 1 = p^2 \cdot 18 \Rightarrow p^2 = \frac{1}{18} \quad \Rightarrow \quad p = \pm\frac{1}{3\sqrt{2}}$$

$$\therefore \qquad \vec{c} = \pm\frac{1(-\hat{j} + \hat{k})}{\sqrt{2}}$$

Therefore (a) is the answer.

18. Given $\vec{a} + \vec{b} + \vec{c} = 0$ (by triangle law)

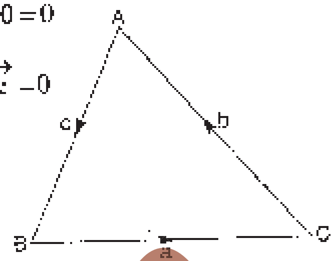
Taking cross product by $\vec{a}, \vec{b}, \vec{c}$

$$\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times 0 = 0$$

$$\Rightarrow \vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = 0$$

$$\Rightarrow \qquad \vec{a} \times \vec{b} = \vec{c} \times \vec{a}$$

$$[\because \vec{a} \times \vec{a} = 0]$$



Similarly, $\qquad \vec{a} \times \vec{b} = \vec{b} \times \vec{c}.$

Therefore, $\quad \vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}.$

19. If θ is the angle between P_1 and P_2.

$$\left.\begin{array}{l} N_1 = \vec{a} \times \vec{b}, \\ N_2 = \vec{c} \times \vec{d}, \end{array}\right\} N_1 \times N_2 = 0.$$

then $\qquad |N_1| \times |N_2| \sin\theta = 0$

or $\sin\theta = 0 \Rightarrow \theta = 0.$

20. $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors, $2\vec{a} - \vec{b}, 2\vec{b} - \vec{c}$ and $2\vec{c} - \vec{a}$ are also coplanar vectors. Thus

$$[2\vec{a} - \vec{b} \;\; 2\vec{b} - \vec{c} \;\; 2\vec{c} - \vec{a}] = 0$$

21. $(\vec{a} + \vec{b} + \vec{c})^2 = \Sigma\vec{a}^2 + 2\Sigma\,\vec{a} \cdot \vec{b} \geq 0$

$$\Rightarrow \qquad 2\Sigma\,\vec{a} \cdot \vec{b} \geq -3$$

Now, $\quad \Sigma|\vec{a} - \vec{b}|^2 = 2\Sigma\vec{a}^2 - 2\Sigma\,\vec{a} \cdot \vec{b}$

$$\leq 2(3) + 3 = 9$$

22. $[\vec{a} \; \vec{b} \; \vec{c}] = \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1-x \\ y & x & 1+x-y \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ y & x & 1+x \end{vmatrix}$

Apply $\qquad C_3 \to C_1 + C_3$

$$= \begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & x & 1 \end{vmatrix} = 1$$

Therefore, it depends on neither x nor y.

23. $5|\vec{a}|^2 + 6\,\vec{a} \cdot \vec{b} - 8|\vec{b}|^2 = 0$

$$6\,\vec{a} \cdot \vec{b} = 3$$

$$\cos\theta = 1/2 \quad \Rightarrow \quad \theta = 60^\circ$$

24. $\vec{V} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{W} = \hat{i} + 3\hat{k}$

$$[\vec{U}\ \vec{V}\ \vec{W}] = \vec{U}\cdot[(2\hat{i}+\hat{j}-\hat{k})\times(\hat{i}+3\hat{k})]$$

$$= \vec{U}\cdot(3\hat{i}-7\hat{j}-\hat{k})$$

which is maximum if angle between $\vec{U}$ and $3\hat{i}-7\hat{j}-\hat{k}$ is 0 and maximum value

$$= |3\hat{i}-7\hat{j}-\hat{k}| = \sqrt{59}.$$

25. We know volume of parallelopiped whose edges are $\vec{a}, \vec{b}, \vec{c}$ $= [\vec{a}\ \vec{b}\ \vec{c}]$.

$$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & a & 1 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = 1 + a^3 - a,$$

to minimise volume.

Let $f(a) = a^3 - a + 1 \Rightarrow f'(a) = 3a^2 - 1$

$\Rightarrow \quad f''(a) = 6a$

Put $f'(a) = 0 \Rightarrow a = \pm\frac{1}{\sqrt{3}}$ which shows $f(a)$ is minimum at $a = \frac{1}{\sqrt{3}}$ and maximum at $a = -\frac{1}{\sqrt{3}}$.

26. As we know

$$\vec{a}\times(\vec{a}\times\vec{b}) = (\vec{a}\cdot\vec{b})\vec{a} - (\vec{a}\cdot\vec{a})\vec{b}$$

$$(\hat{i}+\hat{j}+\hat{k})\times(\hat{j}-\hat{k}) = (\hat{i}+\hat{j}+\hat{k}) - (\sqrt{3})^2\vec{b}$$

$$\Rightarrow \quad -2\hat{i}+\hat{j}+\hat{k} = \hat{i}+\hat{j}+\hat{k} - 3\vec{b}$$

$$\Rightarrow \quad 3\vec{b} = +3\hat{i}$$

$$\Rightarrow \quad \vec{b} = \hat{i}$$

27. As we know, a vector coplanar to $\vec{a}$, $\vec{b}$ and orthogonal to $\vec{c}$ is $\lambda\{(\vec{a}\times\vec{b})\times\vec{c}\}$

$\therefore$ A vector coplanar to $(2\hat{i}+\hat{j}+\hat{k})$, $(\hat{i}-\hat{j}+\hat{k})$ and orthogonal to $3\hat{i}+2\hat{j}+6\hat{k}$

$$= \lambda[\{(2\hat{i}+\hat{j}+\hat{k})\times(\hat{i}-\hat{j}+\hat{k})\}\times(3\hat{i}+2\hat{j}+6\hat{k})]$$

$$= \lambda(21\hat{j}-7\hat{k})$$

$\therefore$ a unit vector is $\pm\dfrac{(\vec{a}\times\vec{b})\times\vec{c}}{|(\vec{a}\times\vec{b})\times\vec{c}|}$

$$\Rightarrow \quad \pm\frac{(21\hat{j}-7\hat{k})}{\sqrt{(21)^2+(7)^2}} = \pm\frac{(3\hat{j}-\hat{k})}{\sqrt{10}}$$

28. As $\vec{b_1} = \vec{b} - \dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}$, $\vec{b_2} = \vec{b} + \dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}$

and $\vec{c_1} = \vec{c} - \dfrac{\vec{c}\cdot\vec{a}}{|\vec{a}|^2}\vec{a} - \dfrac{\vec{c}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}$,

$$\vec{c_2} = \vec{c} - \frac{\vec{c}\cdot\vec{a}}{|\vec{a}|^2}\vec{a} - \frac{\vec{c}\cdot\vec{b_1}}{|\vec{b_1}|^2}\vec{b_1},$$

$$\vec{c_3} = \vec{c} - \frac{\vec{c}\cdot\vec{a}}{|\vec{a}|^2}\vec{a} - \frac{\vec{c}\cdot\vec{b_2}}{|\vec{b_2}|^2}\vec{b_2},\ \vec{c_4} = \vec{a} - \frac{\vec{c}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}.$$

Which shows

$$\vec{a}\cdot\vec{b_1} = 0 = \vec{a}\cdot\vec{c_2} = \vec{b_1}\cdot\vec{c_2}$$

$\therefore \{\vec{a}, \vec{b_1}, \vec{c_2}\}$ are mutually orthogonal vectors.

29. Let vector $\vec{r}$ be coplanar to $\vec{a}$ and $\vec{b}$.

$$\therefore \quad \vec{r} = \vec{a} + t\vec{b}$$

$$\Rightarrow \quad \vec{r} = (\hat{i}+2\hat{j}+\hat{k}) + t(\hat{i}-\hat{j}+\hat{k})$$

$$= \hat{i}(1+t) + \hat{j}(2-t) + \hat{k}(1+t)$$

The projection of $\vec{r}$ on $\vec{c} = \frac{1}{\sqrt{3}}$

$$\Rightarrow \quad \frac{\vec{r}\cdot\vec{c}}{|\vec{c}|} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \frac{|1.(1+t)+1.(2-t)-1.(1+t)|}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad |2-t| = \pm 1$$

$\Rightarrow t = 1$ or 3

when, $t = 1$ we have $\vec{r} = 2\hat{i}+\hat{j}+2\hat{k}$

when, $t = 3$ we have $\vec{r} = 4\hat{i}-\hat{j}+4\hat{k}$

Hence, (a) is the correct answer.

30.
$$\begin{vmatrix} -\lambda^2 & 1 & 1 \\ 1 & -\lambda^2 & 1 \\ 1 & 1 & -\lambda^2 \end{vmatrix} = 0 \Rightarrow \lambda^6 - 3\lambda^2 - 2 = 0$$

$$\Rightarrow \quad (1+\lambda^2)^2(\lambda^2-2) = 0 \Rightarrow \lambda = \pm\sqrt{2}.$$

31. Since $\vec{a}$, $\vec{b}$, $\vec{c}$ are unit vectors and $\vec{a}+\vec{b}+\vec{c} = \vec{0}$, $\vec{a}$, $\vec{b}$, $\vec{c}$ represent an equilateral triangle.

$$\therefore \quad \vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a} \neq \vec{0}.$$

D OBJECTIVE (MORE THAN ONE OPTION)

1. Let the required vector is $\vec{p} = x\hat{i} + y\hat{j} + z\hat{k}$.

This vector is coplanar with $\vec{b}$ and $\vec{c}$, therefore, we must have

$$\begin{vmatrix} x & y & z \\ 1 & 2 & -1 \\ 1 & 1 & -2 \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_3$

$$\Rightarrow \begin{vmatrix} x & y & z \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{vmatrix} = 0$$

$$\Rightarrow x(-2-1) - y(0-1) + z(0-1) = 0$$

$$\Rightarrow -2x + y - z = 0 \quad \ldots (1)$$

Again projection of vector $\vec{p}$ on $\vec{a}$ is given by

$$\cos\theta = \frac{|\vec{a} \cdot \vec{p}|}{|\vec{a}| \cdot |\vec{p}|} = \frac{|(2\hat{i} - \hat{j} + \hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k})|}{\sqrt{4+1+1} \cdot \sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{|2x - y + z|}{\sqrt{6}\sqrt{x^2 + y^2 + z^2}}$$

Therefore, $\sqrt{x^2 + y^2 + z^2} \cdot \cos\theta = \frac{|2x - y + z|}{\sqrt{6}} \quad \ldots (2)$

$= \sqrt{2/3}$ (given)

Hence, $2x - y + z = \pm 2$

Therefore, (a) and (c) satisfy this equation.

2. (a) $\vec{u} \cdot (\vec{v} \times \vec{w})$ is a meaningful operation, therefore, (a) is the answer.

(b) $\vec{u} \cdot (\vec{v} \cdot \vec{w})$ is not meaningful since $\vec{v} \cdot \vec{w}$ is a scalar quantity and for dot product both quantities should be vector. Therefore, (b) is not the answer.

(c) $(\vec{u} \cdot \vec{v})\vec{w}$ is meaningful since it is a simple multiplication of vector and scalar quantity. Therefore, (c) is the answer.

(d) $\vec{u} \times (\vec{v} \cdot \vec{w})$ is not meaningful since $\vec{v} \cdot \vec{w}$ is a scalar quantity and for cross product, both quantity should be vector. Therefore, (d) is not the answer.

Hence (a) and (c) are the answer.

3. Let θ be the angle between $\vec{a}$ and $\vec{b}$. As $\vec{a}$ and $\vec{b}$ are non-collinear, $\theta \neq 0$ and $\theta \neq \pi$.

We have $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$= \cos\theta \quad |\vec{a}| = 1, |\vec{b}| = 1$ given

Now, $\vec{u} = \vec{a} - (\vec{a} \cdot \vec{b})\vec{b}$

Taking modulus

$$|\vec{u}| = |\vec{a} - (\vec{a} \cdot \vec{b})\vec{b}|$$

$$\Rightarrow |\vec{u}|^2 = |\vec{a} - (\vec{a} \cdot \vec{b})\vec{b}|^2$$

$$\Rightarrow |\vec{u}|^2 = |\vec{a} - \cos\theta\, \vec{b}|^2$$

$$\Rightarrow |\vec{u}|^2 = |\vec{a}|^2 + \cos^2\theta|\vec{b}|^2 - 2\cos\theta(\vec{a} \cdot \vec{b})$$

$$\Rightarrow |\vec{a}|^2 = 1 + \cos^2\theta - 2\cos^2\theta$$

$$\Rightarrow |\vec{u}|^2 = 1 - \cos^2\theta$$

$$\Rightarrow |\vec{u}|^2 = \sin^2\theta$$

Also $\vec{v} = \vec{a} \times \vec{b}$ (given)

$$\Rightarrow |\vec{v}| = |\vec{a} \times \vec{b}|$$

$$\Rightarrow |\vec{v}|^2 = |\vec{a} \times \vec{b}|^2$$

$$\Rightarrow |\vec{v}|^2 = |\vec{a}|^2 |\vec{b}|^2 \cdot \sin^2\theta$$

$$\Rightarrow |\vec{v}|^2 = \sin^2\theta$$

$$\therefore |\vec{u}|^2 = |\vec{v}|^2$$

Also, $\vec{u} \cdot \vec{a} = [\vec{a} - (\vec{a} \cdot \vec{b})\vec{b}] \cdot \vec{a}$

$= \vec{a} \cdot \vec{a} - (\vec{a} \cdot \vec{b})(\vec{b} \cdot \vec{a})$

$= (\vec{a})^2 - \cos^2\theta$

$= 1 - \cos^2\theta = \sin^2\theta$

$$\therefore |\vec{u}| + |\vec{u} \cdot \vec{a}| = \sin\theta + \sin^2\theta \neq |\vec{u}|$$

Next $\vec{u} \cdot \vec{b} = (\vec{a} - (\vec{a} \cdot \vec{b}) \cdot \vec{b}) \cdot \vec{b}$

$= \vec{a} \cdot \vec{b} - (\vec{a} \cdot \vec{b})(\vec{b} \cdot \vec{b})$

$= \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{b}|\vec{b}|^2$

$= \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{b} = 0 \quad \ldots (1)$

$$\therefore |\vec{u}| + |\vec{u} \cdot \vec{b}| = |\vec{u}| + 0 = |\vec{u}| = |\vec{v}|$$

Also, $\vec{u}.(\vec{a}+\vec{b}) = \vec{u}.\vec{a}+\vec{u}.\vec{b} = \vec{u}.\vec{a}$

$\Rightarrow \quad |\vec{u}| + \vec{u}.(\vec{a}-\vec{b}) = |\vec{u}| + \vec{u}.\vec{a} \ne |\vec{v}|$

4. Let vector $\vec{AO}$ be parallel to line of intersection of planes P_1 and P_2 through

i.e., $[(2\hat{j}-3\hat{k})\times(4\hat{j}-3\hat{k})]\times[(\hat{j}-\hat{k})\times(3\hat{i}+3\hat{j})]$

$= 54(\hat{j}-\hat{k})$.

$\therefore$ Angle between $54(\hat{j}-\hat{k})$ and $(2\hat{i}+\hat{j}-2\hat{k})$

$$\Rightarrow \quad \cos\theta = \pm\left(\frac{54+108}{3\cdot 54\cdot\sqrt{2}}\right) = \pm\frac{1}{\sqrt{2}}$$

$$\therefore \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

Hence (b) and (d) are correct answer.

E SUBJECTIVE QUESTIONS

1. As, $\vec{OA}_1, \vec{OA}_2, \ldots, \vec{OA}_n$ are all vectors of same magnitude and angle between any two consecutive vectors is same that is $(2\pi/n)$.

$$\therefore \quad \vec{OA}_1\times\vec{OA}_2 = a^2\cdot\sin\frac{2\pi}{n}\cdot\hat{p} \quad \ldots(1)$$

where $\hat{p}$ is perpendicular to plane of polygon.

Now,

$$\sum_{i=1}^{n-1}\vec{OA}_i\times\vec{OA}_{i+1} = \sum_{i=1}^{n-1} a^2\cdot\sin\frac{2\pi}{n}\cdot\hat{p}$$

$$= (n-1)\cdot a^2\cdot\sin\frac{2\pi}{n}\cdot\hat{p}$$

$$= (n-1)[\vec{OA}_1\times\vec{OA}_2]$$

$$= (1-n)[\vec{OA}_2\times\vec{OA}_1] = \text{R.H.S.}$$

2. As, $(\hat{i}+\hat{j}+3\hat{k})x+(3\hat{i}-3\hat{j}+\hat{k})y+(-4\hat{i}+5\hat{j})z$

$$= \lambda(\hat{i}x+\hat{j}y+\hat{k}z)$$

$$\Rightarrow \quad x+3y-4z = \lambda x$$
$$x-3y+5z = \lambda y$$
$$3x+y+0z = \lambda z$$

or
$$(1-\lambda)x+3y-4z = 0$$
$$x-(3+\lambda)y+5z = 0$$
$$3x+y-\lambda z = 0$$

Since, $(x, y, z) \ne (0, 0, 0)$

$\therefore$ non-trivial solution

$\Rightarrow \Delta = 0$.

or
$$\begin{vmatrix} 1-\lambda & 3 & -4 \\ 1 & -(3+\lambda) & 5 \\ 3 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(3\lambda+\lambda^2-5) - 3(-\lambda-15) - 4(1+9+3\lambda) = 0$$

$$\Rightarrow \quad \lambda(\lambda+1)^2 = 0$$

$$\Rightarrow \quad \lambda = 0, -1$$

3. (a) Given: $\vec{A}\perp\vec{B} \Rightarrow \vec{A}\cdot\vec{B} = 0 \quad \ldots(1)$

and $\vec{A}\times\vec{X} = \vec{B} \Rightarrow \vec{A}\cdot\vec{B} = 0$ and $\vec{X}\cdot\vec{B} = 0 \quad \ldots(2)$

Now, consider,

$$[\vec{X}\ \vec{A}\ \vec{A}\times\vec{B}] = \vec{X}\cdot\{\vec{A}\times(\vec{A}\times\vec{B})\}$$

$$= \vec{X}\cdot\{(\vec{A}\cdot\vec{B})\vec{A}-(\vec{A}\cdot\vec{A})\vec{B}\}$$

$$= (\vec{A}\cdot\vec{B})(\vec{X}\cdot\vec{A})-(\vec{A}\cdot\vec{A})(\vec{X}\cdot\vec{B})$$

$$= 0$$

$\Rightarrow \quad \vec{X}, \vec{A}, \vec{A}\times\vec{B}$ are coplanar

So, $\vec{X}$ can be represented as a linear combination of $\vec{A}$ and $\vec{A}\times\vec{B}$.

Let us consider, $\vec{X} = l\vec{A}+m(\vec{A}\times\vec{B})$

As $\quad \vec{A}\times\vec{X} = c$

$\therefore \quad \vec{A}\cdot(l\vec{A}+m(\vec{A}\times\vec{B})) = c$

$\Rightarrow \quad l|\vec{A}|^2+0 = c$

$\Rightarrow \quad l = \dfrac{c}{|\vec{A}|^2}$

also, $\quad \vec{A}\times\vec{X} = \vec{B}$

$\Rightarrow \quad \vec{A}\times(l\vec{A}+m(\vec{A}\times\vec{B})) = \vec{B}$

$\Rightarrow \quad l(\vec{A}\times\vec{A})+m(\vec{A}\times(\vec{A}\times\vec{B})) = \vec{B}$

$\Rightarrow \quad 0 - m|\vec{A}|^2\,\vec{B} = \vec{B}$

$\Rightarrow \quad m = -\dfrac{1}{|\vec{A}|^2}$

$$\therefore \quad \vec{X} = \left(\frac{c}{|\vec{A}|^2}\right)\vec{A} - \left(\frac{1}{|\vec{A}|^2}\right)(\vec{A}\times\vec{B})$$

3. (b) Since vector $\vec{A}$ has component A_1, A_2, A_3 in the coordinate system oxyz.

$$\therefore \quad \vec{A} = A_1\hat{i}+A_2\hat{j}+A_3\hat{k}$$

when the given system is rotated about an angle of $\pi/2$. The new x-axis is along old y-axis and new y-axis is along the old negative x-axis, where as z remains same

Hence, the components of A in the new system are $(A_2, -A_1, A_3)$.

$\therefore \quad \vec{A}$ becomes $(A_2\hat{i} - A_1\hat{j} - A_3\hat{k})$

4. Here, $\overrightarrow{AB} = -\hat{i} - 5\hat{j} - 3\hat{k}$

$\overrightarrow{AC} = -4\hat{i} + 3\hat{j} + 3\hat{k}$

$\overrightarrow{AD} = \hat{i} + 7\hat{j} + (1-\lambda)\hat{k}$

We know that, A, B, C, D lie in a plane if $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar *i.e.*,

$$[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = 0$$

$$\Rightarrow \begin{vmatrix} -1 & -5 & -3 \\ -4 & 3 & 3 \\ 1 & 7 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = \frac{146}{17}$$

5. Let the position vectors of points A, B, C, D be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ respectively.

Then $\overrightarrow{AB} = \vec{b} - \vec{a}, \overrightarrow{BC} = \vec{c} - \vec{b}, \overrightarrow{AD} = \vec{d} - \vec{a}$

$\overrightarrow{BD} = \vec{d} - \vec{b}, \overrightarrow{CA} = \vec{a} - \vec{c}, \overrightarrow{CD} = \vec{d} - \vec{c}$

Now, $|\overrightarrow{AB} \times \overrightarrow{CD} + \overrightarrow{BC} \times \overrightarrow{AD} + \overrightarrow{CA} \times \overrightarrow{BD}|$

$$= |(\vec{b} - \vec{a}) \times (\vec{d} - \vec{c}) + (\vec{c} - \vec{b}) \times (\vec{d} - \vec{a}) + (\vec{a} - \vec{c}) \times (\vec{d} - \vec{b})|$$

$$= |\vec{b} \times \vec{d} - \vec{a} \times \vec{d} - \vec{b} \times \vec{c} + \vec{a} \times \vec{c} + \vec{c} \times \vec{d} - \vec{c} \times \vec{a} - \vec{b} \times \vec{d} + \vec{b} \times \vec{a} + \vec{a} \times \vec{d} - \vec{a} \times \vec{b} - \vec{c} \times \vec{d} + \vec{c} \times \vec{b}|$$

$$= 2|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}| \quad \ldots(1)$$

Also, Area of $\Delta ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$$= \frac{1}{2}|(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})|$$

$$= \frac{1}{2}|\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{a} \times \vec{a}|$$

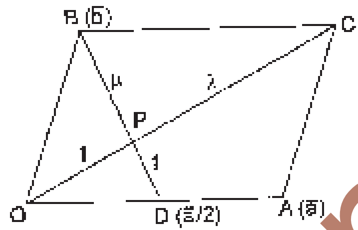
$$= \frac{1}{2}|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}| \quad \ldots(2)$$

$\therefore$ from (1) and (2), we get

$$|\overrightarrow{AB} \times \overrightarrow{CD} + \overrightarrow{BC} \times \overrightarrow{AD} + \overrightarrow{CA} \times \overrightarrow{BD}|$$

$= 2(2 \text{ Area of } \Delta ABC) = 4(\text{area of } \Delta ABC)$

6. $OACB$ is a parallelogram with O as origin let $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OC} = \vec{a} + \vec{b}$



$\overrightarrow{OD} = \frac{\vec{a}}{2}$, as mid-point of $\overrightarrow{OA}$

$\overrightarrow{CO}$ and $\overrightarrow{BD}$ meets at P.

$$\therefore \quad \overrightarrow{OP} = \frac{\lambda \cdot 0 + 1(\vec{a} + \vec{b})}{\lambda + 1} \quad (\text{along } \overrightarrow{OC})$$

$$\Rightarrow \quad \overrightarrow{OP} = \frac{\vec{a} + \vec{b}}{\lambda + 1} \quad \ldots(1)$$

again, $$\overrightarrow{OP} = \frac{\mu\left(\frac{\vec{a}}{2}\right) + 1(\vec{b})}{\mu + 1}, \quad (\text{along } \overrightarrow{BD})$$

$$\Rightarrow \quad \overrightarrow{OP} = \frac{\mu\vec{a} + 2\vec{b}}{2(\mu + 1)} \quad \ldots(2)$$

from (1) and (2), we get

$$\frac{\vec{a} + \vec{b}}{\lambda + 1} = \frac{\mu\vec{a} + 2\vec{b}}{2(\mu + 1)}$$

$$\Rightarrow \quad \frac{1}{\lambda + 1} = \frac{\mu}{2(\mu + 1)} \quad \text{and} \quad \frac{1}{\lambda + 1} = \frac{1}{\mu + 1}$$

on solving, we get

$$\mu = \lambda = 2$$

Thus, required ratio is $2 : 1$.

7. Given that $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors.

$\therefore$ there exists scalars x, y, z not all zero, such that

$$x\vec{a} + y\vec{b} + z\vec{c} = 0 \quad \ldots(1)$$

Taking dot with $\vec{a}$ and $\vec{b}$, we get

$$x(\vec{a} \cdot \vec{a}) + y(\vec{a} \cdot \vec{b}) + z(\vec{a} \cdot \vec{c}) = 0 \quad \ldots(2)$$

$$x(\vec{a} \cdot \vec{b}) + y(\vec{b} \cdot \vec{b}) + z(\vec{c} \cdot \vec{b}) = 0 \quad \ldots(3)$$

Since (1), (2) and (3) represents homogeneous equations with $(x, y, z) \neq (0, 0, 0)$

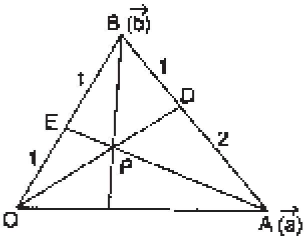
$\Rightarrow$ non-trivial solutions

or $\Delta = 0$

$$\Rightarrow \begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \end{vmatrix} = \vec{0}$$

8. Let O be origin and $\vec{OA} = \vec{a}, \vec{OB} = \vec{b}$

$$\vec{OE} = \frac{\vec{b}}{2} \quad \text{(as } E \text{ being mid-point of } \vec{OB}\text{)}$$



$$\vec{OD} = \frac{\vec{a}\cdot 1 + \vec{b}\cdot 2}{1+2} \quad \text{(as } D \text{ divides } \vec{AB} \text{ in the ratio of } 2:1\text{)}$$

$\Rightarrow$ Equation of $\vec{OD}$ is,

$$\vec{r} = t\left(\frac{\vec{a} + 2\vec{b}}{3}\right)$$

and equation of $\vec{AE}$ is,

$$\vec{r} = \vec{a} + s\left(\frac{\vec{b}}{2} - \vec{a}\right)$$

If $\vec{OD}$ and $\vec{AE}$ intersect at P, then there must be some $\vec{r}$ for which they are equal.

$$\Rightarrow \quad t\left(\frac{\vec{a} + 2\vec{b}}{3}\right) = \vec{a} + s\left(\frac{\vec{b}}{2} - \vec{a}\right)$$

$$\Rightarrow \quad \frac{t}{3} = \frac{1}{5} \quad \text{and} \quad \frac{2t}{3} = \frac{s}{2}$$

$$\Rightarrow \quad t = 3/5 \quad \text{and} \quad s = 4/5$$

$\therefore$ Point P is $\dfrac{\vec{a} + 2\vec{b}}{5}$ ...(1)

Since P divides $\vec{OD}$ in the ratio of $\lambda : 1$

$$\therefore \quad \frac{\lambda\left(\frac{\vec{a} + 2\vec{b}}{3}\right) + 1.0}{\lambda + 1} = \frac{\lambda}{3(\lambda + 1)}(\vec{a} + 2\vec{b}) \quad ...(2)$$

from (1) and (2), we get

$$\frac{\lambda}{3(\lambda + 1)} = \frac{1}{5}$$

or $5\lambda = 3\lambda + 3$

$\Rightarrow \lambda = 3/2$

$$\therefore \quad \frac{OP}{PD} = \frac{3}{2}$$

9. Let, $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\therefore \quad \vec{R} \times \vec{B} = \vec{C} \times \vec{B}$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -3 & 7 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\Rightarrow (y - z)\hat{i} - (x - z)\hat{j} + (x - y)\hat{k} = -10\hat{i} - 3\hat{j} + 7\hat{k}$$

$$\Rightarrow \quad y - z = -10$$
$$z - x = -3$$
$$x - y = 7$$

and $\vec{R}\cdot\vec{A} = 0 \Rightarrow 2x + z = 0$, on solving above equations

$$x = -1, \; y = -8 \quad \text{and} \quad z = 2$$

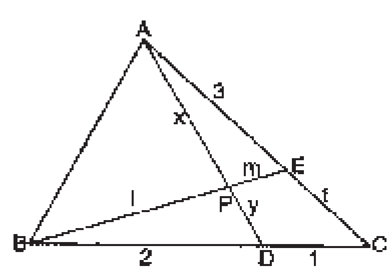
$$\therefore \quad \vec{R} = -\hat{i} - 8\hat{j} + 2\hat{k}$$

10. As, $\vec{a} = cx\hat{i} - 6\hat{j} + 3\hat{k}$ and $\vec{b} = x\hat{i} + 2\hat{j} + 2cx\hat{k}$

makes an obtuse angle $\Rightarrow \vec{a}\cdot\vec{b} < 0$

$$\Rightarrow \quad cx^2 - 12 + 6cx < 0$$
$$\Rightarrow \quad c < 0 \quad \text{and} \quad D < 0$$
$$\Rightarrow \quad c < 0 \quad \text{and} \quad 36c^2 - 4\cdot(-12)c < 0$$
$$\Rightarrow \quad c < 0 \quad \text{and} \quad 12c(3c + 4) < 0$$
$$\Rightarrow \quad c < 0 \quad \text{and} \quad c > -4/3$$
$$\therefore \quad c \in (-4/3, 0)$$

11. Let the position vectors of A, B and C are $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively, since the point D divides BC in the ratio of $2 : 1$, the position vector of D will be



$$D = \left(\frac{2\vec{c} + \vec{b}}{4} \right)$$

and the point E divides AC in the ratio 3 : 1,

therefore, $E = \left(\frac{3\vec{c} + \vec{a}}{4} \right)$

Now, let P divides BE in the ratio $l:m$ and AD in the ratio $x:y$. Hence the position vector of P getting from BE and AD must be the same.

Hence, we have

$$\frac{l\left(\frac{3\vec{c} + \vec{a}}{4} \right) + m\vec{b}}{l+m} = \frac{x\left(\frac{2\vec{c} + \vec{b}}{4} \right) + y\vec{a}}{x+y}$$

$$\Rightarrow \frac{\frac{3l\vec{c} + l\vec{a}}{4} + m\vec{b}}{l+m} = \frac{\frac{2\vec{c}x}{3} + \frac{\vec{b}x}{3} + y\vec{a}}{x+y}$$

$$\Rightarrow \frac{\frac{3l\vec{c}}{4} + \frac{l\vec{a}}{4} + m\vec{b}}{l+m} = \frac{\frac{2\vec{c}x}{3} + \frac{\vec{b}x}{3} + y\vec{a}}{x+y}$$

$$\Rightarrow \frac{3l}{4(l+m)}\vec{c} + \frac{l}{4(l+m)}\vec{a} + \frac{m\vec{b}}{l+m}$$

$$= \frac{2x}{3(x+y)}\vec{c} + \frac{x}{3(x+y)}\vec{b} + \frac{y}{(x+y)}\vec{a}$$

Now, comparing the coefficients, we get

$$\frac{3l}{4(l+m)} = \frac{2x}{3(x+y)} \quad \ldots(1)$$

and $$\frac{l}{4(l+m)} = \frac{y}{x+y} \quad \ldots(2)$$

and $$\frac{m}{l+m} = \frac{x}{3(x+y)} \quad \ldots(3)$$

For solving divide eq. (1) by eq. (3), we get

$$\frac{\frac{3l}{4(l+m)}}{\frac{m}{l+m}} = \frac{\frac{2x}{3(x+y)}}{\frac{x}{3(x+y)}}$$

$$\Rightarrow \frac{3}{4}\cdot\frac{l}{m} = 2$$

$$\Rightarrow \frac{l}{m} = \frac{8}{3} = \frac{BP}{PE}$$

Hence, proved.

12. As $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar $\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] \neq 0$

Also $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} = p\vec{a} + q\vec{b} + r\vec{c}$

Taking dot product with $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively both sides, we get

$$p + q\cos\theta + r\cos\theta = [\vec{a}\ \vec{b}\ \vec{c}] \quad \ldots(1)$$

$$p\cos\theta + q + r\cos\theta = 0 \quad \ldots(2)$$

$$p\cos\theta + q\cos\theta + r = [\vec{a}\ \vec{b}\ \vec{c}] \quad \ldots(3)$$

adding above equations,

$$p + q + r = \frac{2[\vec{a}\ \vec{b}\ \vec{c}]}{2\cos\theta + 1} \quad \ldots(4)$$

multiplying (4) by $\cos\theta$ and subtracting (1), we get

$$p(\cos\theta - 1) = \frac{2[\vec{a}\ \vec{b}\ \vec{c}]\cos\theta}{2\cos\theta + 1} - [\vec{a}\ \vec{b}\ \vec{c}]$$

$$\Rightarrow \quad p = \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{(1-\cos\theta)(2\cos\theta + 1)}$$

Similarly, $q = \frac{-2[\vec{a}\ \vec{b}\ \vec{c}]\cos\theta}{(1+2\cos\theta)(1-\cos\theta)}$

and $r = \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{(1+2\cos\theta)(1-\cos\theta)}$

where $[\vec{a}\ \vec{b}\ \vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}$

$$= \begin{vmatrix} 1 & \cos\theta & \cos\theta \\ \cos\theta & 1 & \cos\theta \\ \cos\theta & \cos\theta & 1 \end{vmatrix}$$

applying $R_1 \to R_1 + R_2 + R_3$

$$= (1+2\cos\theta)\begin{vmatrix} 1 & 1 & 1 \\ \cos\theta & 1 & \cos\theta \\ \cos\theta & \cos\theta & 1 \end{vmatrix}$$

applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$

$$= (1+2\cos\theta)\begin{vmatrix} 1 & 0 & 0 \\ \cos\theta & 1-\cos\theta & 0 \\ \cos\theta & 0 & 1-\cos\theta \end{vmatrix}$$

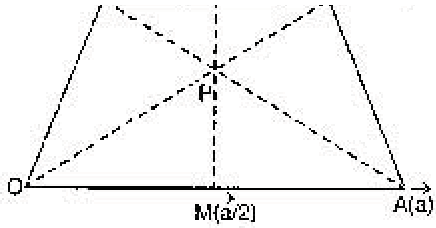
$$= (1+2\cos\theta)\cdot(1-\cos\theta)^2$$

$$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = (\sqrt{1+2\cos\theta})\cdot(1-\cos\theta)$$

$$\therefore \quad p = \frac{1}{\sqrt{1+2\cos 2\theta}}$$

$$q = \frac{-2\cos\theta}{\sqrt{1-2\cos\theta}}$$

and $$r = \frac{1}{\sqrt{1+2\cos\theta}}$$



As $BC \parallel OA$, $\vec{BC} = \alpha\, \vec{OA} = \alpha\, \vec{a}$ for some constant α.

equation of OC is $\vec{r} = t(\vec{b} + \alpha\vec{a})$

equation of AB is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

Let P be the point of intersection of OC and AB. Then at point P, $t(\vec{b} + \alpha\vec{a}) = \vec{a} + \lambda(\vec{b} - \vec{a})$ for some values of t and λ.

$\Rightarrow \quad (t\alpha - 1 + \lambda)\vec{a} = (\lambda - t)\vec{b}$

Since $\vec{a}$ and $\vec{b}$ are non-parallel vectors, we must have

$t\alpha - 1 + \lambda = 0$ and $\lambda = t$

$\Rightarrow \quad t = 1/(\alpha + 1)$

Thus, position vector of P is

$$\vec{r}_1 = \frac{1}{\alpha+1}(\vec{b} + \alpha\vec{a})$$

Equation of MN is

$$\vec{r} = \frac{1}{2}\vec{a} + k\left[\vec{b} + \frac{1}{2}(\alpha - 1)\vec{a}\right] \quad ...(1)$$

For $k = 1/(\alpha + 1)$ {which is the coefficient of $\vec{b}$ in $\vec{r}_1$}, we get

$$\vec{r} = \frac{1}{2}\vec{a} + \frac{1}{\alpha+1}\left[\vec{b} + \frac{1}{2}(\alpha - 1)\vec{a}\right]$$

$$= \frac{1}{(\alpha+1)}\vec{b} + \frac{1}{2}(\alpha - 1).\frac{1}{\alpha+1}\vec{a} + \frac{1}{2}\vec{a}$$

$$= \frac{1}{(\alpha+1)}\vec{b} + \frac{1}{2(\alpha+1)}(\alpha - 1 + \alpha + 1)\vec{a}$$

$$= \frac{1}{(\alpha+1)}(\vec{b} + \alpha\vec{a}) = \vec{r}_1$$

$\Rightarrow P$ lies on MN.

14. $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ (considering first part)

Let $\vec{c} \times \vec{d} = \vec{e}$

$$(\vec{a} \times \vec{b}) \times \vec{e} = (\vec{a}.\vec{e})\vec{b} - (\vec{b}.\vec{e})\vec{a}$$

$$= [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a} \quad ...(1)$$

Similarly,

$$(\vec{a} \times \vec{c}) \times (\vec{d} \times \vec{b}) = [\vec{a}\ \vec{d}\ \vec{b}]\vec{c} - [\vec{c}\ \vec{d}\ \vec{b}]\vec{a}$$

$$= [\vec{a}\ \vec{d}\ \vec{b}]\vec{c} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a} \quad ...(2)$$

Also, $(\vec{a} \times \vec{d}) \times (\vec{b} \times \vec{c}) = -(\vec{b} \times \vec{c}) \times (\vec{a} \times \vec{d})$

$$= (\vec{b} \times \vec{c}) \times (\vec{d} \times \vec{a})$$

$$= [\vec{b}\ \vec{d}\ \vec{a}]\vec{c} - [\vec{c}\ \vec{d}\ \vec{a}]\vec{b}$$

$$= [\vec{a}\ \vec{d}\ \vec{b}]\vec{c} - [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} \quad ...(3)$$

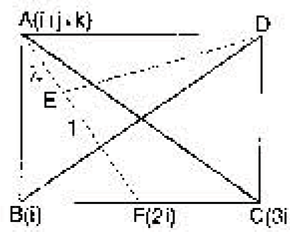
From (1), (2) and (3), we get

$$\vec{u} = (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) + (\vec{a} \times \vec{c}) \times (\vec{d} \times \vec{b}) + (\vec{a} \times \vec{d}) \times (\vec{b} \times \vec{c})$$

$$= [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a} + [\vec{a}\ \vec{d}\ \vec{b}]\vec{c} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a} - [\vec{a}\ \vec{d}\ \vec{b}]\vec{c} - [\vec{a}\ \vec{c}\ \vec{d}]\vec{b}$$

$$= -2[\vec{b}\ \vec{c}\ \vec{d}]\vec{a}$$

15. F is mid-point of BC i.e., $F \equiv \frac{\hat{i} + 3\hat{i}}{2} = 2\hat{i}$ and $AE \perp DE$ (given)



Let E divides AF in $\lambda : 1$. The position vector of E is given by

$$\frac{2\lambda\hat{i} + 1(\hat{i} + \hat{j} + \hat{k})}{\lambda + 1} = \left(\frac{2\lambda + 1}{\lambda + 1}\right)\hat{i} + \frac{1}{\lambda+1}\hat{j} + \frac{1}{\lambda+1}\hat{k}$$

Now, volume of the tetrahedron

$= \frac{1}{3}$ (area of the base) (height)

$\Rightarrow \quad \frac{2\sqrt{2}}{3} = \frac{1}{3}$ (area of the ΔABC) (DE)

But area of the $\Delta ABC = \frac{1}{2}|\vec{BC} \times \vec{BA}|$

$$= \frac{1}{2}|2\hat{i} \times (\hat{j} + \hat{k})| = |\hat{i} \times \hat{j} + \hat{i} \times \hat{k}|$$

$$= |\hat{k} - \hat{j}| = \sqrt{2}$$

Therefore, $\frac{2\sqrt{2}}{3}=\frac{1}{3}(\sqrt{2})(DE)$

$\Rightarrow \quad DE=2$

Since ΔADE is a right angle triangle,

$$AD^2=AE^2+DE^2$$

$\Rightarrow \quad (4)^2=AE^2+(2)^2$

$\Rightarrow \quad AE^2=12$

But $\quad \vec{AE}=\frac{2\lambda+1}{\lambda+1}\hat{i}+\frac{1}{\lambda+1}\hat{j}+\frac{1}{\lambda+1}\hat{k}-(\hat{i}+\hat{j}+\hat{k})$

$$=\frac{\lambda}{\lambda+1}\hat{i}-\frac{\lambda}{\lambda+1}\hat{j}-\frac{\lambda}{\lambda+1}\hat{k}$$

$\Rightarrow \quad |\vec{AE}|^2=\frac{1}{(\lambda+1)^2}[\lambda^2+\lambda^2+\lambda^2]=\frac{3\lambda^2}{(\lambda+1)^2}$

Therefore, $12=\frac{3\lambda^2}{(\lambda+1)^2}$

$\Rightarrow \quad 4(\lambda+1)^2=\lambda^2$

$\Rightarrow \quad 4\lambda^2+4+8\lambda=\lambda^2$

$\Rightarrow \quad 3\lambda^2+8\lambda+4=0$

$\Rightarrow \quad 3\lambda^2+6\lambda+2\lambda+4=0$

$\Rightarrow \quad 3\lambda(\lambda+2)+2(\lambda+2)=0$

$\Rightarrow \quad (3\lambda+2)(\lambda+2)=0$

$\Rightarrow \quad \lambda=-2/3, \lambda=-2$

Therefore, when $\lambda=-2/3$, position vector of E is given by

$$\left(\frac{2\lambda+1}{\lambda+1}\right)\hat{i}+\frac{1}{\lambda+1}\hat{j}+\frac{1}{\lambda+1}\hat{k}$$

$$=\frac{2.(-2/3)+1}{-2/3+1}\hat{i}+\frac{1}{-2/3+1}\hat{j}+\frac{1}{-2/3+1}\hat{k}$$

$$=\frac{-4/3+1}{\frac{-2+3}{3}}\hat{i}+\frac{1}{\frac{-2+3}{3}}\hat{j}+\frac{1}{\frac{-2+3}{3}}\hat{k}$$

$$=\frac{\frac{-4+3}{3}}{1/3}\hat{i}+\frac{1}{1/3}\hat{j}+\frac{1}{1/3}\hat{k}$$

$$=-\hat{i}+3\hat{j}+3\hat{k}$$

and when $\lambda=-2$,

position vector of E is given by,

$$\frac{2\times(-2)+1}{-2+1}\hat{i}+\frac{1}{-2+1}\hat{j}+\frac{1}{-2+1}\hat{k}=\frac{-4+1}{-1}\hat{i}-\hat{j}-\hat{k}$$

$$=3\hat{i}-\hat{j}-\hat{k}$$

Therefore, $-\hat{i}+3\hat{j}+3\hat{k}$ and $+3\hat{i}-\hat{j}-\hat{k}$ are the answer.

16. We have $(\vec{A}+\vec{B})\times(\vec{A}+\vec{C})$

$$=\vec{A}\times\vec{A}+\vec{B}\times\vec{A}+\vec{A}\times\vec{C}+\vec{B}\times\vec{C}$$

$$=\vec{B}\times\vec{A}+\vec{A}\times\vec{C}+\vec{B}\times\vec{C} \quad [\because \vec{A}\times\vec{A}=0]$$

Now, $[(\vec{A}+\vec{B})\times(\vec{A}+\vec{C})]\times(\vec{B}\times\vec{C})$

$$=[\vec{B}\times\vec{A}+\vec{A}\times\vec{C}+\vec{B}\times\vec{C}]\times(\vec{B}\times\vec{C})$$

$$=(\vec{B}\times\vec{A})\times(\vec{B}\times\vec{C})+(\vec{A}\times\vec{C})\times(\vec{B}\times\vec{C})$$

$$=\{(\vec{B}\times\vec{A}).\vec{C}\}\vec{B}-\{(\vec{B}\times\vec{A}).\vec{B}\}\vec{C}$$

$$+\{(\vec{A}\times\vec{C}).\vec{C}\}\vec{B}-\{(\vec{A}\times\vec{C}).\vec{B}\}\vec{C}$$

$[\because (\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}]$

$$=[\vec{B}\;\vec{A}\;\vec{C}]\vec{B}-[\vec{A}\;\vec{C}\;\vec{B}]\vec{C}$$

$[\because [\vec{a}\;\vec{b}\;\vec{c}]=0$ if any two of $\vec{a},\vec{b},\vec{c}$ are equal$]$

$$=[\vec{A}\;\vec{C}\;\vec{B}](\vec{B}-\vec{C})$$

Now, $[(\vec{A}+\vec{B})\times(\vec{A}\times\vec{C})]\times(\vec{B}\times\vec{C}).(\vec{B}+\vec{C})$

$$=([\vec{A}\;\vec{C}\;\vec{B}](\vec{B}-\vec{C})).(\vec{B}+\vec{C})$$

$$=[\vec{A}\;\vec{C}\;\vec{B}]\{(\vec{B}-\vec{C}).(\vec{B}+\vec{C})\}$$

$$=[\vec{A}\;\vec{C}\;\vec{B}]\{|\vec{B}|^2-|\vec{C}|^2\}=0$$

$[\because |\vec{B}|=|\vec{C}|$ given$]$

17. (a) We have $\vec{u}.\vec{v}=|\vec{u}||\vec{v}|\cos\theta$

and $\quad \vec{u}\times\vec{v}=|\vec{u}||\vec{v}|\sin\theta\,\hat{n}$

where θ is the angle between $\vec{u}$ and $\vec{v}$ and $\hat{n}$ is unit vector perpendicular to the plane of $\vec{u}$ and $\vec{v}$.

Again $|\vec{u}.\vec{v}|^2=|\vec{u}|^2|\vec{v}|^2\cos^2\theta$ and

$$|\vec{u}\times\vec{v}|^2=|\vec{u}|^2|\vec{v}|^2\sin^2\theta=|\vec{u}|^2.|\vec{v}|^2\sin^2\theta$$

$\therefore \quad |\vec{u}.\vec{v}|^2+|\vec{u}\times\vec{v}|^2=|\vec{u}|^2.|\vec{v}|^2(\cos^2\theta+\sin^2\theta)$

$$=|\vec{u}|^2.|\vec{v}|^2 \qquad \ldots(1)$$

(b) $|\vec{u}+\vec{v}+(\vec{u}\times\vec{v})|^2$

$$=|\vec{u}+\vec{v}|^2+|\vec{u}\times\vec{v}|^2+2(\vec{u}+\vec{v}).(\vec{u}\times\vec{v})$$

$$=|\vec{u}|^2+|\vec{v}|^2+2\vec{u}.\vec{v}+|\vec{u}\times\vec{v}|^2+0$$

$[\because \vec{u}\times\vec{v}$ is perpendicular to the plane of $\vec{u}$ and $\vec{v}]$

$+1 \cdot 2\vec{u}\cdot\vec{v} + |\vec{u}\cdot\vec{v}|^2$

$= |\vec{u}|^2 + |\vec{v}|^2 + 1 + |\vec{u}|^2 |\vec{v}|^2$ [from (1)]

$= |\vec{u}|^2 (1+|\vec{v}|^2) + (1+|\vec{v}|^2)$

$= (1+|\vec{v}|^2)(1+|\vec{u}|^2)$

18. Given equation is $\vec{w} + (\vec{w}\times\vec{u}) = \vec{v}$ ...(1)

taking cross product with $\vec{u}$, we get

$$\vec{u}\times[\vec{w} + (\vec{w}\times\vec{u})] = \vec{u}\times\vec{v}$$

$\Rightarrow$ $\vec{u}\times\vec{w} + \vec{u}\times(\vec{w}\times\vec{u}) = \vec{u}\times\vec{v}$

$\Rightarrow$ $\vec{u}\times\vec{w} + (\vec{u}\cdot\vec{u})\vec{w} - (\vec{u}\cdot\vec{w})\vec{u} = \vec{u}\times\vec{v}$

$\Rightarrow$ $\vec{u}\times\vec{w} + \vec{w} - (\vec{u}\cdot\vec{w})\vec{u} = \vec{u}\times\vec{v}$ ...(2)

Now, doing scalar product of (1) with $\vec{u}$, we get

$$\vec{u}\cdot\vec{w} + \vec{u}\cdot(\vec{w}\times\vec{u}) = \vec{u}\cdot\vec{v}$$

$\Rightarrow$ $\vec{u}\cdot\vec{w} = \vec{u}\cdot\vec{v}$ [$\because \vec{u}\cdot(\vec{w}\times\vec{v}) = 0$] ...(3)

Now, taking scalar product of (1) with $\vec{v}$, we get

$$\vec{v}\cdot\vec{w} + \vec{v}\cdot(\vec{w}\times\vec{u}) = \vec{v}\cdot\vec{v}$$

$\Rightarrow$ $\vec{v}\cdot\vec{w} + [\vec{v}\ \vec{w}\ \vec{u}] = 1$

$\Rightarrow$ $\vec{v}\cdot\vec{w} + [\vec{v}\ \vec{w}\ \vec{u}] - 1 = 0$

$\Rightarrow$ $-(\vec{u}\times\vec{v})\cdot\vec{w} - \vec{v}\cdot\vec{w} + 1 = 0$

$\Rightarrow$ $1 - \vec{v}\cdot\vec{w} = (\vec{u}\times\vec{v})\cdot\vec{w}$ ...(4)

Taking scalar product of (2) with $\vec{w}$, we get

$$(\vec{u}\times\vec{w})\cdot\vec{w} + \vec{w}\cdot\vec{w} - (\vec{u}\cdot\vec{w})(\vec{u}\cdot\vec{w})$$

$= (\vec{u}\times\vec{v})\cdot\vec{w}$ ...(5)

$\Rightarrow 0 + |\vec{w}|^2 - (\vec{u}\cdot\vec{w})^2 = (\vec{u}\times\vec{v})\cdot\vec{w}$

$\Rightarrow$ $(\vec{u}\times\vec{v})\cdot\vec{w} = |\vec{w}|^2 - (\vec{u}\cdot\vec{w})^2$

Taking scalar product of (1) with $\vec{w}$, we get

$\Rightarrow$ $|\vec{w}|^2 = 1 - (\vec{u}\times\vec{v})\cdot\vec{w}$ (by eq. (4))

again from (5), we get

$$(\vec{u}\times\vec{v})\cdot\vec{w} = |\vec{w}|^2 - (\vec{u}\cdot\vec{w})^2$$

$= 1 - (\vec{u}\times\vec{v})\cdot\vec{w} - (\vec{u}\cdot\vec{w})^2$

$\Rightarrow$ $2(\vec{u}\times\vec{v})\cdot\vec{w} = 1 - (\vec{u}\cdot\vec{v})^2$ (by eq. (3))

Now taking modulus of both sides

Thus, $\left|(\vec{u}\times\vec{v})\cdot\vec{w}\right| = \frac{1}{2}\left|1 - (\vec{u}\cdot\vec{v})^2\right|$

$\le \frac{1}{2}$ [$\because (\vec{u}\cdot\vec{v})^2 \ge 0$]

The equality holds if and only if $\vec{u}\cdot\vec{v} = 0$ or iff $\vec{u}$ is perpendicular to $\vec{v}$.

19. Let the position vector of A, B, C be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively and that of P, Q, R be $\vec{p}, \vec{q}$ and $\vec{r}$ respectively. Let $\vec{h}$ be the position vector of the orthocentre H of the ΔPQR. We have $HP \perp QR$. Equation of straight line passing through A and perpendicular to QR *i.e.*, Parallel to $\vec{HP} = \vec{p} - \vec{h}$ is

$$\vec{r} = \vec{a} + t_1(\vec{p} - \vec{h}) \quad ...(1)$$

where t_1 is a parameter.

Similarly, equation of straight line through B and perpendicular to RP is

$$\vec{r} = \vec{b} + t_2(\vec{q} - \vec{h}) \quad ...(2)$$

Again, equation of straight line through C and perpendicular to PQ is

$$\vec{r} = \vec{c} + t_3(\vec{r} - \vec{h}) \quad ...(3)$$

If the lines (1), (2) and (3) are concurrent, then there exists a point D with position vector $\vec{d}$ which lies on all of them, that is for some values of t_1, t_2 and t_3,

which implies that

$$\frac{1}{t_1}\vec{d} = \frac{1}{t_1}\vec{a} + \vec{p} - \vec{h} \quad ...(4)$$

$$\frac{1}{t_2}\vec{d} = \frac{1}{t_2}\vec{b} + \vec{q} - \vec{h} \quad ...(5)$$

$$\frac{1}{t_3}\vec{d} = \frac{1}{t_3}\vec{c} + \vec{r} - \vec{h} \qquad \text{...(6)}$$

From (4) and (5), we get

$$\left(\frac{1}{t_1}-\frac{1}{t_2}\right)\vec{d} = \frac{1}{t_1}\vec{a} - \frac{1}{t_2}\vec{b} + \vec{p} - \vec{q} \qquad \text{...(7)}$$

and from (5) and (6), we get

$$\left(\frac{1}{t_2}-\frac{1}{t_3}\right)\vec{d} = \frac{1}{t_2}\vec{b} - \frac{1}{t_3}\vec{c} + \vec{q} - \vec{r} \qquad \text{...(8)}$$

eliminating $\vec{d}$ from (7) and (8), we get

$$\left(\frac{1}{t_2}-\frac{1}{t_3}\right)\left[\frac{1}{t_1}\vec{a} - \frac{1}{t_2}\vec{b} + \vec{p} - \vec{q}\right] = \left(\frac{1}{t_1}-\frac{1}{t_2}\right)\left[\frac{1}{t_2}\vec{b} - \frac{1}{t_3}\vec{c} + \vec{q} - \vec{r}\right]$$

$$\Leftrightarrow \quad (t_3-t_2)[t_2\vec{a} - t_1\vec{b} + t_1t_2(\vec{p} - \vec{q})] = (t_2-t_1)[t_3\vec{b} - t_2\vec{c} + t_2t_3(\vec{q} - \vec{r})]$$

(multiplying both sides by $t_1\, t_2\, t_3$)

$$\Leftrightarrow \quad t_2(t_3-t_2)\vec{a} + t_2(t_1-t_3)\vec{b} + t_2(t_2-t_1)\vec{c} + t_1t_2(t_3-t_2)\vec{p} + t_2^2(t_1-t_3)\vec{q} + t_2t_3(t_2-t_1)\vec{r} = \vec{0}$$

Thus, (1), (2), and (3) are concurrent is equivalent to say that there exist scalars t_1, t_2 and t_3 such that

$$(t_2-t_3)\vec{a} + (t_3-t_1)\vec{b} + (t_1-t_2)\vec{c} + t_1(t_2-t_3)\vec{p} + t_2(t_3-t_1)\vec{q} + t_3(t_1-t_2)\vec{r} = \vec{0}$$

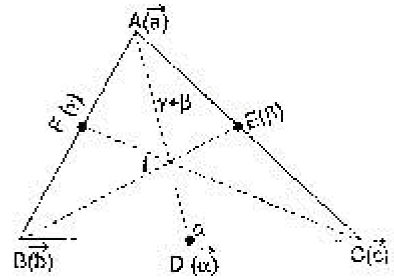
Dividing by $t_1\, t_2\, t_3$, we get

$$(\lambda_2-\lambda_3)\vec{p} + (\lambda_3-\lambda_1)\vec{q} + (\lambda_1-\lambda_2)\vec{r} + \lambda_1(\lambda_2-\lambda_3)\vec{a} + \lambda_2(\lambda_3-\lambda_1)\vec{b} + \lambda_3(\lambda_1-\lambda_2)\vec{c} = 0$$

where $\lambda_i = \frac{1}{t_i}$ for $i = 1, 2, 3$

So this is the condition that the lines from P perpendicular to BC, from Q perpendicular to CA and from R perpendicular to AB are concurrent. (by changing ABC and PQR simultaneously).

20. Let AD be the angular bisector of angle A. Let BC, AC and AB are α, β and γ respectively. Then $\frac{BD}{DC} = \frac{\gamma}{\beta}$. Hence position vector of $D = \frac{\gamma\vec{c} + \beta\vec{b}}{\gamma+\beta}$. On AD, there lies a point I which divides it in ratio $\gamma+\beta : \alpha$.



Now, position vector of $I = \frac{\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}}{\alpha+\beta+\gamma}$

which is symmetric in $\vec{a}$, $\vec{b}$, $\vec{c}$, α, β and γ. Hence I lies on every angle bisector. Hence, angle bisectors are concurrent.

Here, $\alpha = |\vec{b} - \vec{c}|$, $\beta = |\vec{a} - \vec{c}|$,

$\gamma = |\vec{a} - \vec{b}|$.

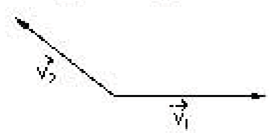
21. We have $|\vec{v}_1| = 2$, $|\vec{v}_2| = \sqrt{2}$ and $|\vec{v}_3| = \sqrt{29}$

If θ is the angle between $\vec{v}_1$ and $\vec{v}_2$, then

$2\sqrt{2}\cos\theta = -2 \Rightarrow \cos\theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = 135°$

Since any two vectors are always coplanar and data is not sufficient, so we can assume $\vec{v}_1$ and $\vec{v}_2$ in $x-y$ plane

and $\vec{v}_1 = 2\hat{i}$ (let)



and $\vec{v}_2 = -\hat{i} + \hat{j}$

and $\vec{v}_3 = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$

since $\vec{v}_3 \cdot \vec{v}_1 = 6 = 2\alpha \Rightarrow \alpha = 3$

Also $\vec{v}_3 \cdot \vec{v}_2 = -5 = -\alpha + \beta \Rightarrow \beta = \pm 2$

and $\vec{v}_3 \cdot \vec{v}_3 = 29 = \alpha^2 + \beta^2 + \gamma^2 \Rightarrow \gamma = \pm 4$

Hence, $\vec{v}_3 = 3\hat{i} + 2\hat{j} \pm 4\hat{k}$

22. $\vec{A}(t)$ is parallel to $\vec{B}(t)$ for some $t \in [0,1]$ if and only if $\frac{f_1(t)}{g_1(t)} = \frac{f_2(t)}{g_2(t)}$ for some $t \in [0,1]$

or $f_1(t) \cdot g_2(t) = f_2(t) \cdot g_1(t)$ for some $t \in [0,1]$

Let $h(t) = f_1(t) \cdot g_2(t) - f_2(t) g_1(t)$

$h(0) = 2\times 2 - 3\times 3 = -5 < 0$

$h(1) = f_1(1) \cdot g_2(1) - f_2(1) \cdot g_1(1) = 6\times 6 - 2\times 2 = 32 > 0$

Since, h is a continuous function and $h(0) \cdot h(1) < 0$

$\Rightarrow$ there is some $t \in [0,1]$ for which $h(t) = 0$

i.e., $\vec{A}(t)$ and $\vec{B}(t)$ are parallel vectors for this t.

Now, $L = \frac{(a_1 + a_2 + a_3) + (b_1 + b_2 + b_3) + (c_1 + c_2 + c_3)}{3} \geq$

$[(a_1 + a_2 + a_3)(b_1 + b_2 + b_3)(c_1 + c_2 + c_3)]^{1/3}$

{using A.M. ≥ G.M.}

$\therefore L^3 \geq [(a_1 + a_2 + a_3)(b_1 + b_2 + b_3)$

$(c_1 + c_2 + c_3)]$ ...(2)

Now, $(a_1 + a_2 + a_3)^2$

$= a_1^2 + a_2^2 + a_3^2 + 2a_1a_2 + 2a_1a_3 + 2a_2a_3$

$> a_1^2 + a_2^2 + a_3^2.$

$\Rightarrow \quad (a_1 + a_2 + a_3) \geq \sqrt{a_1^2 + a_2^2 + a_3^2}$

Similarly, $(b_1 + b_2 + b_3) \geq \sqrt{b_1^2 + b_2^2 + b_3^2}$

and $(c_1 + c_2 + c_3) \geq \sqrt{c_1^2 + c_2^2 + c_3^2}$

$\therefore$ from (1) and (2),

$L^3 \geq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)(c_1^2 + c_2^2 + c_3^2)^{\frac{1}{3}}$

$\geq V$

24.
$$\left.\begin{aligned} \vec{x} &= \frac{\vec{u} + \vec{v}}{|\vec{u} + \vec{v}|} = \frac{1}{2}\sec\frac{\alpha}{2}(\vec{u} + \vec{v}) \\ \vec{y} &= \frac{\vec{w} + \vec{v}}{|\vec{v} + \vec{w}|} = \frac{1}{2}\sec\frac{\beta}{2}(\vec{v} + \vec{w}) \\ \vec{z} &= \frac{\vec{w} + \vec{u}}{|\vec{w} + \vec{u}|} = \frac{1}{2}\sec\frac{\gamma}{2}(\vec{w} + \vec{u}) \end{aligned}\right\} \quad ...(1)$$

as $[\vec{x} \times \vec{y} \;\; \vec{y} \times \vec{z} \;\; \vec{z} \times \vec{x}] = [\vec{x}\ \vec{y}\ \vec{z}]$ (using (1))

$= \frac{1}{64}\sec^2\frac{\alpha}{2}.\sec^2\frac{\beta}{2}.\sec^2\frac{\gamma}{2}$

$[\vec{u} + \vec{v} \;\; \vec{v} + \vec{w} \;\; \vec{w} + \vec{u}]^2$ ...(2)

Now, $[\vec{u} + \vec{v} \;\; \vec{v} + \vec{w} \;\; \vec{w} + \vec{u}] = 2[\vec{u} + \vec{v}\ \vec{w}]$ ...(3)

$\Rightarrow [\vec{x} \times \vec{y} \;\; \vec{y} \times \vec{z} \;\; \vec{z} \times \vec{x}]$

$= \frac{1}{16}[\vec{u}\ \vec{v}\ \vec{w}]\sec\frac{\alpha}{2}.\sec\frac{\beta}{2}.\sec\frac{\gamma}{2}$

25. Here, $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$

$\Rightarrow \quad \vec{a} \times \vec{b} - \vec{a} \times \vec{c} = \vec{c} \times \vec{d} - \vec{b} \times \vec{d}$

$\Rightarrow \quad \vec{a} \times (\vec{b} - \vec{c}) = (\vec{c} - \vec{b}) \times \vec{d}$

$\Rightarrow \quad \vec{a} \times (\vec{b} - \vec{c}) - (\vec{c} - \vec{b}) \times \vec{d} = 0$

$\Rightarrow \quad \vec{a} \times (\vec{b} - \vec{c}) - \vec{d} \times (\vec{b} - \vec{c}) = 0$

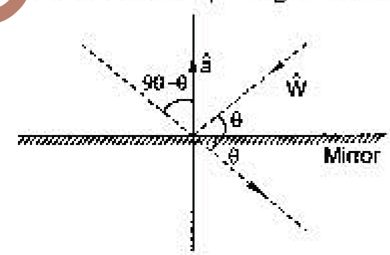
$\Rightarrow \quad (\vec{a} - \vec{d}) \times (\vec{b} - \vec{c}) = 0$

or $\quad (\vec{a} - \vec{d}) \,||\, (\vec{b} - \vec{c})$

$\therefore \quad (\vec{a} - \vec{d}).(\vec{b} - \vec{c}) \neq 0$

$\vec{a}.\vec{b} + \vec{d}.\vec{c} \neq \vec{d}.\vec{b} + \vec{a}.\vec{c}$

26. Since $\hat{v}$ is unit vector along the incident ray and $\hat{w}$ is the unit vector along the reflected ray.
Hence $\hat{a}$ is a unit vector along the external bisector of $\hat{v}$ and $\hat{w}$. Hence $\hat{w} - \hat{v} = \lambda\hat{a}$, squaring both sides



$\Rightarrow \quad 1 + 1 - \hat{w}\cdot\hat{v} = \lambda^2$

or $\quad 2 - 2\cos 2\theta = \lambda^2$

or $\quad \lambda = 2\sin\theta$

where 2θ is the angle between $\hat{v}$ and $\hat{w}$.

Hence $\hat{w} - \hat{v} = 2\sin\theta \cdot \hat{a}$

$= 2\cos(90° - \theta)\,\hat{a} = -(2\,\hat{a}\cdot\hat{v})\,\hat{a}$

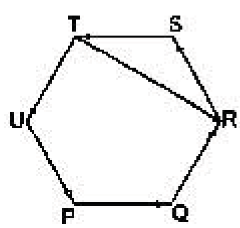
$\hat{w} = \hat{v} - 2(\hat{a}\cdot\hat{v})\,\hat{a}$

G ASSERTION AND REASON

1. Since, $\vec{PQ}$ is not parallel to $\vec{TR}$

$\because$ $\vec{TR}$ is resultant of $\vec{SR}$ and $\vec{ST}$ vectors.

$\Rightarrow \vec{PQ} \times (\vec{RS} + \vec{ST}) \neq \vec{0}.$



But for statement II, we have $\vec{PQ} \times \vec{RS} = \vec{0}$

which is not possible as $\vec{PQ}$ not parallel to $\vec{RS}$.

Hence, statement I is true and statement II is false.