Introduction to Digital Signal Processing

1.1 What is DSP?

DSP is a technique of performing the mathematical operations on the signals in digital domain. As real time signals are analog in nature we need first convert the analog signal to digital, then we have to process the signal in digital domain and again converting back to analog domain. Thus ADC is required at the input side whereas a DAC is required at the output end. A typical DSP system is as shown in figure 1.1.

Fig 1.1: A Typical DSP System

1.2 Need for DSP

Analog signal Processing has the following drawbacks:

- They are sensitive to environmental changes
- Aging
- Uncertain performance in production units
- Variation in performance of units
- Cost of the system will be high
- Scalability

If Digital Signal Processing would have been used we can overcome the above shortcomings of ASP.

1.3 A Digital Signal Processing System

A computer or a processor is used for digital signal processing. Antialiasing filter is a LPF which passes signal with frequency less than or equal to half the sampling frequency in order to avoid Aliasing effect. Similarly at the other end, reconstruction filter is used to reconstruct the samples from the staircase output of the DAC (Figure 1.2).

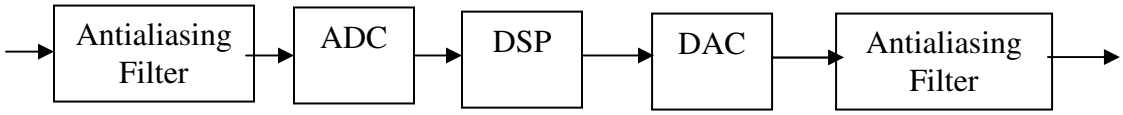

Fig 1.2 The Block Diagram of a DSP System

Signals that occur in a typical DSP are as shown in figure 1.3.

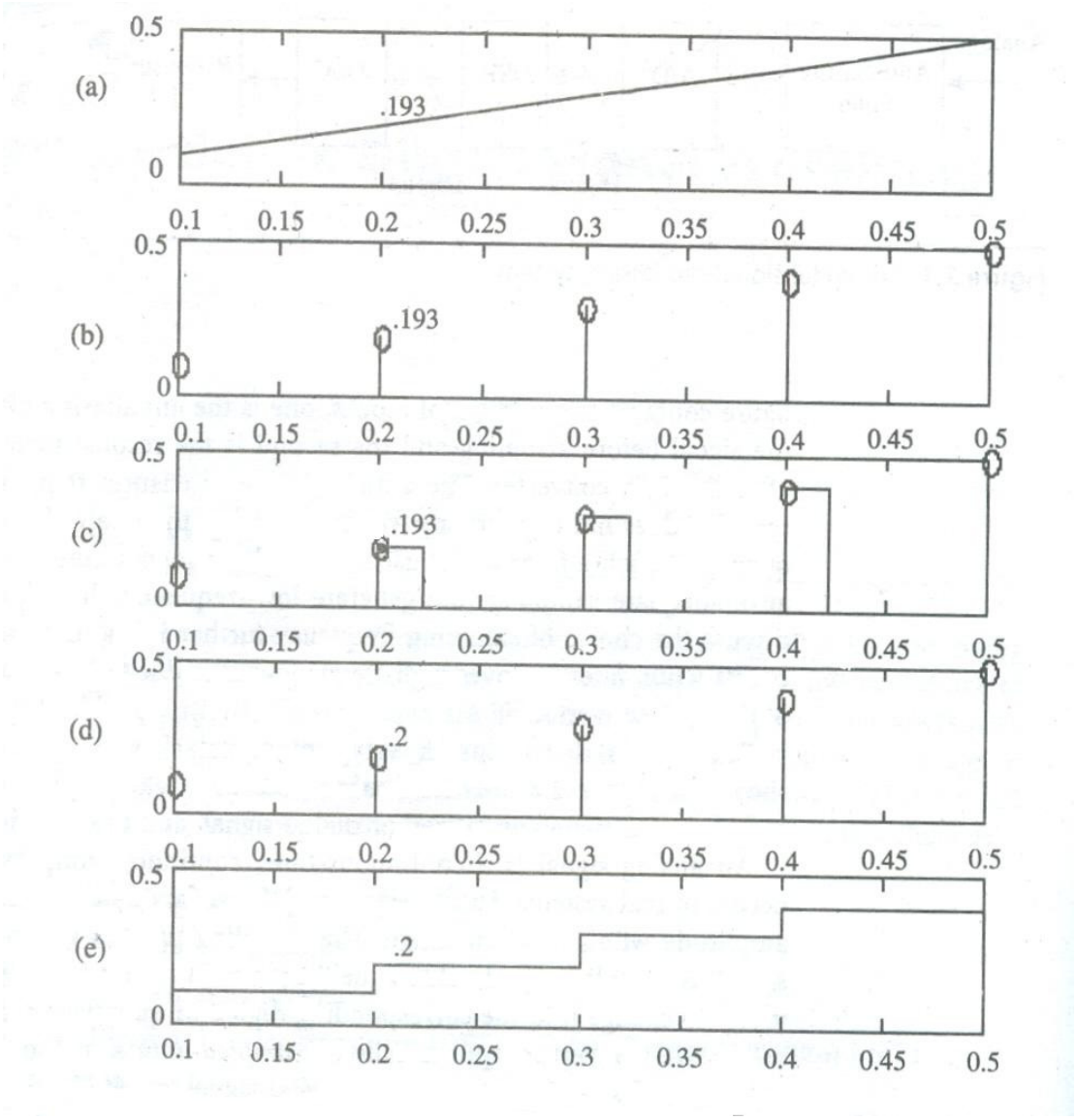

Fig 1.3: (a) Continuous time signal (b) Sampled Signal (c) Sampled Data Signal (d) Quantized Signal (e) DAC Output

1.4 The Sampling Process

ADC process involves sampling the signal and then quantizing the same to a digital value. In order to avoid Aliasing effect, the signal has to be sampled at a rate atleast equal to the Nyquist rate. The condition for Nyquist Criterion is as given below,

$$fs = 1/T \geq 2\ fm$$

where fs is the sampling frequency

fm is the maximum frequency component in the message signal

If the sampling of the signal is carried out with a rate less than the Nyquist rate, the higher frequency components of the signal cannot be reconstructed properly. The plots of the reconstructed outputs for various conditions are as shown in figure 1.4.

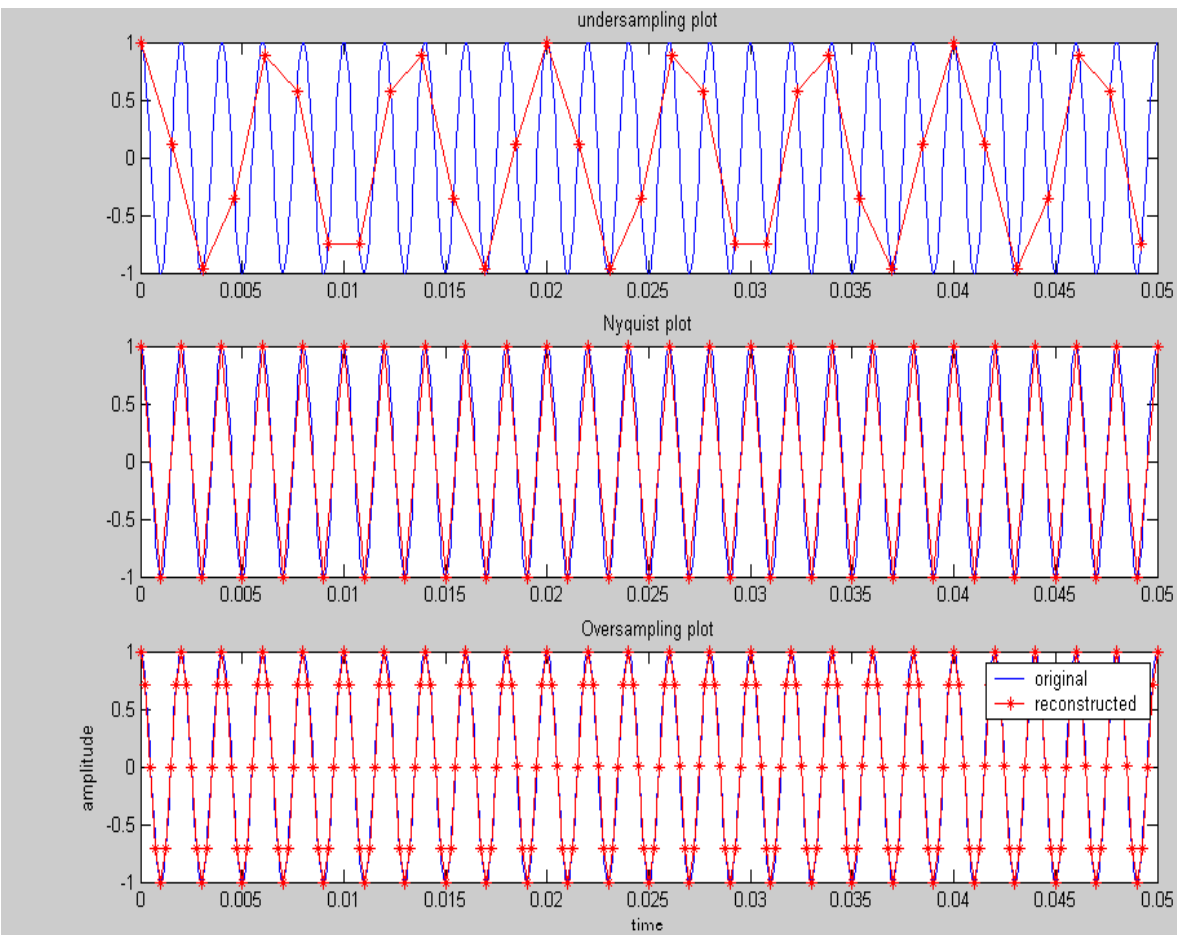

Fig 1.4 Verification of Sampling Theorem

1.5 Discrete Time Sequences

Consider an analog signal x(t) given by,

$x(t)= A \cos (2\pi ft)$

If this signal is sampled at a Sampling Interval T, in the above equation replacing t by nT we get,

$x (nT) = A \cos (2\pi fnT)$ where n= 0,1, 2,..etc

For simplicity denote x (nT) as x (n)

$\therefore x (n) = A \cos (2\pi fnT)$ where n= 0,1, 2,..etc

We have fs=1/T also $\theta = 2\pi fnT$

$\therefore x (n) = A \cos (2\pi fnT)= A \cos (2\pi fn/fs) = A \cos \theta n$

The quantity θ is called as digital frequency.

$\theta = 2\pi fT = 2\pi f/fs$ radians

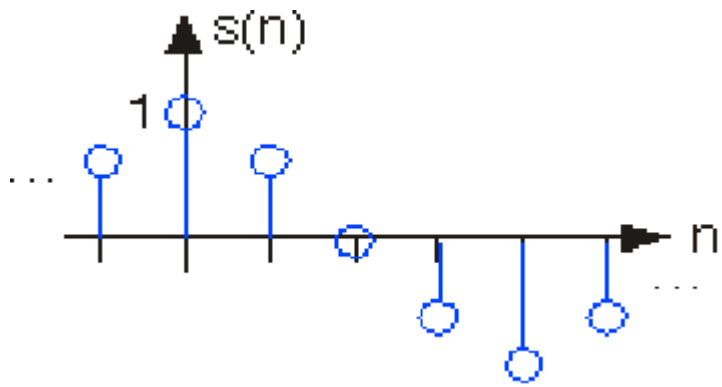

Fig 1.5 A Cosine Waveform

A sequence that repeats itself after every period N is called a periodic sequence. Consider a periodic sequence x (n) with period N

x (n)=x (n+N) n=……..,-1,0,1,2,……..

Frequency response gives the frequency domain equivalent of a discrete time sequence. It is denoted as $X (e^{j\theta})$

$X(e^{j\theta})=\Sigma x(n) e^{-jn\theta}$

Frequency response of a discrete sequence involves both magnitude response and phase response.

1.6 Discrete Fourier Transform and Fast Fourier Transform

1.6.1 DFT Pair

DFT is used to transform a time domain sequence x (n) to a frequency domain sequence X (K).

The equations that relate the time domain sequence x (n) and the corresponding frequency domain sequence X (K) are called DFT Pair and is given by,

$DFT(FFT)$:

$$X(k)=\sum_{n=0}^{N-1} x(n)\cdot e^{-j\left(\frac{2\pi}{N}\right)nk} \quad (k=0,1,\ldots,N-1)$$

$IDFT(IFFT)$:

$$x(n)=\frac{1}{N}\sum_{k=0}^{N-1} X(k)\cdot e^{j\left(\frac{2\pi}{N}\right)nk} \quad (n=0,1,\ldots,N-1)$$

1.6.2 The Relationship between DFT and Frequency Response

We have

$X(e^{j\theta})=\Sigma x(n)\, e^{-jn\theta}$

Also

$X(K)=\Sigma x(n)\, e^{-j2\pi nk/N}$

$\therefore X(K)= X(e^{j\theta})$ at $\theta = 2\pi k/N$

From the above expression it is clear that we can use DFT to find the Frequency response of a discrete signal.

Spacing between the elements of X(k) is given as

$\Delta f=f_s/N=1/NT=1/T_0$

Where T_0 is the signal record length

It is clear from the expression of Δf that, in order to minimize the spacing between the samples N has to be a large value.

Although DFT is an efficient technique of obtaining the frequency response of a sequence, it requires more number of complex operations like additions and multiplications. Thus many improvements over DFT were proposed. One such technique is to use the periodicity property of the twiddle factor $e^{-j2\pi/N}$. Those algorithms were

called as Fast Fourier Transform Algorithms. The following table depicts the complexity involved in the computation using DFT algorithms.

Table 1.1 Complexity in DFT algorithm

Operations	**Number of Computations**
Complex Multiplications	N^2
Complex Additions	N (N-1)
Real Multiplications	$4N^2$
Real Additions	2N (2N-1)
Trigonometric Functions	$2N^2$

FFT algorithms are classified into two categories viz

1. Decimation in Time FFT
2. Decimation in Frequency FFT

In decimation in time FFT the sequence is divided in time domain successively till we reach the sequences of length 2. Whereas in Decimation in Frequency FFT, the sequence X(K) is divided successively. The complexity of computation will get reduced considerably in case of FFT algorithms.

1.7 Linear Time Invariant Systems

A system which satisfies superposition theorem is called as a linear system and a system that has same input output relation at all times is called a Time Invariant System. Systems, which satisfy both the properties, are called LTI systems.

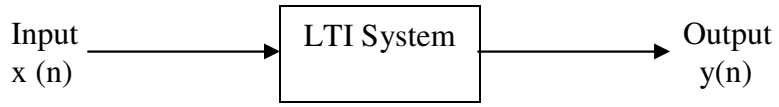

Fig 1.6 An LTI System

LTI systems are characterized by its impulse response or unit sample response in time domain whereas it is characterized by the system function in frequency domain.

1.7.1 Convolution

Convolution is the operation that related the input output of an LTI system, to its unit sample response. The output of the system y (n) for the input x (n) and the impulse response of the system being h (n) is given as

$y(n) = x(n) * h(n) = \Sigma\, x(k)\, h(n-k)$

x(n) is the input of the system

h(n) is the impulse response of the system

y(n) is the output of the system

1.7.2 Z Transformation

Z Transformations are used to find the frequency response of the system. The Z Transform for a discrete sequence x (n) is given by,

$X(Z) = \Sigma\ x(n)\, Z^{-n}$

1.7.3 The System Function

An LTI system is characterized by its System function or the transfer function. The system function of a system is the ratio of the Z transformation of its output to that of its input. It is denoted as H (Z) and is given by

$H(Z) = Y(Z) / X(Z)$

The magnitude and phase of the transfer function H (Z) gives the frequency response of the system. From the transfer function we can also get the poles and zeros of the system by solving its numerator and denominator respectively.

1.8 Digital Filters

Filters are used to remove the unwanted components in the sequence. They are characterized by the impulse response h (n). The general difference equation for an Nth order filter is given by,

$y(n) = \Sigma\, a_k y(n-k) + \Sigma\, b_k x(n-k)$

A typical digital filter structure is as shown in figure 1.7.

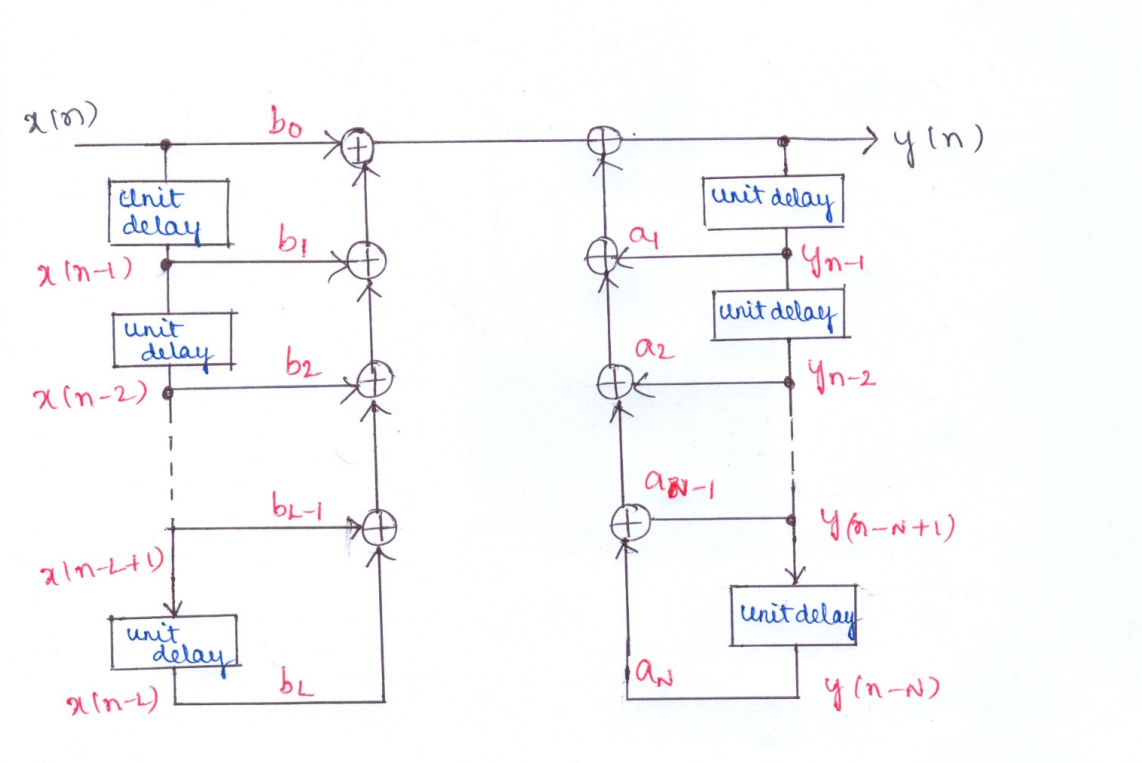

Fig 1.7 Structure of a Digital Filter

Values of the filter coefficients vary with respect to the type of the filter. Design of a digital filter involves determining the filter coefficients. Based on the length of the impulse response, digital filters are classified into two categories viz Finite Impulse Response (FIR) Filters and Infinite Impulse Response (IIR) Filters.

1.8.1 FIR Filters

FIR filters have impulse responses of finite lengths. In FIR filters the present output depends only on the past and present values of the input sequence but not on the previous output sequences. Thus they are non recursive hence they are inherently stable.

FIR filters possess linear phase response. Hence they are very much applicable for the applications requiring linear phase response.

The difference equation of an FIR filter is represented as

$y(n) = \Sigma\, b_k x(n-k)$

The frequency response of an FIR filter is given as

$H(e^{j\theta}) = \Sigma b_k\, e^{-jk\theta}$

Also

$H(Z)=\Sigma b_k Z^{-k}$

The major drawback of FIR filters is, they require more number of filter coefficients to realize a desired response as compared to IIR filters. Thus the computational time required will also be more.

1. Find the magnitude and phase response of an FIR filter represented by the difference equation

$y(n)= 0.5\, x(n) + 0.5\, x(n-1)$

As

$Y(n)= 0.5\, x(n) + 0.5\, x(n-1)$

$h(n)= 0.5\, \delta(n) + 0.5\, \delta(n-1) = [0.5 \quad 0.5]$

$H(Z)= 0.5+0.5Z^{-1}$

$H(e^{j\theta})= 0.5+0.5\, e^{-j\theta}$

$= 0.5+0.5 \cos\theta - j0.5 \sin\theta$

$=0.5(1+\cos\theta) - j0.5 \sin\theta$

$= [0.5*2*\cos^2(\theta/2)]-j[0.5*2*\sin(\theta/2)*\cos(\theta/2)]$

$= \cos^2(\theta/2) - j[\sin(\theta/2)*\cos(\theta/2)]$

$\text{mag}(H(e^{j\theta})) = \text{sqrt}(\cos^4(\theta/2) + \sin^2(\theta/2)\cos^2(\theta/2))$

$= \text{sqrt}[\cos^2(\theta/2)(\cos^2(\theta/2)+\sin^2(\theta/2))]$

$= \cos(\theta/2)$

Similarly,

$\text{Phase}(H(e^{j\theta})) = \tan^{-1}[-(\sin(\theta/2)\cos(\theta/2))/\cos^2(\theta/2)]$

$= \tan^{-1}[-\tan(\theta/2)]$

$= -(\theta/2)$

The magnitude and phase response curves of the designed FIR filter is as shown in figure 1.8.

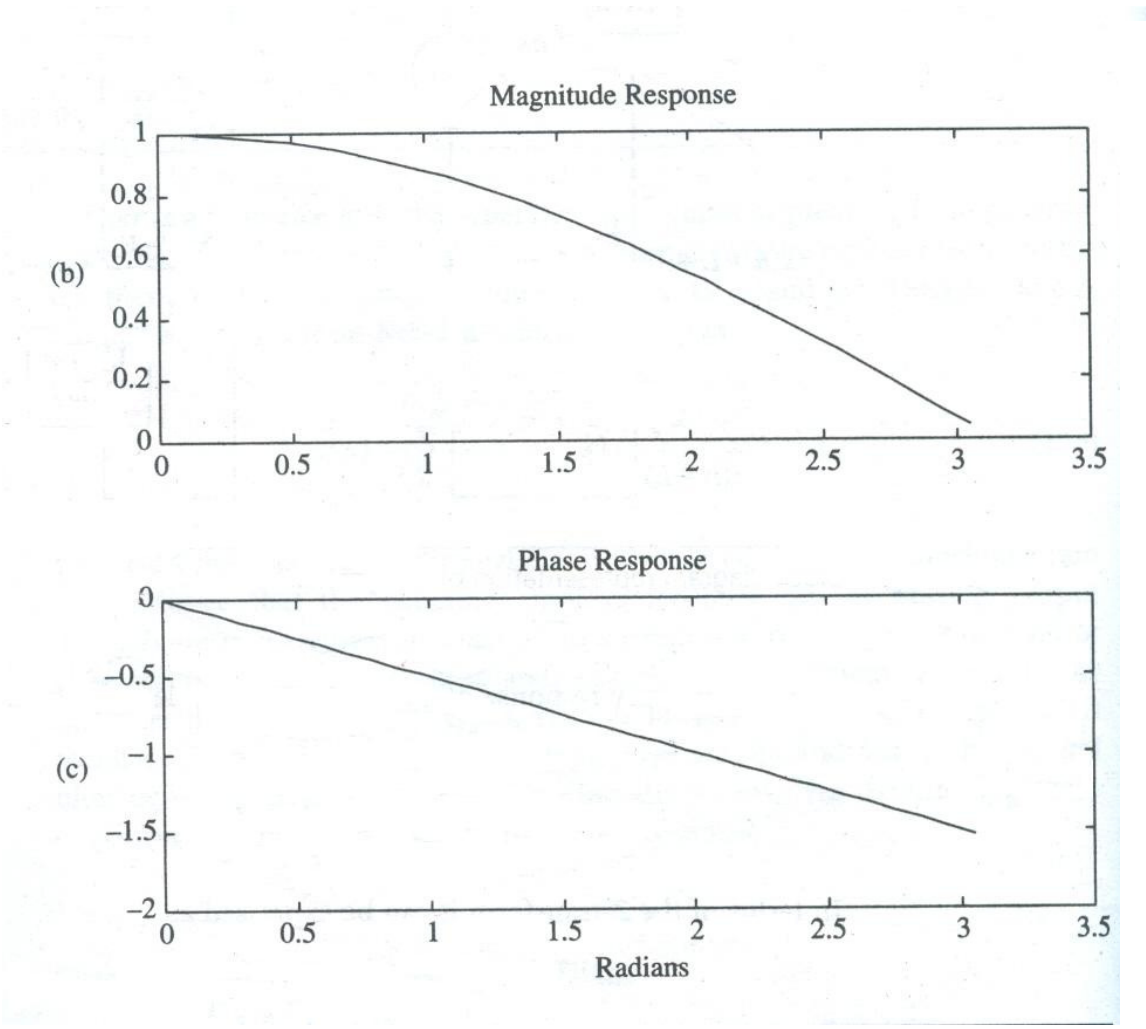

Fig 1.8 Frequency Response

1.8.2 IIR Filters

Unlike FIR filters, IIR filters have infinite number of impulse response samples. They are recursive filters as the output depends not only on the past and present inputs but also on the past outputs. They generally do not have linear phase characteristics. Typical system function of such filters is given by,

$$H(Z) = (b_0+b_1z^{-1}+b_2z^{-2}+\ldots\ldots\ldots\ldots b_Lz^{-L}) / (1-a_1z^{-1}-a_2z^{-2}-\ldots\ldots\ldots a_Nz^{-N})$$

Stability of IIR filters depends on the number and the values of the filter coefficients.

The major advantage of IIR filters over FIR is that, they require lesser coefficients compared to FIR filters for the same desired response, thus requiring less computation time.

2 Obtain the transfer function of the IIR filter whose difference equation is given by

y (n)= 0.9y (n-1)+0.1x (n)

y (n)= 0.9y (n-1)+0.1x (n)

Taking Z transformation both sides

$Y(Z) = 0.9\, Z^{-1}\, Y(Z) + 0.1\, X(Z)$

$Y(Z)\,[\,1 - 0.9\, Z^{-1}] = 0.1\, X(Z)$

The transfer function of the system is given by the expression,

$H(Z) = Y(Z)/X(Z)$

$= 0.1/\,[\,1 - 0.9\, Z^{-1}]$

Realization of the IIR filter with the above difference equation is as shown in figure 1.9.

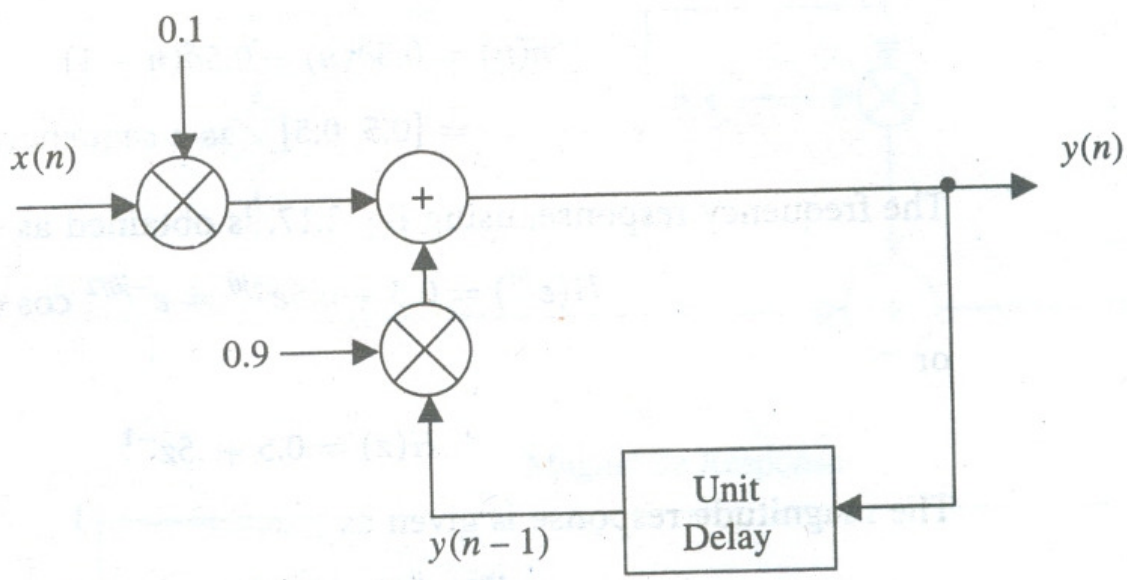

Fig 1.9 IIR Filter Structure

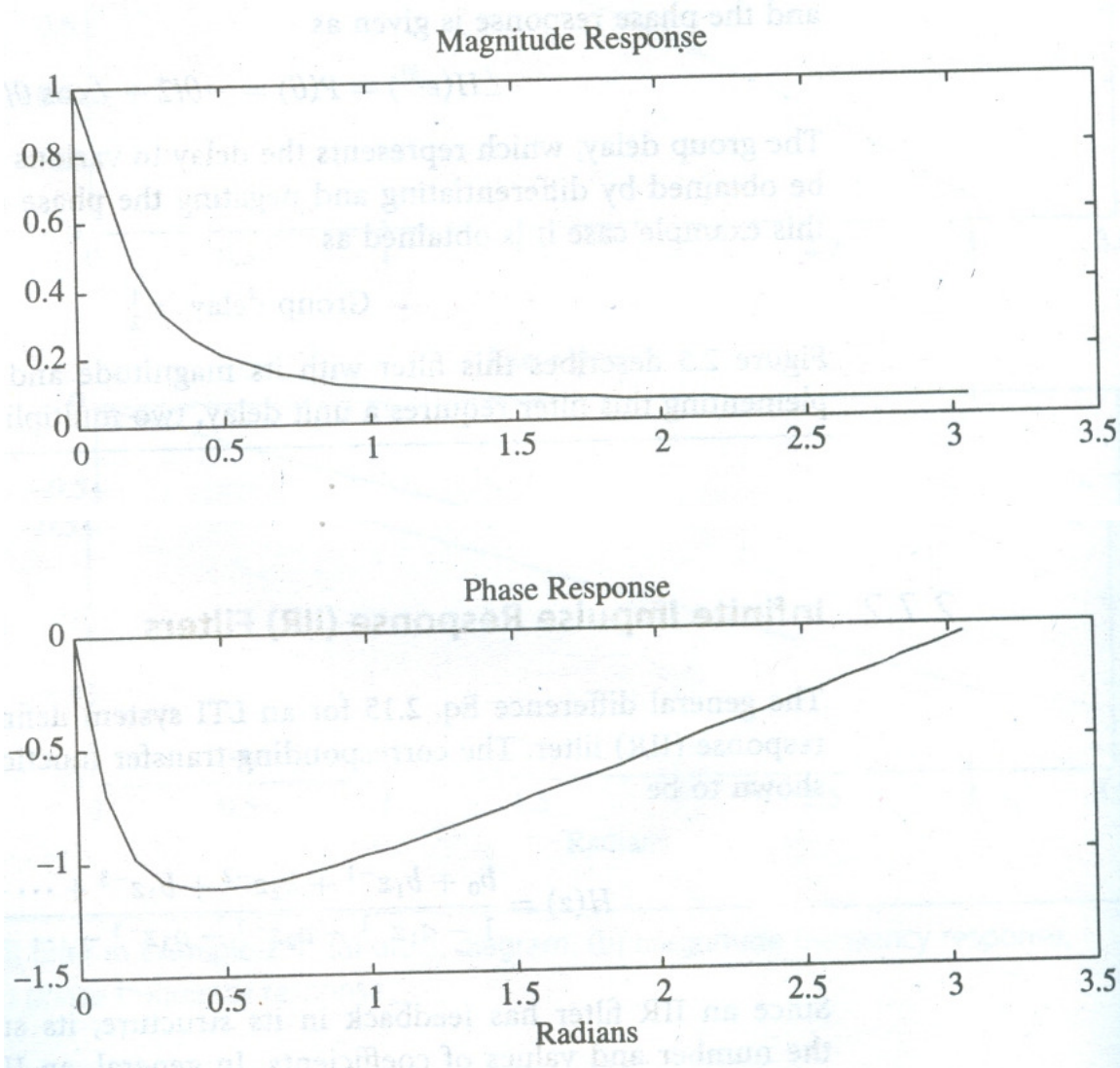

Fig 1.10 Frequency Response of the IIR Filter

1.8.3 FIR Filter Design

Frequency response of an FIR filter is given by the following expression,

$H(e^{j\theta}) = \Sigma b_k e^{-jk\theta}$

Design procedure of an FIR filter involves the determination of the filter coefficients b_k.

$b_k = (1/2\pi) \int H(e^{j\theta}) e^{-jk\theta} d\theta$

1.8.4 IIR Filter Design

IIR filters can be designed using two methods viz using windows and direct method.

In this approach, a digital filter can be designed based on its equivalent analog filter. An analog filter is designed first for the equivalent analog specifications for the given digital specifications. Then using appropriate frequency transformations, a digital filter can be obtained.

The filter specifications consist of passband and stopband ripples in dB and Passband and Stopband frequencies in rad/sec.

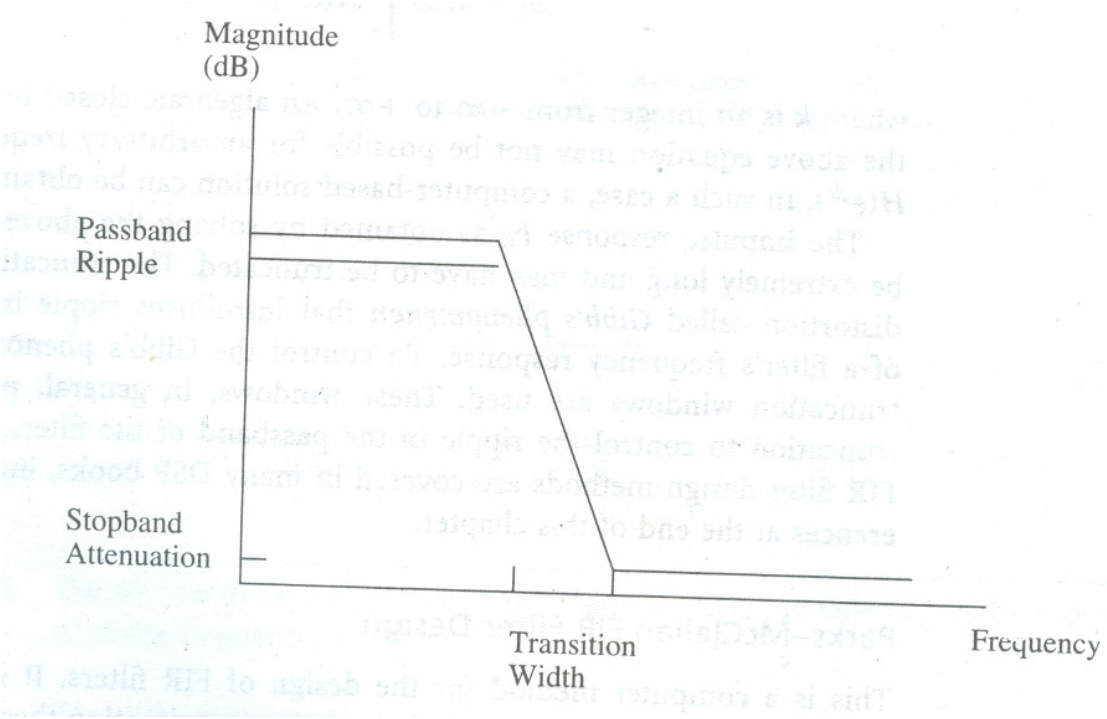

Fig 1.11 Lowpass Filter Specifications

Direct IIR filter design methods are based on least squares fit to a desired frequency response. These methods allow arbitrary frequency response specifications.

1.9 Decimation and Interpolation

Decimation and Interpolation are two techniques used to alter the sampling rate of a sequence. Decimation involves decreasing the sampling rate without violating the sampling theorem whereas interpolation increases the sampling rate of a sequence appropriately by considering its neighboring samples.

1.9.1 Decimation

Decimation is a process of dropping the samples without violating sampling theorem. The factor by which the signal is decimated is called as decimation factor and it is denoted by M. It is given by,

$$y(m)=w(mM)= \Sigma\, b_k\, x(mM-k)$$

where $w(n)= \Sigma\, b_k\, x(n-k)$

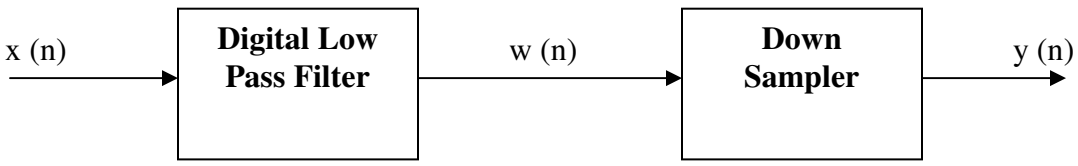

Fig 1.12 Decimation Process

3. Let x(n)=[3 2 2 4 1 0 –3 –2 –1 0 2 3] be decimated with a factor of 2. Let the filtered sequence be w(n)=[2.1 2 3.9 1.5 0.1 –2.9 –2 –1.1 0.1 1.9 2.9]. Obtain the decimated sequence y(m)

Sequence y(m) can be obtained by dropping every alternative sample of w (n).

y (m) = [2 1.5 -2.9 -1.1 1.9]

1.9.2 Interpolation

Interpolation is a process of increasing the sampling rate by inserting new samples in between. The input output relation for the interpolation, where the sampling rate is increased by a factor L, is given as,

$$y(m)= \Sigma\, b_k\, w(m-k)$$

where $w(n)= x(m/L), \quad m=0,\pm L, \pm 2L\ldots\ldots$
$\qquad\qquad\qquad 0 \qquad\qquad \text{Otherwise}$

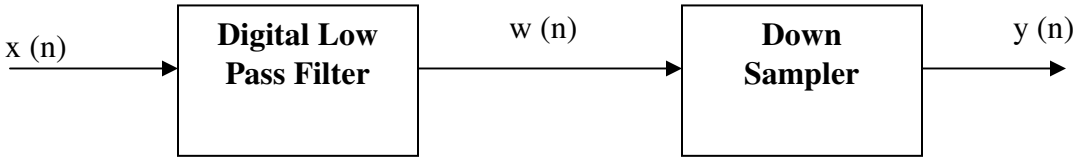

Fig 1.13 Interpolation Process

4. Let x(n)= [0 3 6 9 12] be interpolated with L=3. If the filter coefficients of the filters are b_k=[1/3 2/3 1 2/3 1/3], obtain the interpolated sequence

After inserting zeros,

w (m) = [0 0 0 3 0 0 6 0 0 9 0 0 12]

b_k=[1/3 2/3 1 2/3 1/3]

We have,

$y(m)= \Sigma\, b_k\, w(m-k) = b_{-2}\, w(m+2)+ b_{-1}\, w(m+1)+ b_0\, w(m)+ b_1\, w(m-1)+ b_2\, w(m-2)$

Substituting the values of m, we get

$y(0)= b_{-2}\, w(2)+ b_{-1}\, w(1)+ b_0\, w(0)+ b_1\, w(-1)+ b_2\, w(-2)= 0$

$y(1)= b_{-2}\, w(3)+ b_{-1}\, w(2)+ b_0\, w(1)+ b_1\, w(0)+ b_2\, w(-1)=1$

$y(2)= b_{-2}\, w(4)+ b_{-1}\, w(3)+ b_0\, w(2)+ b_1\, w(1)+ b_2\, w(0)=2$

Similarly we get the remaining samples as,

y (n) = [0 1 2 3 4 5 6 7 8 9 10 11 12]