BASIC CONCEPTS
AND
CONVENTIONAL METHODS
OF
STUCTURAL ANALYSIS
(LECTURE NOTES)

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ACKNOWLEDGEMENTS

I realize the profound truth that He who created all things inert as well as live is GOD. But many things in this world are created through knowledge of some living beings and these living beings are groomed by their teachers and through their own effort of self study and practice.

Those who are gifted by God are exceptions and those who are gifted by their teachers are lucky. I am thankful to God as He has been grateful to me much more than I deserve and to my all teachers from school level to University heights as I am product of their efforts and guidance and hence this work.

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The present work is the result of inspirations of my teachers and also any future work that I may accomplish.

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The inspiration and knowledge gained from my teachers and literature have motivated me to condense the conventional methods of structural analysis in this work so that a reader can get the quick insight into the essence of the subject of Structural Mechanics.

In the end I wish to acknowledge specifically the efforts of Mrs. Jyoti Bhatia, who faithfully, sincerely and conscientiously typed the manuscript and perfected it as far as possible by reviewing and removing the errors.

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LECTURE NOTES ON STRUCTURAL ANALYSIS

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BASIC CONCEPTS AND CONVENTIONAL METHODS OF STRUCTURAL ANALYSIS

1 INTRODUCTION

The structural analysis is a mathematical algorithm process by which the response of a structure to specified loads and actions is determined. This response is measured by determining the internal forces or stress resultants and displacements or deformations throughout the structure.

The structural analysis is based on engineering mechanics, mechanics of solids, laboratory research, model and prototype testing, experience and engineering judgment. The basic methods of structural analysis are flexibility and stiffness methods. The flexibility method is also called force method and compatibility method. The stiffness method is also called displacement method and equilibrium method. These methods are applicable to all type of structures; however, here only skeletal systems or framed structures will be discussed. The examples of such structures are beams, arches, cables, plane trusses, space trusses, plane frames, plane grids and space frames.

The skeletal structure is one whose members can be represented by lines possessing certain rigidity properties. These one dimensional members are also called bar members because their cross sectional dimensions are small in comparison to their lengths. The skeletal structures may be determinate or indeterminate.

2 CLASSIFICATIONS OF SKELETAL OR FRAMED STRUCTURES

They are classified as under.

1) **Direct force structures** such as pin jointed plane frames and ball jointed space frames which are loaded and supported at the nodes. Only one internal force or stress resultant that is axial force may arise. Loads can be applied directly on the members also but they are replaced by equivalent nodal loads. In the loaded members additional internal forces such as bending moments, axial forces and shears are produced.

The plane truss is formed by taking basic triangle comprising of three members and three pin joints and then adding two members and a pin node as shown in Figure 2.1. Sign

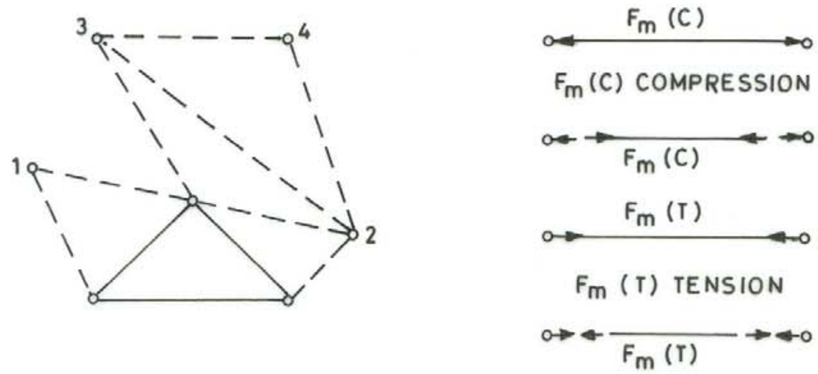


FIG. 2·1. FORMATION OF PLANE TRIANGULATED TRUSS AND SIGN CONVENTION FOR INTERNAL MEMBER FORCES.

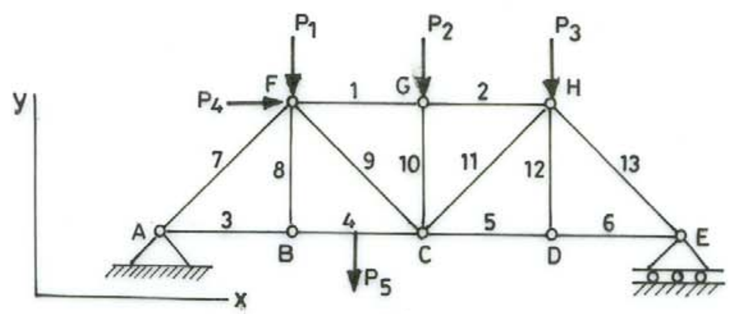


FIG. 2·2. PIN JOINTED PLANE TRUSS SUBJECTED TO MEMBER AND NODAL LOADS.

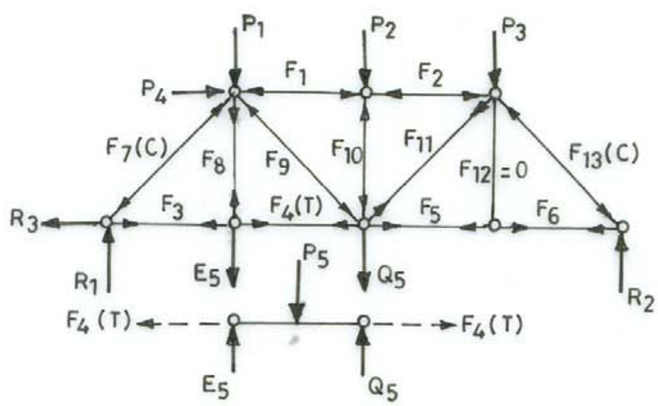


FIG. 2·3. EQUIVALENT NODAL LOADS AND FREE BODY OF LOADED MEMBER AS BEAM.

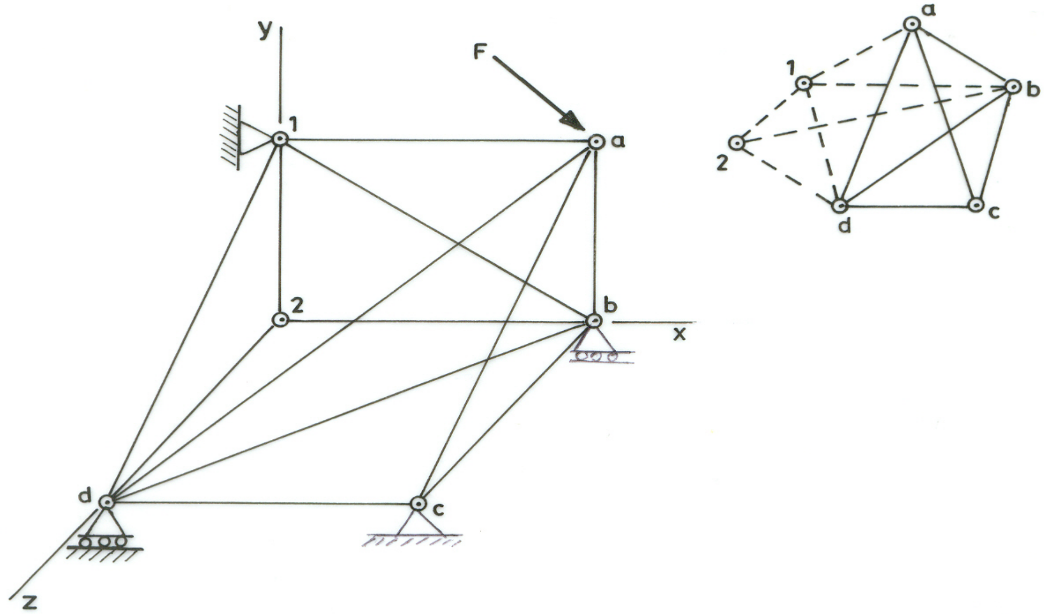


FIG. 2·4. BALL AND SOCKET (UNIVERSAL) JOINTED SPACE TRUSS WITH NODAL LOADING.

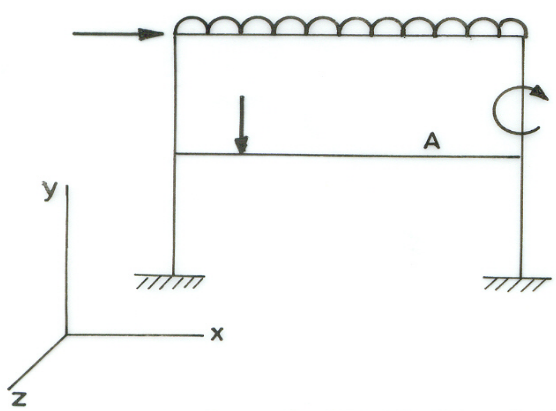


FIG. 2·5. PLANE FRAME SUBJECTED TO IN PLANE EXTERNAL LOADING.

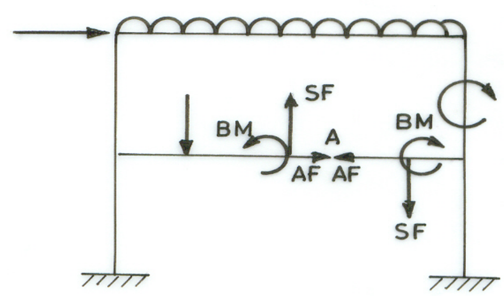


FIG. 2·6. INTERNAL FORCES DEVELOPED AT SECTION A DUE TO APPLIED LOADING.

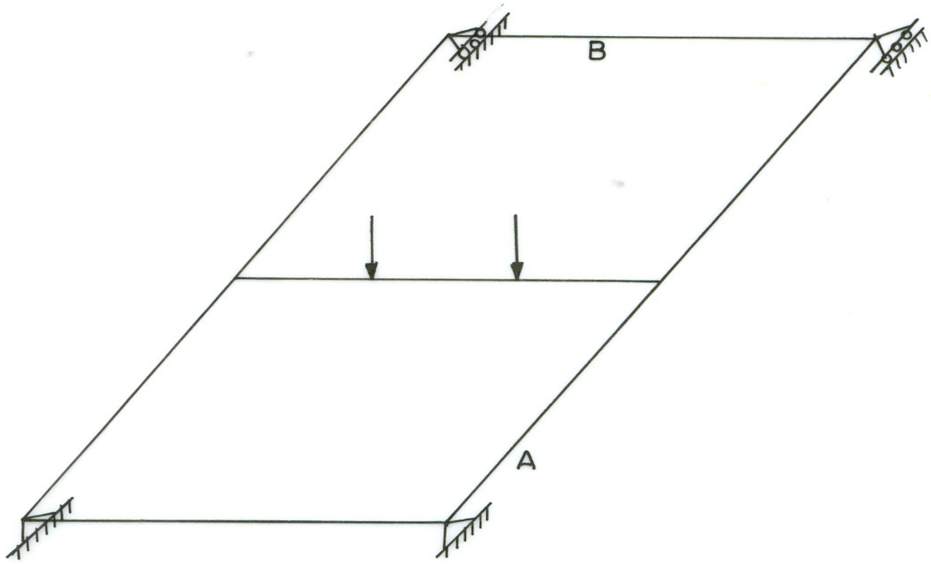


FIG. 2·7. PLANE GRID SUBJECTED TO NORMAL TO PLANE LOADING

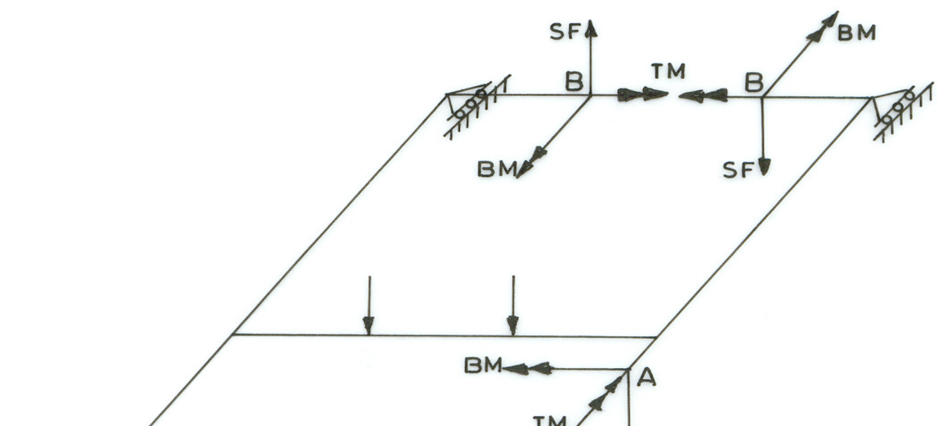


FIG. 2·8. INTERNAL STRESS RESLTANTS DEVELOPED IN MEMBERS AT A AND B DUE TO APPLIED LOADING.

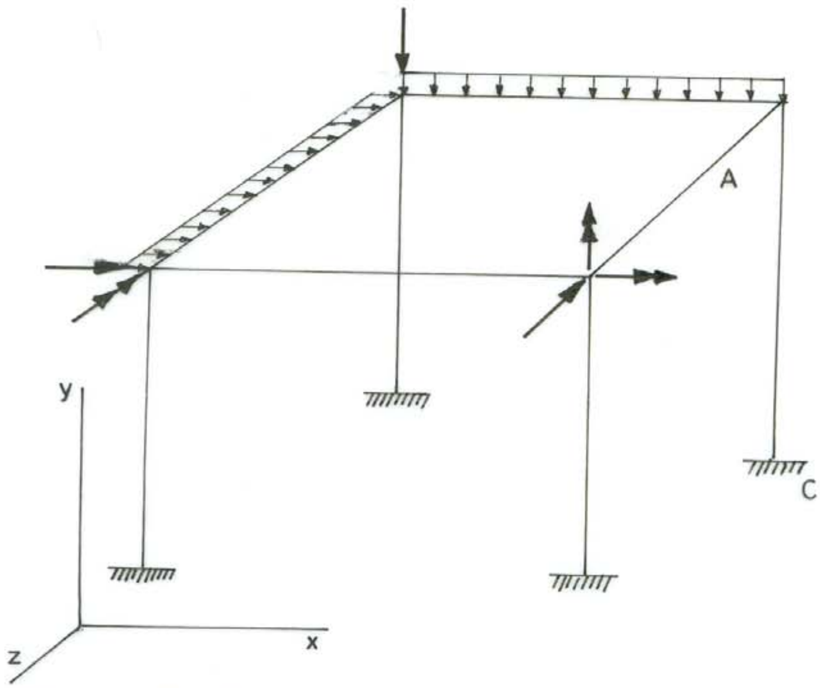


FIG. 2·9. SPACE FRAME SUBJECTED TO GENERAL EXTERNAL LOADING.

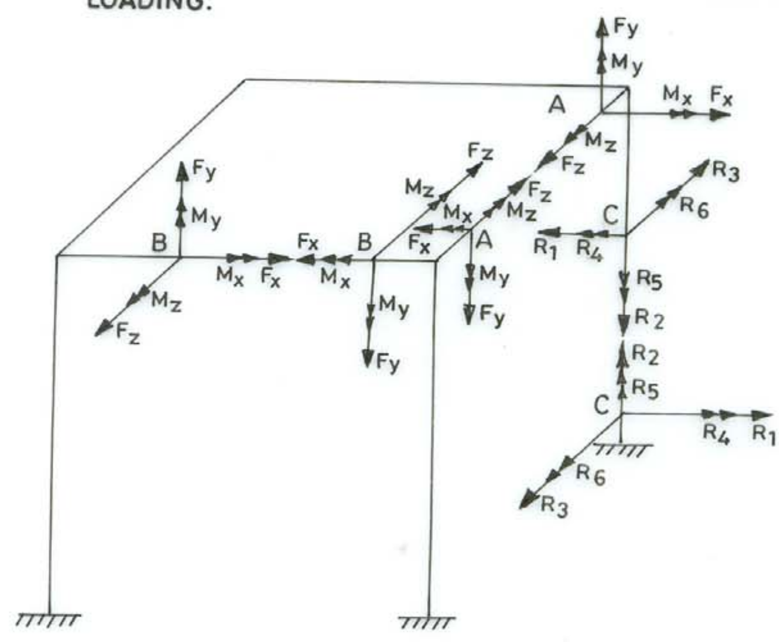


FIG. 2·10. INTERNAL FORCES GENERATED AT A AND B AND REACTIONS DEVELOPED AT C DUE TO EXTERNAL LOADING.

Convention for internal axial force is also shown. In Fig.2.2, a plane triangulated truss with joint and member loading is shown. The replacement of member loading by joint loading is shown in Fig.2.3. Internal forces developed in members are also shown.

The space truss is formed by taking basic prism comprising of six members and four ball joints and then adding three members and a node as shown in Fig.2.4.

2) **Plane frames** in which all the members and applied forces lie in same plane as shown in Fig.2.5. The joints between members are generally rigid. The stress resultants are axial force, bending moment and corresponding shear force as shown in Fig.2.6.

3) **Plane frames** in which all the members lay in the same plane and all the applied loads act normal to the plane of frame as shown in Fig.2.7. The internal stress resultants at a point of the structure are bending moment, corresponding shear force and torsion moment as shown in Fig.2.8.

4) **Space frames** where no limitations are imposed on the geometry or loading in which maximum of six stress resultants may occur at any point of structure namely three mutually perpendicular moments of which two are bending moments and one torsion moment and three mutually perpendicular forces of which two are shear forces and one axial force as shown in figures 2.9 and 2.10.

3 INTERNAL **LOADS DEVELOPED IN STRUCTURAL MEMBERS**

External forces including moments acting on a structure produce at any section along a structural member certain internal forces including moments which are called stress resultants because they are due to internal stresses developed in the material of member.

The maximum number of stress resultants that can occur at any section is six, the three Orthogonal moments and three orthogonal forces. These may also be described as the axial force F1 acting along x – axis of member, two bending moments F_5 and F_6 acting about the principal y and z axes respectively of the cross section of the member, two corresponding shear forces F_3 and F2 acting along the principal z and y axes respectively and lastly the torsion moment F_4 acting about x – axis of member. The stress resultants at any point of centroidal axis of member are shown in Fig. 3.1 and can be represented as follows.

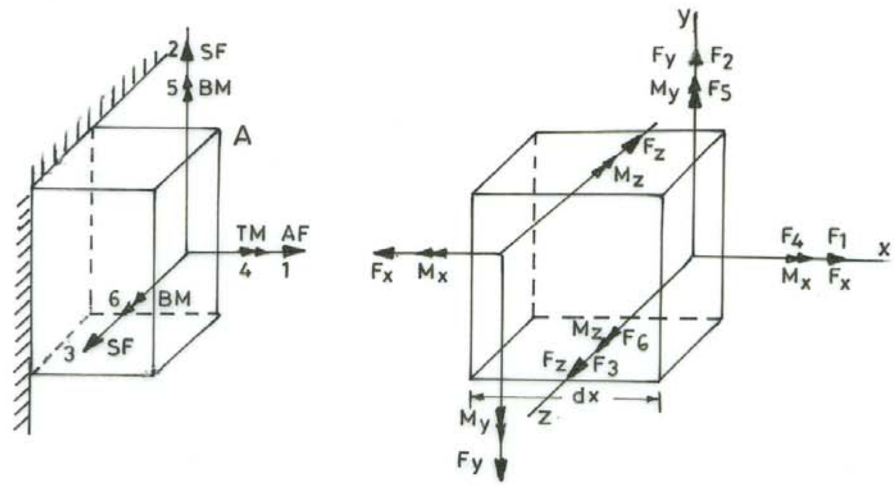


FIG. 3·1. SIX INTERNAL FORCES AT A SECTION OF MEMBER UNDER GENERAL LOADING.

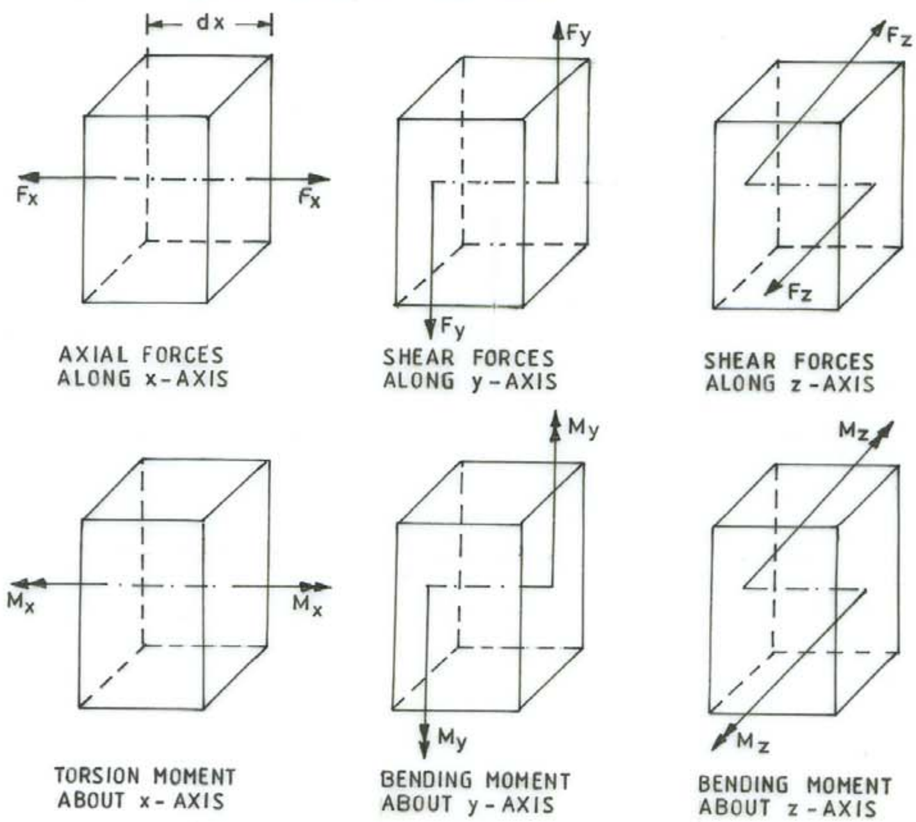


FIG. 3·2. VARIOUS BIACTIONS AT A SECTION ON AN ELEMENT

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} \quad \text{OR} \quad \begin{Bmatrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{Bmatrix}$$

Numbering system is convenient for matrix notation and use of electronic computer. Each of these actions consists essentially of a pair of opposed actions which causes deformation of an elemental length of a member. The pair of torsion moments cause twist of the element, pair of bending moments cause bending of the element in corresponding plane, the pair of axial loads cause axial deformation in longitudinal direction and the pair of shearing forces cause shearing strains in the corresponding planes. The pairs of biactions are shown in Fig.3.2.

Primary and secondary internal forces.

In many frames some of six internal actions contribute greatly to the elastic strain energy and hence to the distortion of elements while others contribute negligible amount. The material is assumed linearly elastic obeying Hooke's law. In direct force structures axial force is primary force, shears and bending moments are secondary. Axial force structures do not have torsional resistance. The rigid jointed plane grid under normal loading has bending moments and torsion moments as primary actions and axial forces and shears are treated secondary.

In case of plane frame subjected to in plane loading only bending moment is primary action, axial force and shear force are secondary. In curved members bending moment, torsion and thrust (axial force) are primary while shear is secondary. In these particular cases many a times secondary effects are not considered as it is unnecessary to complicate the analysis by adopting general method.

4 TYPES OF STRUCTURAL LOADS

For the analysis of structures various loads to be considered are: dead load, live load, snow load, rain load, wind load, impact load, vibration load, water current, centrifugal force, longitudinal forces, lateral forces, buoyancy force, earth or soil pressure, hydrostatic pressure, earthquake forces, thermal forces, erection forces, straining forces etc. How to consider these loads is described in loading standards of various structures. These loads are idealized for the purpose of analysis as follows.

Concentrated loads: They are applied over a small area and are idealized as point loads.

Line loads: They are distributed along narrow strip of structure such as the wall load or the self weight of member. Neglecting width, load is considered as line load acting along axis of member.

Surface loads: They are distributed over an area. Loads may be static or dynamic, stationary or moving. Mathematically we have point loads and concentrated moments. We have distributed forces and moments, we have straining and temperature variation forces.

5 DETERMINATE AND INDETERMINATE STRUCTURAL SYSTEMS

If skeletal structure is subjected to gradually increasing loads, without distorting the initial geometry of structure, that is, causing small displacements, the structure is said to be stable. Dynamic loads and buckling or instability of structural system are not considered here. If for the stable structure it is possible to find the internal forces in all the members constituting the structure and supporting reactions at all the supports provided from statical equations of equilibrium only, the structure is said to be determinate. If it is possible to determine all the support reactions from equations of equilibrium alone the structure is said to be externally determinate else externally indeterminate. If structure is externally determinate but it is not possible to determine all internal forces then structure is said to be internally indeterminate. Therefore a structural system may be:

(1) Externally indeterminate but internally determinate

(2) Externally determinate but internally indeterminate

(3) Externally and internally indeterminate

(4) Externally and internally determinate

These systems are shown in figures 5.1 to 5.4.

A system which is externally and internally determinate is said to be determinate system.

A system which is externally or internally or externally and internally indeterminate is said to be indeterminate system.

Let: v = Total number of unknown internal and support reactions

s = Total number of independent statical equations of equilibrium.

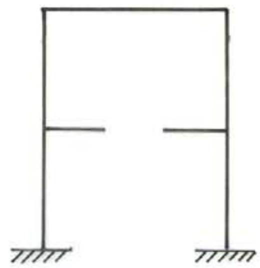


FIG. 5·1. EXTERNALLY INDETERMINATE BUT INTERNALLY DETERMINATE PLANE SYSTEM

FIG. 5·2. EXTERNALLY DETERMINATE BUT INTERNALLY INDETERMINATE PLANE SYSTEM

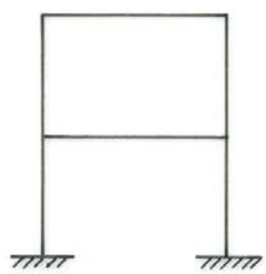


FIG. 5·3. EXTERNALLY AND INTERNALLY INDETERMINATE PLANE SYSTEM

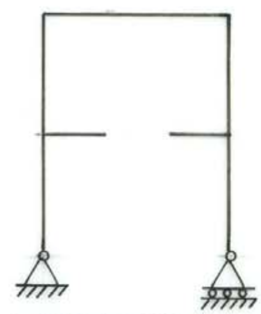


FIG. 5·4. EXTERNALLY AND INTERNALLY DETERMINATE PLANE SYSTEM

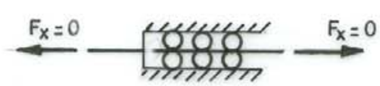


FIG. 5·5.

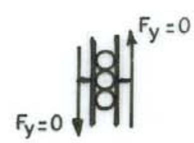


FIG. 5·6.

FIG. 5·7.

FIG. 5·8.

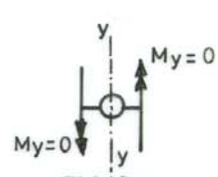


FIG. 5·9.

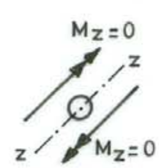


FIG. 5·10

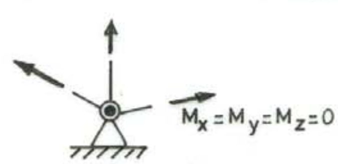


FIG. 5·11.

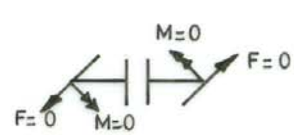


FIG. 5·12.

FIG. 5·5 – 5·12. RELEASES AT A SECTION FOR VARIOUS FORCES

Then if: $v = s$ the structure is determinate

$v > s$ the structure is indeterminate

$v < s$ the structure is unstable

Total indeterminacy of structure = Internal indeterminacy + External indeterminacy

Equations of equilibrium

Space frames arbitrarily loaded

$\sum F_x = 0$ $\sum M_x = 0$
$\sum F_y = 0$ $\sum M_y = 0$
$\sum F_z = 0$ $\sum M_z = 0$

For space frames number of equations of equilibrium is 6. Forces along three orthogonal axes should vanish and moments about three orthogonal axes should vanish.

Plane frames with in plane loading

$\sum F_x = 0$ $\sum F_y = 0$ $\sum M_z = 0$

There are three equations of equilibrium. Forces in x and y directions should vanish and moment about z axis should vanish.

Plane frames with normal to plane loading

There are three equations of equilibrium.

$\sum F_y = 0,$ $\sum M_x = 0,$ $\sum M_z = 0$

Sum of forces in y direction should be zero. Sum of moments about x and z axes be zero.

Release and constraint

A release is a discontinuity which renders a member incapable of transmitting a stress resultant across that section. There are six releases corresponding to the six stress resultants at a section as shown below by zero elements in the vectors. Various releases are shown in figures 5.5 to 5.12.

Release for Axial Force (AF) F_x: $\begin{Bmatrix} 0 \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{Bmatrix}$

Release for Shear Force (SF) F_y: $\begin{Bmatrix} F_x \\ 0 \\ F_z \\ M_x \\ M_y \\ M_z \end{Bmatrix}$

Release for Shear Force (SF) F_z: $\begin{Bmatrix} F_x \\ F_y \\ 0 \\ M_x \\ M_y \\ M_z \end{Bmatrix}$

Release for Torsion Moment (TM) M_x: $\begin{Bmatrix} F_x \\ F_y \\ F_z \\ 0 \\ M_y \\ M_z \end{Bmatrix}$

Release for Bending Moment (BM) M_y: $\begin{Bmatrix} F_x \\ F_y \\ F_z \\ M_x \\ 0 \\ M_z \end{Bmatrix}$

Release for Bending Moment (BM) M_z: $\begin{Bmatrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ 0 \end{Bmatrix}$

The release may be represented by zero elements of forces

Universal joint (Ball and socket joint) $F = \begin{Bmatrix} F_x \\ F_y \\ F_z \\ 0 \\ 0 \\ 0 \end{Bmatrix}$, Cut $F = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$

A release does not necessarily occur at a point, but may be continuous along whole length of member as in chain for BM. On the other hand a constraint is defined as that which prevents any relative degree of freedom between two adjacent nodes connected by a member or when a relative displacement of the nodes does not produce a stress resultant in the member.

6 INDETERMINACY **OF STRUCTURAL SYSTEM**

The indeterminacy of a structure is measured as statical ($\propto_s$) or kinematical ($\propto_k$) indeterminacy.

$$\propto_s = P\,(M - N + 1) - r = PR - r$$

$$\propto_k = P\,(N - 1) + r - c$$

$\propto_s + \propto_k = PM - c$

P = 6 for space frames subjected to general loading

P = 3 for plane frames subjected to in plane or normal to plane loading.

N = Number of nodes in structural system.

M = Number of members of completely stiff structure which includes foundation as singly connected system of members. In completely stiff structure there is no release present. In singly connected system of rigid foundation members there is only one route between any two points in which tracks are not retraced. The system is considered comprising of closed rings or loops.

R = Number of loops or rings in completely stiff structure.

r = Number of releases in the system.

c = Number of constraints in the system.

R = (M – N + 1)

For plane and space trusses $\propto_s$ reduces to:

$\propto_s = \overline{M} - (NDOF)\ \overline{N} + P$

$\overline{M}$ = Number of members in completely stiff truss.

P = 6 and 3 for space and plane truss respectively

$\overline{N}$ = Number of nodes in truss.

NDOF = Degrees of freedom at node which is 2 for plane truss and 3 for space truss.

For space truss $\propto_s = \overline{M} - 3\ \overline{N} + 6$

For plane truss $\propto_s = \overline{M} - 2\ \overline{N} + 3$

<u>Test for static indeterminacy of structural system</u>

If $\propto_s > 0$	Structure is statically indeterminate
If $\propto_s = 0$	Structure is statically determinate

and if $\propto_s < 0$ Structure is a mechanism.

It may be noted that structure may be mechanism even if $\propto_s > 0$ if the releases are present in such a way so as to cause collapse as mechanism. The situation of mechanism is unacceptable.

Statical Indeterminacy

It is difference of the unknown forces (internal forces plus external reactions) and the equations of equilibrium.

Kinematic Indeterminacy

It is the number of possible relative displacements of the nodes in the directions of stress resultants.

Computation of static and kinematic indeterminacies

It is possible to compute mentally the static and kinematic inderminacies of structures. Consider a portal frame system shown in Fig.6.1. It is space structure with five members and three clamps at foundation. There is one internal space hinge in member BC. Foundation is replaced with two stiff members to give entire system as shown in Fig.6.2. So we have completely stiff structure with seven members and forms two rings which are statically indeterminate to twelve degrees as shown in Fig.6.3. There are three releases in member BC because of ball and socket (universal) joint. Three moments are zero at this section. Therefore $\propto_s = 9$. There are three joints E, B and C which can move. Being space system degree of freedom per node is 6. There will be three rotations at universal joint. Therefore total dof is (3 x 6 + 3) or $\propto_k = 21$. Joints F, A and D can not have any displacement that is degree of freedom is zero at these nodes.

Using formula:

$\propto_s = P (M – N + 1) – r$

$\propto_k = P (N – 1) + r – c$

P = 6, M = 7, N = 6

c = 12 (Foundation members are rigid), r = 3

$\propto_s = P (M – N + 1) – r = 6 (7 – 6 + 1) – 3 = 9$

$\propto_k = P (N – 1) + r – c = 6 (6 – 1) + 3 – 12 = 21$

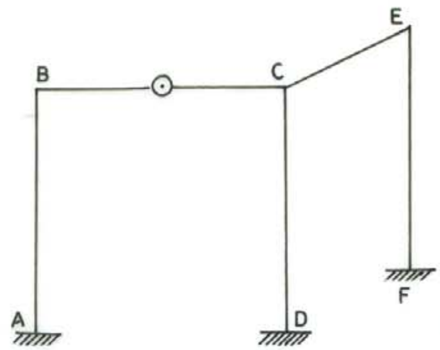


FIG. 6·1. GIVEN SPACE STRUCTURE

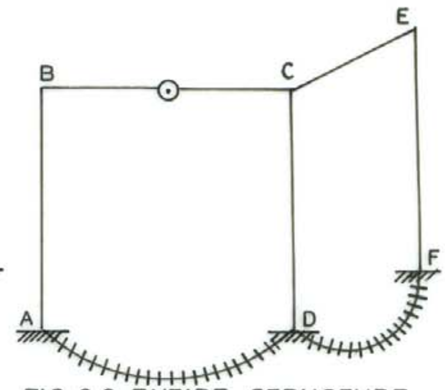


FIG. 6·2. ENTIRE STRUCTURE

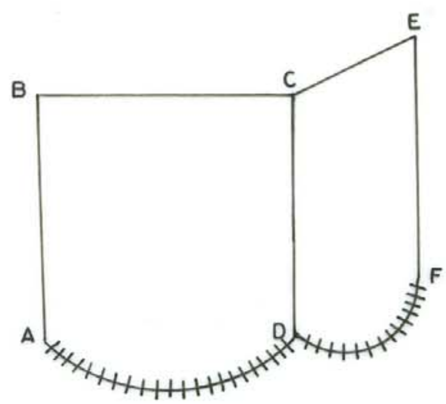


FIG.6·3. COMPLETELY STIFF STRUCTURE

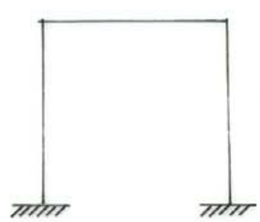


FIG. 6·4. GIVEN PLANE STRUCTURE

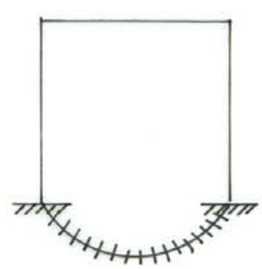
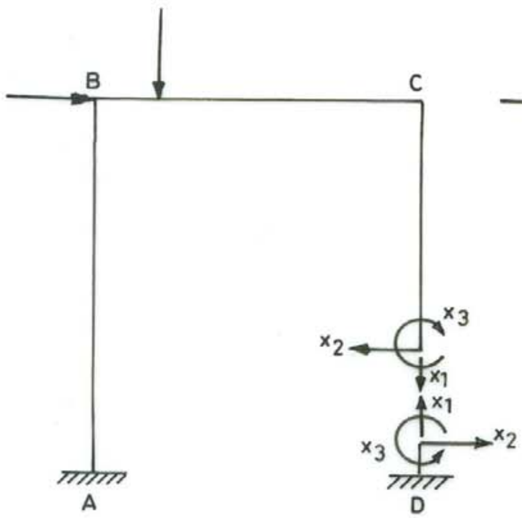


FIG. 6·5. ENTIRE STRUCTURE

FIG. 6·6. COMPLETELY STIFF STRUCTURE



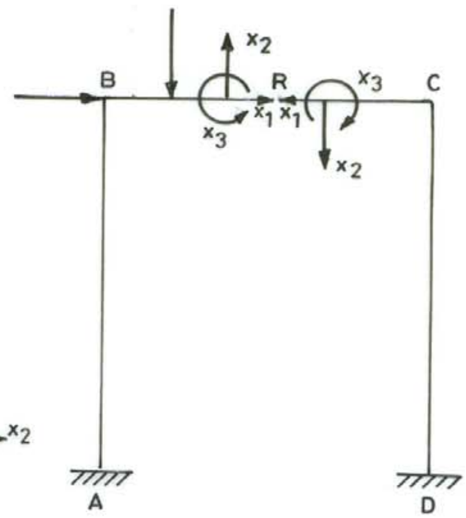
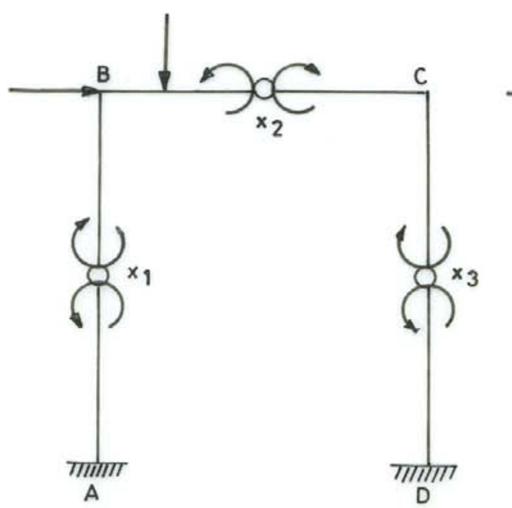


FIG. 6·7. CUTTING CLAMP/SUPPORT FIG. 6·8. CUT IN BEAM



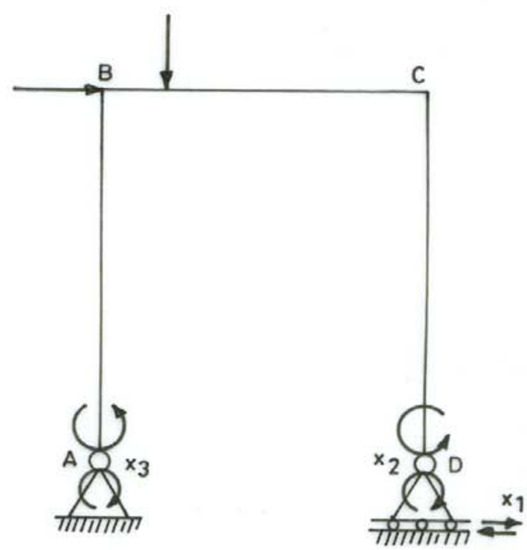


FIG. 6·9. INSERTING ONE HINGE IN EACH MEMBER

FIG. 6·10. REPLACING CLAMPS WITH ROLLER AND HINGE SUPPORTS

FIG. 6.7 – 6·10. DIFFERENT WAYS OF INSERTING RELEASES

$\propto_s + \propto_k = PM - c = 6 \times 7 - 12 = 30$

Static indeterminacy can also be determined by introducing releases in the system and rendering it a stable determinate system. The number of biactions corresponding to releases will represent static indeterminacy. Consider a portal frame fixed at support points as shown in Fig.6.4. The entire structure is shown in Fig.6.5 and completely stiff structure in Fig.6.6.

$\propto_s = P\,(M - N + 1) - r$

$\propto_k = P\,(N - 1) + r - c$

$P = 3,\ M = 4,\ N = 4,\ c = 3,\ r = 0$

$\propto_s = 3\,(4 - 4 + 1) - 0 = 3$

$\propto_k = 3\,(4 - 1) + 0 - 3 = 6$

$\propto_s + \propto_k = 3 + 6 = 9$

The structure can be made determinate by introducing in many ways three releases and thus destroying its capacity to transmit internal forces X1, X_2, X_3 at the locations of releases.

In figure 6.7. a cut is introduced just above clamp D that is clamp is removed. It becomes tree or cantilever structure with clamp at A. At this cut member was transmitting three forces X_1, X_2 and X_3 (Two forces and one moment). Therefore $\propto_s = 3$. This is external static indeterminacy.

In figure 6.8. a cut is introduced at point R on member BC. We have two trees or cantilevers with clamps at A and D. We have three internal unknown forces X_1, X_2, and X_3. Thus $\propto_s = 3$.

In figure 6.9. three hinges are introduced. We have determinate and stable system and there are three unknown moments X_1, X_2 and X_3. Thus $\propto_s = 3$.

In figure 6.10. one roller cum hinge and one hinge is introduced. We have one unknown force X_1 and two unknown moments X_2 and X_3 at these releases. Thus $\propto_s = 3$.

The static and kinematic indeterminacies of a few structures are computed in Table 1.

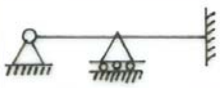
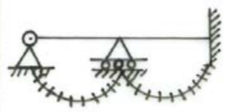
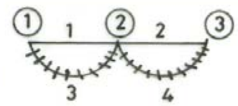
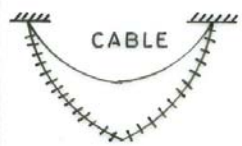
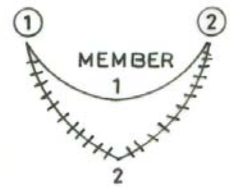
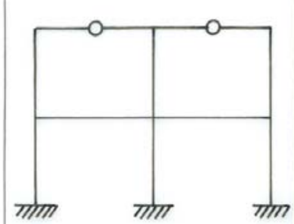
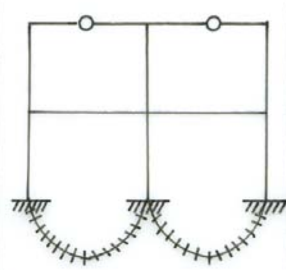
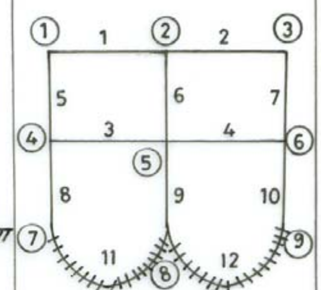
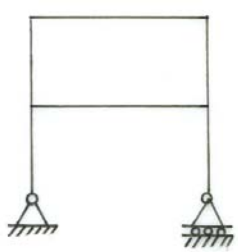
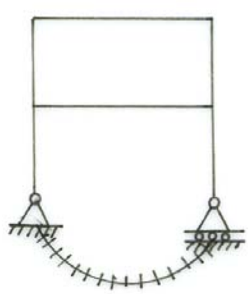
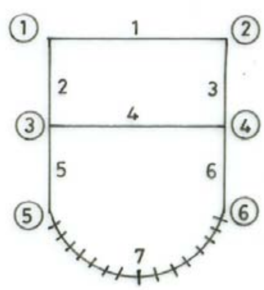
FIG. NO.	GIVEN SYSTEM	ENTIRE SYSTEM	COMPLETELY STIFF SYSTEM
6·11.	 CONTINUOUS BEAM		
6·12.	 CABLE		
6·13.	 ARCH		
6·14.	 PLANE FRAME		
6·15.	 PLANE FRAME		

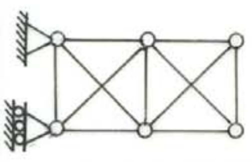
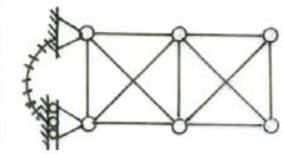
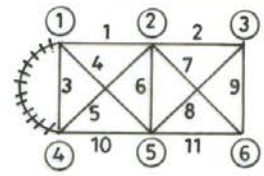
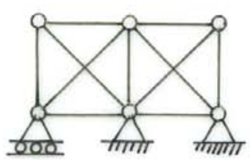
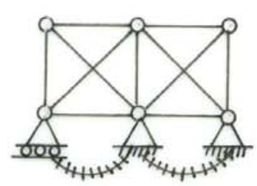
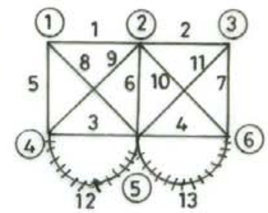
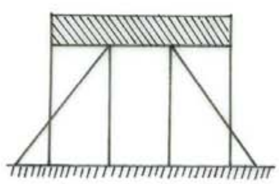
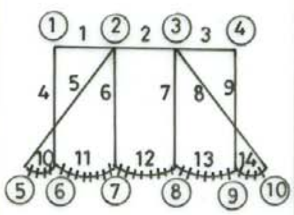
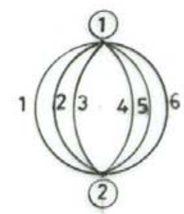
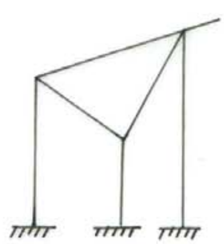
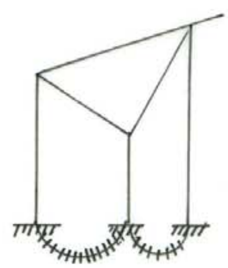
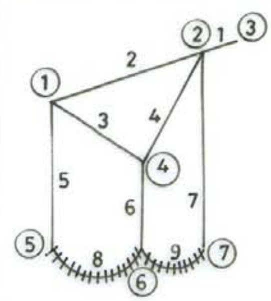
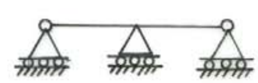
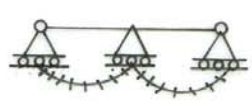
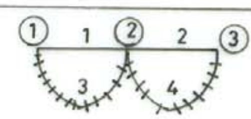
FIG. NO.	GIVEN SYSTEM	ENTIRE SYSTEM	COMPLETELY STIFF SYSTEM
6 · 16.	 CANTILEVER TRUSS		
6 · 17.	 SUPPORTED TRUSS		
6 · 18.	 PILE SYSTEM		
6 · 19.	 SPACE STRUCTURE		
6 · 20.	 CONTINUOS BEAM		

TABLE 1. Examples on static and kinematic indeterminacies.

Example No:	Figure No:	P	M	N	R	c	r	$\propto_s$	$\propto_k$
1	6.11	3	4	3	2	6	3	3	3
2	6.12	3	2	2	1	5	2	1	0
3	6.13	3	2	2	1	3	1	2	1
4	6.14	3	12	9	4	6	2	10	20
5	6.15	3	7	6	2	3	3	3	15
6	6.16	3	12	6	7	25	19	2	9
7	6.17	3	13	6	8	28	20	4	7
8	6.18	3	6 14	2 10	5 5	0 24	0 0	15 15	3 3
9	6.19	6	9	7	3	12	0	18	24
10	6.20	3	4	3	2	6	6	0	5

7 FLEXIBILITY AND STIFFNESS METHODS

These are the two basic methods by which an indeterminate skeletal structure is analyzed. In these methods flexibility and stiffness properties of members are employed. These methods have been developed in conventional and matrix forms. Here conventional methods are discussed.

Flexibility Method

The given indeterminate structure is first made statically determinate by introducing suitable number of releases. The number of releases required is equal to statical indeterminacy $\propto_s$. Introduction of releases results in displacement discontinuities at these releases under the externally applied loads. Pairs of unknown biactions (forces and moments) are applied at these releases in order to restore the continuity or compatibility of structure. The computation of these unknown biactions involves solution of linear simultaneous equations. The number of these equations is equal to statical indeterminacy $\propto_s$. After the unknown biactions are computed all the internal forces can be computed in the entire structure using equations of equilibrium and free bodies of members. The required displacements can also be computed using methods of displacement computation.

In flexibility method since unknowns are forces at the releases the method is also called force method. Since computation of displacement is also required at releases for imposing conditions of compatibility the method is also called compatibility method. In computation of displacements use is made of flexibility properties, hence, the method is also called flexibility method.

Stiffness Method

The given indeterminate structure is first made kinematically determinate by introducing constraints at the nodes. The required number of constraints is equal to degrees of freedom at the nodes that is kinematic indeterminacy $\propto_k$. The kinematically determinate structure comprises of fixed ended members, hence, all nodal displacements are zero. These results in stress resultant discontinuities at these nodes under the action of applied loads or in other words the clamped joints are not in equilibrium. In order to restore the equilibrium of stress resultants at the nodes the nodes are imparted suitable unknown displacements. The number of simultaneous equations representing joint equilibrium of forces is equal to kinematic indeterminacy $\propto_k$. Solution of these equations gives unknown nodal displacements. Using stiffness properties of members the member end forces are computed and hence the internal forces throughout the structure.

Since nodal displacements are unknowns, the method is also called displacement method. Since equilibrium conditions are applied at the joints the method is also called equilibrium method. Since stiffness properties of members are used the method is also called stiffness method.

8 ANALYSIS OF STATICALLY DETERMINATE STRUCTURES

Following are the steps for analyzing statically determinate structures.

(1) Obtain the reactions at the supports of structure applying appropriate equations of equilibrium.

(2) Separate the members at the joints as free bodies and apply equations of equilibrium to each member to obtain member end forces.

(3) Cut the member at a section where internal forces are required. Apply equations of equations to any of the two segments to compute unknown forces at this section.

Example 8.1

Compute reactions for the beam AB loaded as shown in figure 8.1. Also find internal forces at mid span section C.

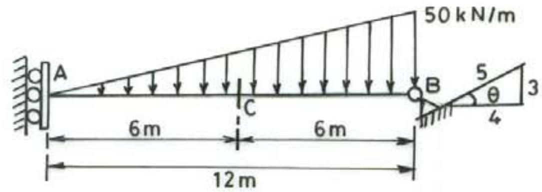


FIG. 8·1. STRUCTURAL SYSTEM AND LOADING

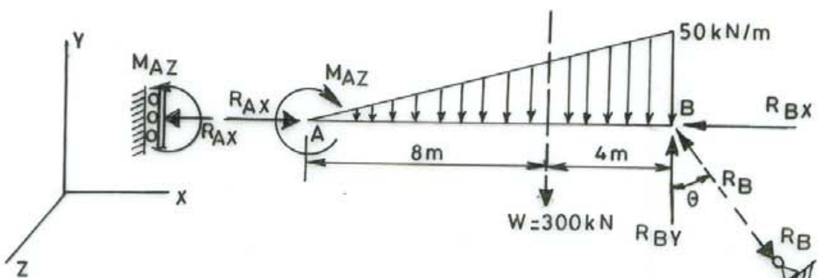


FIG. 8·2. FREE BODY DIAGRAMS WITH UNKNOWN FORCES.

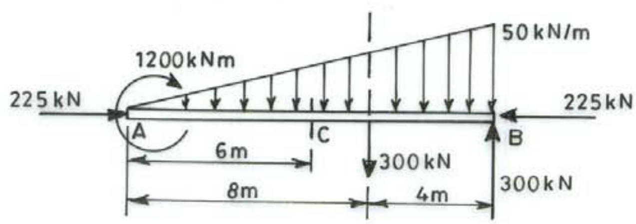


FIG. 8·3. FREE BODY DIAGRAM WITH KNOWN FORCES

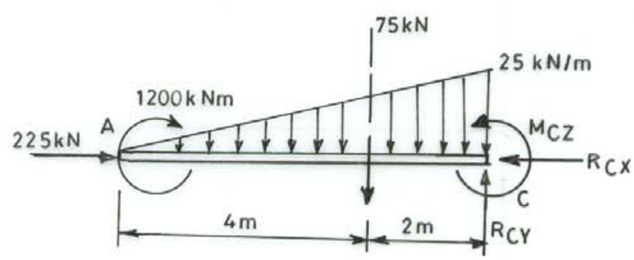


FIG. 8·4. FREE BODY DIAGRAM WITH UNKNOWN FORCES

Detach the beam from supports and show unknown reactions as shown in Fig.8.2

The reaction R_B which is perpendicular to rolling surface is replaced with its horizontal and vertical components R_{BX} and B_{BY}.

$$R_{BX} = R_B \text{ Sin } \theta = \frac{3}{5} R_B, R_{BY} = R_B \text{ Cos } \theta = \frac{4}{5} R_B$$

At A reaction in vertical direction is zero and other components are R_{AX} and M_{AZ}. Resultant of triangular load W is shown acting at 8m from A and 4m from B that is through CG of triangular loading. The free body diagram with known forces is shown in Fig.8.3.

$$W = \frac{1}{2} \text{ x } 50 \text{ x } 12 = 300 \text{ kN}$$

The equations of equilibrium for the member are:

$$\sum F_x = 0, \sum F_y = 0 \text{ and } \sum M_z = 0$$

Alternatively, $\sum F_x = 0$, $\sum_A M_z = 0$, $\sum_B M_z = 0$

$$\sum F_x = 0 \text{ gives : } R_{AX} = R_{BX}$$

$$\sum F_y = 0 \text{ gives : } R_{BY} = 300 \text{ kN}$$

$$\sum_B M_z = 0 \text{ gives : } M_{AZ} = 4 \text{ x } 300 = 1200 \text{ kNm}$$

$$\sum_A M_z = 0 \text{ gives : } 12 R_{BY} = 8 \text{ x } 300 + M_{AZ}, R_{BY} = \frac{2400+1200}{12} = 300 \text{ (check)}$$

$$R_B = \frac{5}{4} \text{ x } 300 = 375, R_{BX} = \frac{3}{5} \text{ x } 375 = 225.$$

Now the beam is cut at mid span and left segment is considered as a free body.

The free body diagram of segment AC with unknown forces is shown in Fig.8.4.

Total triangular load = $\frac{1}{2}$ x 6 x 25 = 75 kN. It acts at 4m from A and 2m from C.

$\sum F_x = 0$ gives, $R_{CX} = 225$ kN

$\sum F_y = 0$ gives, $R_{CY} = 75$ kN

$\sum_C M_z = 0$ gives, $M_{CZ} = 1200 - 75 \times 2 = 1050$ kNm

Example 8.2

Determine the reactions for the three hinged arched frame ABC loaded as shown in Fig. 8.5. Show free body diagrams for members AB and BC and segments BD and DC.

We have three equations of equilibrium and four unknown reactions. The structure is determinate despite four unknown reactions as the moment at hinge B is zero. The free body diagrams of members AB and BC are shown in Fig.8.6 and Fig.8.7.

The equations of equilibrium of free body AB are

$\sum F_x = 0$, $R_{AX} - R_{BX} = 0$ ………. (1)

$\sum F_y = 0$, $R_{AY} + R_{BY} = 40$ ……… (2)

$\sum_A M_z = 0$, $3\ R_{BX} + 4\ R_{BY} = 40 \times 2.5 = 100$ ……… (3)

The equations of equilibrium of free body BC are :

$\sum F_x = 0$, $R_{BX} + R_{CX} = 5$ ………. (4)

$\sum F_y = 0$, $-R_{BY} + R_{CY} = 10$ ………. (5)

$\sum_C M_z = 0$, $4\ R_{BX} - 3\ R_{BY} = -25 + 10 \times 1.5 + 5 \times 2 = 0$ ……… (6)

These equations are solved for the unknown forces.

E_{qn} (3) x 4, $12\ R_{BX} + 16\ R_{BY} = 400$ …….. (7)

E_{qn} (6) x 3, $12\ R_{BX} - 9\ R_{BY} = 0$ ……... (8)

E_{qn} (7) – E_{qn} (8), $25\ R_{BY} = 400$, $R_{BY} = 16$

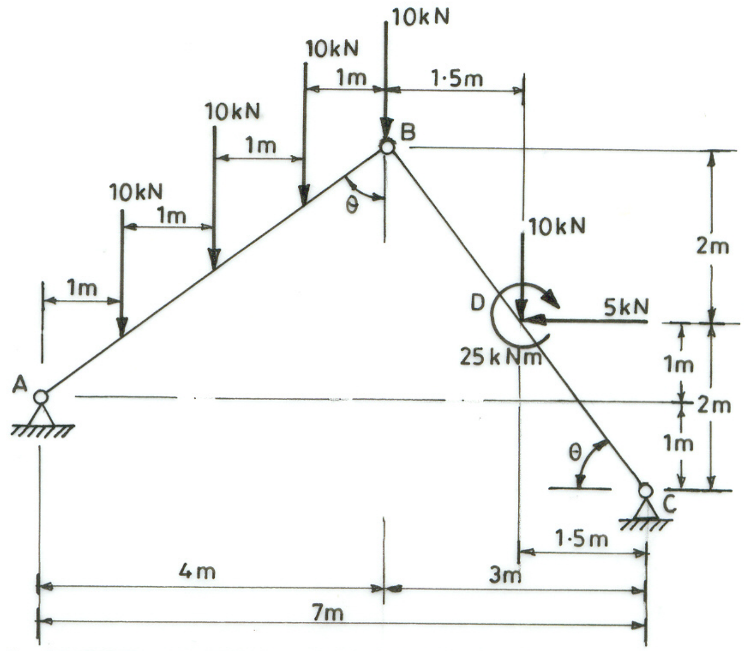


FIG. 8·5. THREE HINGED ARCHED STRUCTURAL SYSTEM AND LOADING.

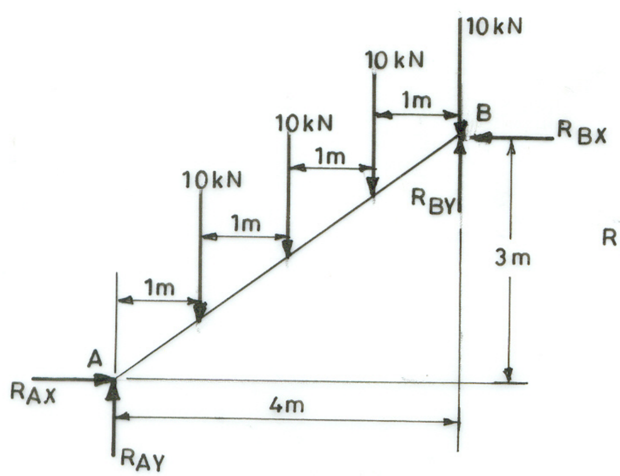


FIG. 8·6. FREE BODY DIAGRAM OF MEMBER AB WITH UNKNOWN FORCES

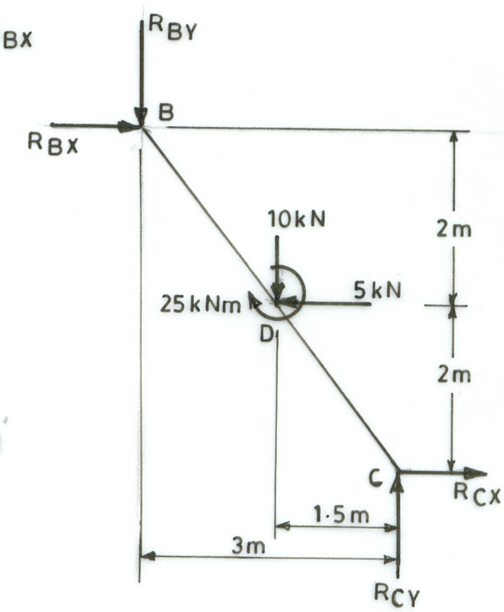


FIG. 8·7. FREE BODY DIAGRAM OF MEMBER BC WITH UNKNOWN FORCES

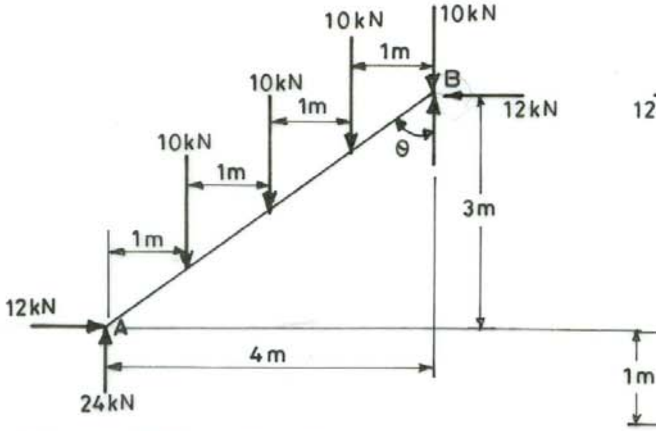


FIG. 8·8. FREE BODY DIAGRAM OF MEMBER AB WITH KNOWN FORCES

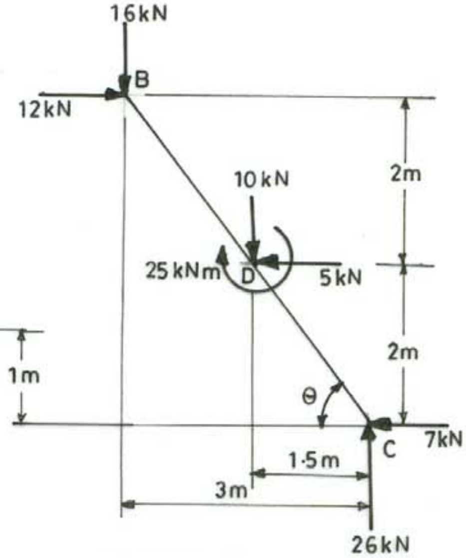


FIG. 8·9. FREE BODY DIAGRAM OF MEMBER BC WITH KNOWN FORCES

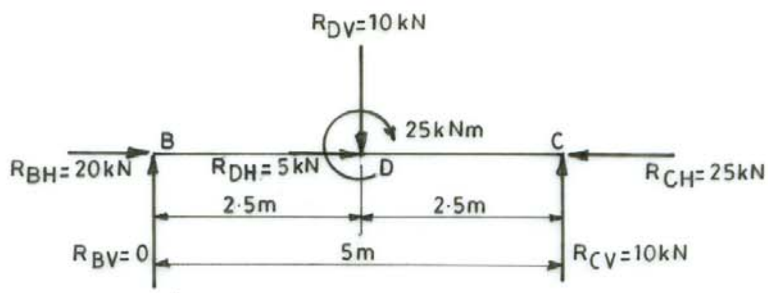


FIG. 8·10. FREE BODY DIAGRAM OF MEMBER BC (SHOWN HORIZONTALLY)

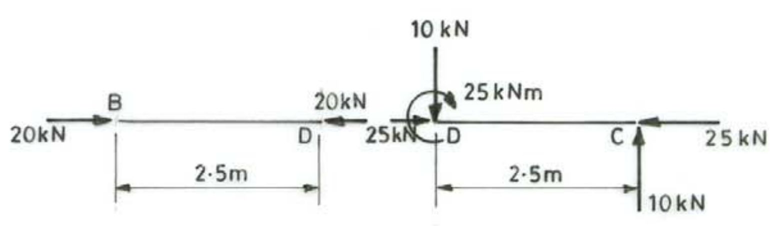


FIG. 8·11. FREE BODY DIAGRAMS OF SEGMENTS BD AND CD (SHOWN HORIZONTALLY)

From (2), $R_{AY} = 40 - 16 = 24$

From (7), $R_{BX} = \frac{1}{12}\,[400 - 16 \times 16] = \frac{144}{12} = 12$

From (8), $R_{BX} = \frac{9 \times 16}{12} = 12$ (check)

From (1), $R_{AX} = 12$

From (4), $R_{CX} = 5 - 12 = -7$

From (5), $R_{CY} = 10 + 16 = 26$

The free body diagrams of members AB and BC with known forces are shown in Figures 8.8 and 8.9.

Member BDC is shown horizontally and the forces are resolved along the axis of member (suffix H) and normal to it (suffix V) as shown in figure 8.10.

<u>At B</u> : $R_{BH} = 12 \cos\theta + 16 \sin\theta = 12 \times \frac{3}{5} + 16 \times \frac{4}{5} = 20$

$R_{BV} = 12 \sin\theta - 16 \cos\theta = 12 \times \frac{4}{5} - 16 \times \frac{3}{5} = 0$

<u>At D</u> : $R_{DH} = 10 \sin\theta - 5 \cos\theta = 10 \times \frac{4}{5} - 5 \times \frac{3}{5} = 5$

$R_{DV} = -5 \sin\theta - 10 \cos\theta = -5 \times \frac{4}{5} - 10 \times \frac{3}{5} = -10$

<u>At C</u> : $R_{CH} = +7 \cos\theta + 26 \sin\theta = +7 \times \frac{3}{5} + 26 \times \frac{4}{5} = +25$

$R_{CV} = -7 \sin\theta + 26 \cos\theta = -7 \times \frac{4}{5} + 26 \times \frac{3}{5} = 10$

It can easily be verified that equations of equilibrium are satisfied in this configuration. By cutting the member just to left of D the free body diagrams of segments are shown in Fig. 8.11.

9 ANALYSIS OF DETERMINATE TRUSSES

The trusses are classified as determinate and indeterminate. They are also classified as simple, compound and complex trusses. We have plane and space trusses. The joints of the trusses are idealized for the purpose of analysis. In case of plane trusses the joints are assumed to be hinged or pin connected. In case of space trusses ball and socket joint is assumed which is called universal joint. If members are connected to a hinge in a plane or universal joint in space, the system is equivalent to m members rigidly connected at the node with hinges or socketed balls in (m-1) number of members at the nodes as shown in figure 9.1. In other words it can be said that the members are allowed to rotate freely at the nodes. The degree of freedom at node is 2 for plane truss (linear displacements in x and y directions) and 3 for space truss (linear displacements in x,y and z directions). The plane truss requires supports equivalent of three reactions and determinate space truss requires supports equivalent of six reactions in such a manner that supporting system is stable and should not turn into a mechanism. For this it is essential that reactions should not be concurrent and parallel so that system will not rotate and move. As regards loads they are assumed to act on the joints or points of concurrency of members. If load is acting on member it is replaced with equivalent loads applied to joints to which it is connected. Here the member discharges two functions that is function of direct force member in truss and flexural member to transmit its load to joints. For this member the two effects are combined to obtain final internal stress resultants in this member.

The truss is said to be just rigid or determinate if removal of any one member destroys its rigidity and turns it into a mechanism. It is said to be over rigid or indeterminate if removal of member does not destroy its rigidity.

Relation between number of members and joints for just rigid truss.

Let m = Number of members and j = Number of joints

Space truss

Number of equivalent links or members or reactive forces to constrain the truss in space is 6 corresponding to equations of equilibrium in space ($\sum F_x = 0$, $\sum F_y = 0$, $\sum F_Z = 0$, $\sum M_x = 0$, $\sum M_y = 0$, $\sum M_Z = 0$). For ball and socket (universal) joint the minimum number of links or force components for support or constraint of joint in space is 3 corresponding to equations of equilibrium of concurrent system of forces in space ($\sum F_x = 0$, $\sum F_y = 0$, $\sum F_Z = 0$). Each member is equivalent to one link or force.

Total number of links or members or forces which support j number of joints in space truss is (m + 6). Thus total number of unknown member forces and reactions is (m + 6). The equations of equilibrium corresponding to j number of joints is 3j. Therefore for determinate space truss system: (m + 6) = 3j.

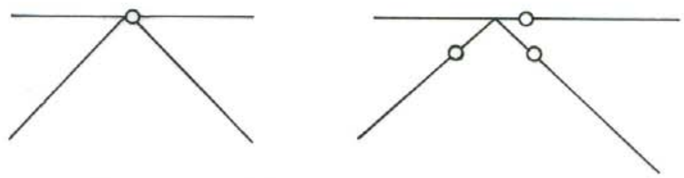


FIG. 9·1. HINGED JOINT OF A TRUSS

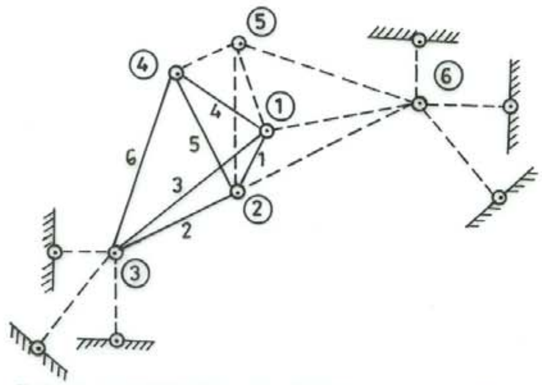


FIG. 9·2. METHOD OF FORMING SPACE TRUSS

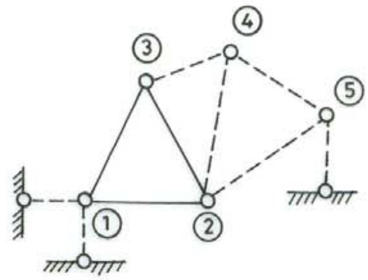


FIG. 9·3. METHOD OF FORMING PLANE TRUSS

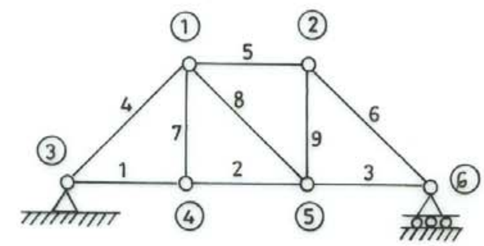


FIG. 9·4. JUST RIGID TRUSS

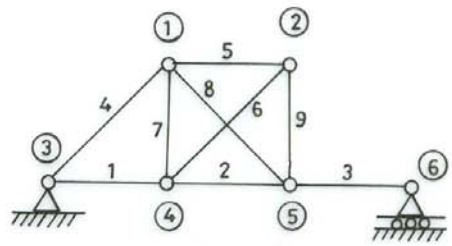


FIG. 9·5. MECHANISM

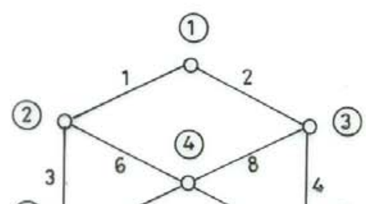


FIG. 9·6. JUST RIGID TRUSS

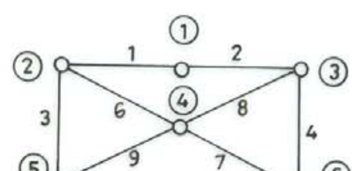


FIG. 9·7. PARTIAL MECHANISM

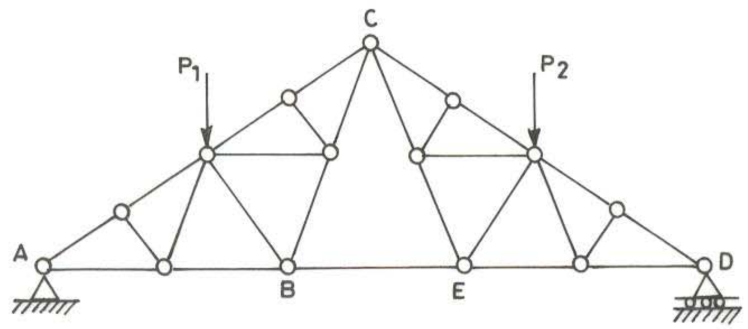


FIG. 9·8. COMPOUND TRUSS

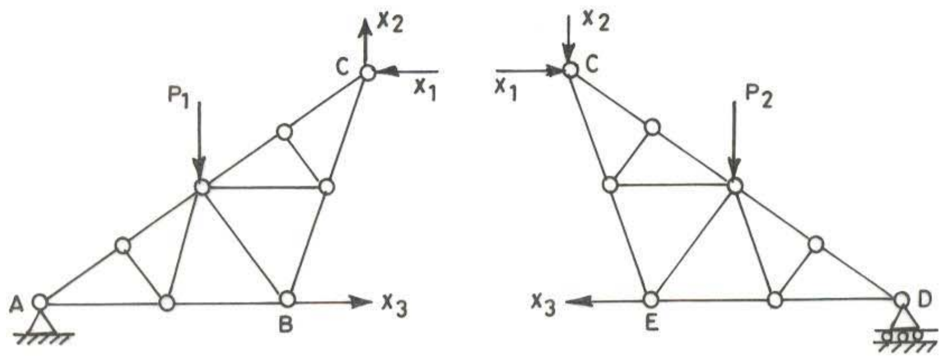


FIG. 9·9. TWO SIMPLE TRUSSES FORMING COMPOUND TRUSS BY HINGE AND MEMBER

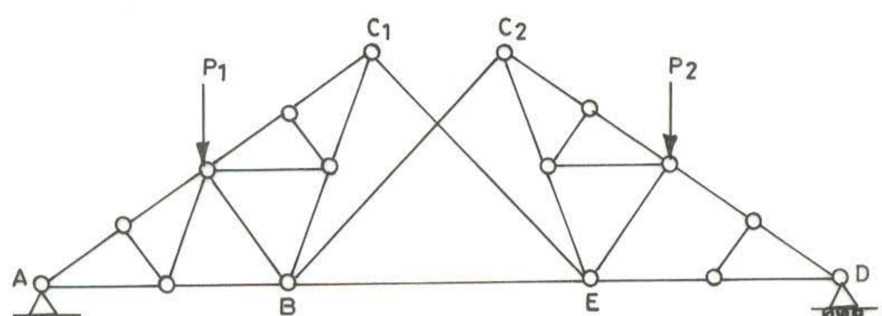


FIG. 9·10. COMPOUND TRUSS

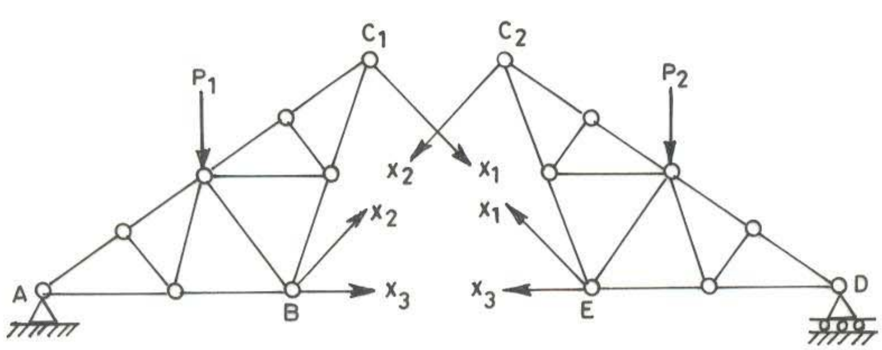


FIG. 9·11. TWO SIMPLE TRUSSES FORMING COMPOUND TRUSS BY THREE MEMBERS

$$m = (3j - 6)$$

Minimum just rigid or stable space truss as shown in Fig.9.2. is a tetrahedron for which m = 6 and j = 4. For this relation between members and joints is satisfied.

$$m = 3 \times 4 - 6 = 6 \text{ (ok)}$$

By adding one node and three members the truss is expanded which can be supported on support system equivalent of six links or forces neither parallel nor concurrent. We get determinate and stable system. As can be seen joints 5 and 6 are added to starting stable and just rigid tetrahedron truss. Three links at each of two joints 3 and 6 corresponding to ball and socket joint are provided.

Plane truss

The stable and just rigid or determinate smallest plane truss as shown in Fig.9.3. comprises of a triangle with three nodes and three members. Two members and a pin joint are added to expand the truss. Total number of non-parallel and non-concurrent links or reactive forces required to support j number of joints is 3. Total number of unknowns is number of member forces and reactions at the supports. Number of available equations is 2j. Therefore for determinate plane truss system:

$$(m + 3) = 2j$$

$$m = (2j - 3)$$

Hinge support is equivalent of two reactions or links and roller support is equivalent of one reaction or link.

Exceptions

Just rigid or simple truss is shown in figure 9.4, m = 9, j = 6, m = (2j – 3) = (2 x 6 – 3) = 9. The member no 6 is removed and connected to joints 2 and 4. As can be seen in figure 9.5. the condition of m = (2j – 3) is satisfied but configuration of truss can not be completed by starting with a triangle and adding two members and a joint. The system is mechanism and it is not a truss.

The stable and just rigid or determinate truss is shown in figure 9.6, m = 9, j = 6, m = 2 j – 3 = 2 x 6 – 3 = 9. The relation between members and joints will also be satisfied if arched part is made horizontal as shown in Fig.9.7. The system has partial constraint at C as there is nothing to balance vertical force at pin C. The two members must deflect to support vertical load at C. In fact the rule for forming determinate simple truss is violated as joint 1 is formed by members 1 and 2 by putting them along same line because these are the only two members at that joint.

Compound truss

Compound plane truss is formed by joining together two simple plane trusses by three nonparallel and nonconcurrent members or one hinge and the member. Compound truss shown in figure 9.8 is formed by combining two simple trusses ABC and CDE by hinge at C and member BE. It is shown supported at A and B. For purpose of analysis after determining reactions at supports the two trusses are separated and unknown forces X_1, X_2 and X_3 are determined by applying equations of equilibrium to any one part. There-after each part is analyzed as simple truss. This is shown in Fig.9.9.

Compound truss shown in figure 9.10 is formed by combining the two simple trusses by three nonparallel and nonconcurrent members. The truss is supported by two links corresponding to hinge support at A and one link corresponding to roller at B. By cutting these three members the two parts are separated and the unknown forces X_1, X_2 and X_3 in these members are determined by equations of equilibrium and each part is analyzed as simple truss. This is shown in Fig.9.11.

In case of compound space truss six members will be required to connect two simple space trusses in stable manner so that connecting system does not turn into a mechanism. Alternatively one common universal ball and socket joint and three members will be required. The method of analysis will be same as in plane truss case.

Complex truss

A complex truss is one which satisfies the relation between number of members and number of joints but can not be configured by rules of forming simple truss by starting with triangle or tetrahedron and then adding two members or three members and a node respectively for plane and space truss. A complex truss is shown in figure 9.12.

$$M = 9,\ j = 6,\ m = 2j - 3 = 2 \times 6 - 3 = 9$$

Method of analysis of determinate trusses.

There are two methods of analysis for determining axial forces in members of truss under point loads acting at joints. The forces in members are tensile or compressive. The first step in each method is to compute reactions. Now we have system of members connected at nodes and subjected to external nodal forces. The member forces can be determined by following methods.

(1) Method of joints

(2) Method of sections

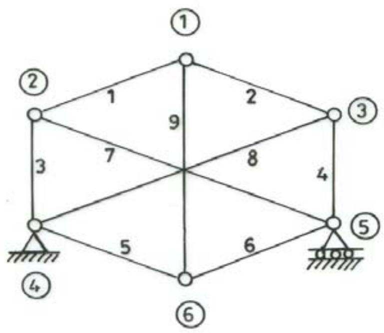


FIG. 9·12. COMPLEX TRUSS

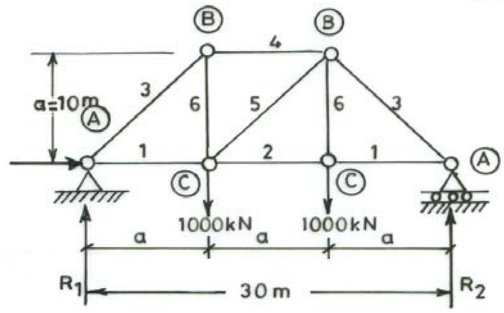


FIG. 9·13. SYMMETRIC SIMPLE TRUSS

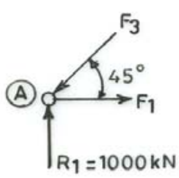


FIG. 9·14. FREE BODY OF JOINT A

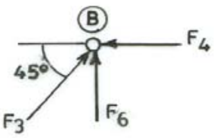


FIG. 9·15. FREE BODY OF JOINT B

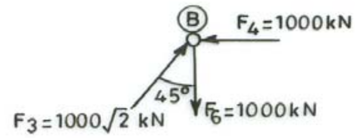


FIG. 9·16. EVALUATED FORCES AT JOINT B

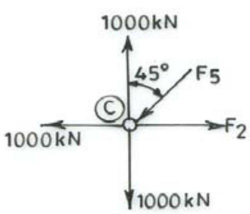


FIG. 9·17. FREE BODY OF JOINT C

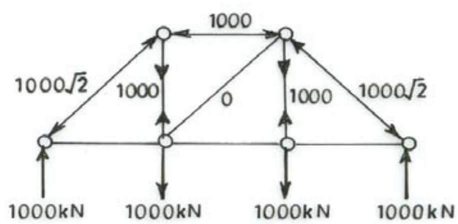
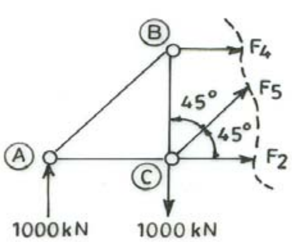


FIG. 9·18. EVALUATED FORCES AND REACTIONS FOR TRUSS



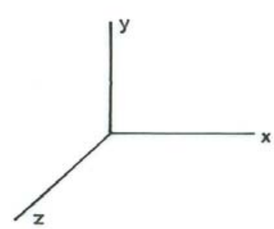


FIG. 9·19. METHOD OF SECTIONS

The method of joints is used when forces in all the members are required. A particular joint is cut out and its free body diagram is prepared by showing unknown member forces. Now by applying equations of equilibrium the forces in the members meeting at this joint are computed. Proceeding from this joint to next joint and thus applying equations of equilibrium to all joints the forces in all members are computed. In case of space truss the number of unknown member forces at a joint should not be more than three. For plane case number of unknowns should not be more than two.

Equations for space ball and socket joint equilibrium: $\sum F_x = 0,\ \sum F_y = 0,\ \sum F_Z = 0$

Equations for xy plane pin joint equilibrium : $\sum F_x = 0,\ \sum F_y = 0$

Method of sections

This method is used when internal forces in some members are required. A section is passed to cut the truss in two parts exposing unknown forces in required members. The unknowns are then determined using equations of equilibrium. In plane truss not more than 3 unknowns should be exposed and in case of space truss not more than six unknowns should be exposed.

Equations of equilibrium for space truss using method of sections: $\sum F_x = 0,\ \sum F_y = 0,\ \sum F_Z = 0$
$\sum M_x = 0,\ \sum M_y = 0,\ \sum M_Z = 0$

Equations of equilibrium for xy-plane truss using method of sections: $\sum F_x = 0,\ \sum F_y = 0,\ \sum M_Z = 0$

Example 9.1

Determine forces in all the members of plane symmetric truss loaded symmetrically as shown in figure 9.13 for all members by method of joints and in members 2,4 and 5 by method of sections.

$\sum F_x = 0$ gives, $R_3 = 0$

$\sum_A M_Z = 0$ gives, $30\ R_2 = 1000 \times 10 + 1000 \times 20 = 30{,}000$, R2 = 1000 kN

$\sum F_y = 0$ gives, $R_2 = 1000 + 1000 - R_1 = 2000 - 1000 = 1000$ kN

Method of joints

Joint A

Free body is shown in figure 9.14. Force in member 1 is assumed tensile and in member 3 compressive. Actions on pin at A are shown.

$$\sum_A F_X = 0 : F_1 - F_3 \cos 45^0 = 0,$$

$$\sum_A F_Y = 0 : - F_3 \sin 45^0 + 1000 = 0,\ F_3 = 1000\ \sqrt{2}\ = 1414\ \text{kN},$$

$$F_1 = 1000\ \sqrt{2}\ \text{x}\ \frac{1}{\sqrt{2}}\ = 1000\ \text{kN}$$

Since positive results are obtained the direction and nature of forces F_1 and F_3 assumed are correct. At joint C there will be three unknowns, hence, we proceed to joint B where there are only two unknowns.

Joint B

The free body diagram of joint B is shown in figure 9.15.

$$\sum_B F_X = 0 \text{ gives, } F_3 \cos 45^0 - F_4 = 0,\ F_4 = 1000\ \sqrt{2}\ \text{x}\ \frac{1}{\sqrt{2}}\ = 1000\ \text{kN}$$

$$\sum_B F_y = 0, \text{ gives : } F_6 + F_3 \cos 45^0 = 0,\ \ F_6 = -\ 1000\ \sqrt{2}\ \text{x}\ \frac{1}{\sqrt{2}}\ = -\ 1000\ \text{kN}$$

The negative sign indicates that direction of force assumed is wrong and it would be opposite. It is desirable to reverse the direction of F_6 here it self and then proceed to joint C, else the value will have to be substituted in subsequent calculation with negative sign and there are more chances of making mistakes in calculations. The corrected free body diagram of joint B is shown in figure 9.16.

Joint C

The free body diagram for joint C is now prepared and is shown in figure 9.17.

$$\sum_C F_Y = 0 \text{ gives, } F_5 \cos 45^0 = 0,\ F_5 = 0$$

$$\sum_C F_X = 0 \text{ gives, } F_2 = 1000\ \text{kN}$$

The results are shown in figure 9.18. The arrows shown at the ends of members are forces actually acting on pin joints. The reactive forces from joints onto members will decide whether it is tension or compression in the members. The sign convention was explained in theory.

Method of sections

Now a section is passed cutting through members 2, 4 and 5 and left segment is considered as a free body as shown in Fig.9.19. The unknown member forces are assumed tensile. However, if it is possible to predict correct nature, the correct direction should be assumed so as to obtain positive result. A critical observation of free body indicates that $F_5 = 0$ as its vertical component can not be balanced as remaining resultant nodal forces in vertical direction vanish. Now equilibrium in horizontal direction indicates that $F_4 = - F_2$. The segment is subjected to clockwise moment of 10,000 kNm, hence, F_2 and F_4 should form counter clockwise couple to balance this moment. This also indicates force F_4 should have opposite direction but same magnitude. Since arm is 10 m, $F_2 \times 10 = 10{,}000$, hence, $F_2 = 1000$ kN. and $F_4 = - 1000$ kN. By method of sections we proceed as follows:

$$\sum_D M_Z = 0 \text{ gives : } F_2 \times 10 + 1000 \times 10 - 1000 \times 20 = 0,\ F_2 = 1000 \text{ kN}$$

$$\sum_C M_Z = 0 \text{ gives : } - F_4 \times 10 - 1000 \times 10 = 0,\ F_4 = - 1000 \text{ kN}$$

$$\sum F_y = 0 \text{ gives : } - F_5 \times \frac{1}{\sqrt{2}} - 1000 + 1000 = 0,\ F_5 = 0$$

$$\sum F_X = 0 \text{ gives : } - F_5 \times \frac{1}{\sqrt{2}} + F_2 + F_4 = 0,\ F_5 = 0$$

Method of tension coefficients for space truss

Consider a member AB of space truss, arbitrarily oriented in space as shown in figure 9.20.

x_A, y_A, z_A = coordinates of end A

x_B, y_B, z_B = coordinates of end B

L_{AB} = length of member AB

l_{AB}, m_{AB}, n_{AB} = direction cosines of member AB.

$\theta_x, \theta_y, \theta_z$ = angle that axis of member AB makes with x, y and z axis respectively.

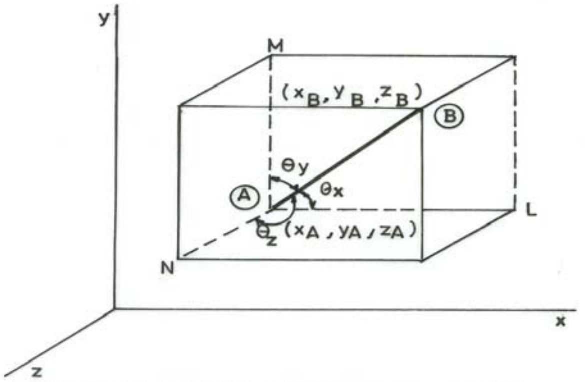


FIG. 9·20. TRUSS MEMBER IN SPACE

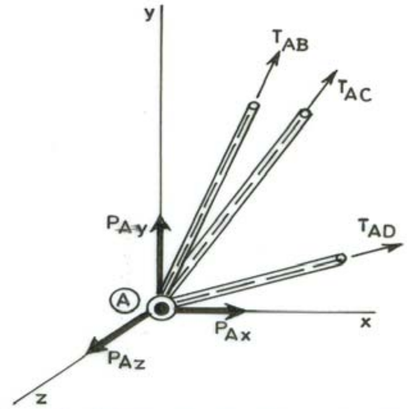


FIG. 9·21. FREE BODY OF A JOINT OF A SPACE TRUSS

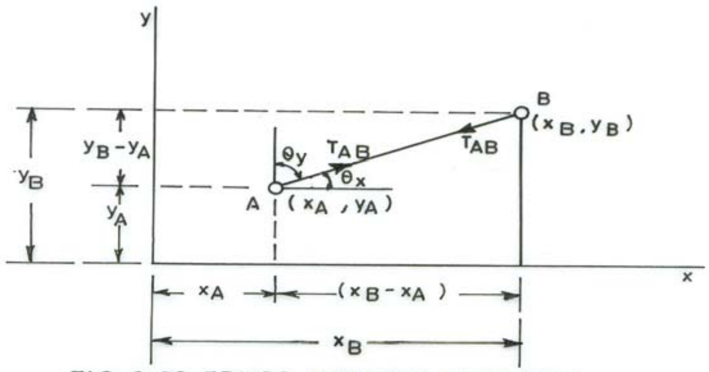


FIG. 9·22. TRUSS MEMBER IN PLANE

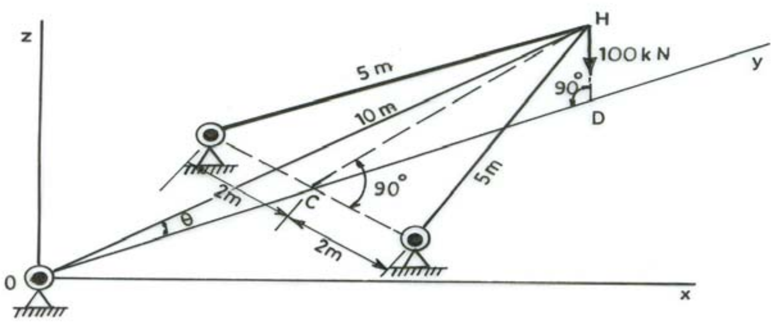


FIG. 9·23. SHEAR LEG ANALYSIS

$L_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$

$l_{AB} = \cos\theta_x$, $m_{AB} = \cos\theta_y$, $n_{AB} = \cos\theta_z$

$AL = (x_B - x_A) = l_{AB}\, L_{AB}$

$AM = (y_B - y_A) = m_{AB}\, L_{AB}$

$AN = (z_B - z_A) = n_{AB}\, L_{AB}$

Tension coefficient t for a member is defined as tensile force T in the member divided by its length L.

$t = \frac{T}{L}$, $t_{AB} = \frac{T_{AB}}{L_{AB}}$ = tension coefficient for member AB.

Components of force T_{AB} in member AB in x, y and z directions are obtained as follows.

$T_{AB} \cos\theta_x = T_{AB} \frac{(x_B - x_A)}{L_{AB}} = t_{AB}\,(x_B - x_A)$

$T_{AB} \text{ con } \theta_y = T_{AB} \frac{(y_B - y_A)}{L_{AB}} = t_{AB}\,(y_B - y_A)$

$T_{AB} \cos\theta_z = T_{AB} \frac{(z_B - z_A)}{L_{AB}} = t_{AB}\,(z_B - z_A)$

P_A = External force acting at joint A of space truss shown in Fig.9.21.

Q_A = Resultant of known member forces at joint A

P_{AX}, P_{AY}, P_{AZ} = Components of force P_A in x,y and z directions

Q_{AX}, Q_{AY}, Q_{AZ} = Components of force Q_A in x,y and z directions

T_{AB}, T_{AC}, T_{AD} = Unknown tensile forces acting on members AB, AC and AD at joint A.

The three equations of equilibrium for joint A are written as follows.

$t_{AB}\ (x_B - x_A) + t_{AC}\ (x_C - x_A) + t_{AD}\ (x_D - x_A) + Q_{AX} + P_{AX} = 0$

$t_{AB}\ (y_B - y_A) + t_{AC}\ (y_C - y_A) + t_{AD}\ (y_D - y_A) + Q_{AY} + P_{AY} = 0$

$t_{AB}\ (z_B - z_A) + t_{AC}\ (z_C - z_A) + t_{AD}\ (z_D - z_A) + Q_{AZ} + P_{AZ} = 0$

These equations can be written in compact form by identifying any member with far and near ends.

x_F, y_F, z_F = coordinates of far end of a member

x_N, y_N, z_N = coordinates of near end of a member

$\sum t\ (x_F - x_N) + Q_{AX} + P_{AX} = 0$

$\sum t\ (y_F - y_N) + Q_{AY} + P_{AY} = 0$

$\sum t\ (z_F - z_N) + Q_{AZ} + P_{AZ} = 0$

Method of tension coefficients for plane trusses

Plane truss member AB in tension is shown in Fig.9.22.

Component of pull T_{AB} in x-direction = $T_{AB} \cos \theta_x = T_{AB} \dfrac{(x_B - x_A)}{L_{AB}} = t_{AB}\ (x_B - x_A)$

Component of pull T_{AB} in y-direction = $T_{AB} \cos \theta_y = T_{AB} \dfrac{(y_B - y_A)}{L_{AB}} = t_{AB}\ (y_B - y_A)$

Positive tension coefficient t will indicate tension

Negative tension coefficient t will indicate compression

$L_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$

Compact form of equations of equilibrium at joint A is:

$\sum t\ (x_F - x_N) + Q_{AX} + P_{AY} = 0$
$\sum t\ (y_F - y_N) + Q_{AY} + P_{AZ} = 0$

Example 9.2

For the shear leg system shown in figure 9.23 determine the axial forces in legs and tie for vertical load of 100 kN at the apex (head). Length of each leg is 5 m and spread of legs is 4 m. The distance from foot of guy rope to center of spread is 7 m. Length of guy rope is 10 m.

OC = 7 m, AB = 4 m, AC = BC = 2 m, OH = 10 m, AH = BH = 5 m.

θ = angle guy makes with y axis

CH = $\sqrt{5^2 - 2^2} = \sqrt{21} = 4.5826$ m

From triangle OCH

$$\text{Cos}\,\theta = \frac{\left(OH^2 + OC^2 - CH^2\right)}{\left(2OC \times OH\right)} = \frac{\left(10^2 + 7^2 - 21\right)}{\left(2 \times 7 \times 10\right)}$$

Cos θ = 0.9143

$\theta = 23.9^0$, Sin θ = 0.4051

OD = 10 Cos θ = 9.143 m

HD = 10 Sin θ = 4.051 m

CD = 9.143 – 7 = 2.143 m

Coordinates of nodes O, H, A and B are

Node	x	y	z
O	0	0	0
H	0	9.143	4.051
A	-2	7	0
B	2	7	0

Equations of equilibrium at H are:

$t_{HA}\ (x_A - x_H) + t_{HB}\ (x_B - x_H) + t_{HO}\ (x_O - x_H) = 0$ ______ (1)

$t_{HA}\ (y_A - y_H) + t_{HB}\ (y_B - y_H) + t_{HO}\ (y_O - y_H) = 0$ ______ (2)

$t_{HA}\ (z_A - z_H) + t_{HB}\ (z_B - z_H) + t_{HO}\ (z_O - z_H) + P_{HY} = 0$ _____ (3)

$-2\ t_{HA} + 2\ t_{HB} = 0$ _______ (1)

$-2.143\ t_{HA} - 2.143\ t_{HB} - 9.143\ t_{HO} = 0$ _______ (2)

$-4.051\ (t_{HA} + t_{HB} + t_{HO}) - 100 = 0$ _______ (3)

From eqn (1): $t_{HA} = t_{HB}$

From eqn (2): $-2 \times 2.143\ t_{HA} = 9.143\ t_{HO}$, $\therefore\ t_{HA} = -2.1332\ t_{HO}$

From Eqn (3): $-4.051\ (-2.1332 - 2.1332 - 1)\ t_{HO} = 100$, $t_{HO} = 7.5573$

$t_{HA} = t_{HB} = -2.1332 \times 7.5573 = 16.1213$

$T_{HO} = t_{HO}\ L_{HO} = 7.5573 \times 10 = 75.57$ kN

C_{HA}, C_{HB} = thrust in shear legs HA and HB

$C_{HA} = C_{HB} = 16.1213 \times 5 = 80.61$ kN

10 CABLES AND ARCHES

10.1 Cables

Cable is a very efficient structural form as it is almost perfectly flexible. Cable has no flexural and shear strength. It has also no resistance to thrust, hence, it carries loads by simple tension only. Cable adjusts its shape to equilibrium link polygon of loads to which it is subjected. A cable has a shape of catenary under its own weight. If a large point load W compared to its own weight is applied to the cable its shape changes to two straight segments. If W is small compared to its own weight the change in shape is insignificant as shown in figure 10.1. From equilibrium point of view a small segment of horizontal length dx shown in Fig.10.2 should satisfy two equations of equilibrium $\sum F_x = 0$ and $\sum F_y = 0$. The cable maintains its equilibrium by changing its tension and slope that is shape. One unknown cable tension T can not satisfy two equilibrium equations, hence, one additional unknown of slope θ is required. The cables are used in suspension and cable stayed bridges, cable car systems, radio towers and guys in derricks and chimneys. By assuming the shape of cable as parabolic, analysis is greatly simplified.

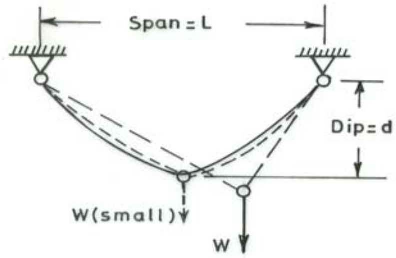


FIG. 10·1. CABLE.

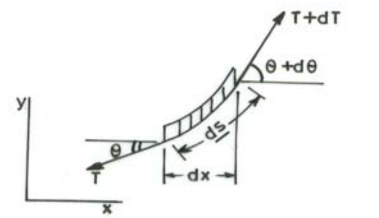


FIG. 10·2. INFINITESIMAL ELEMENT OF CABLE

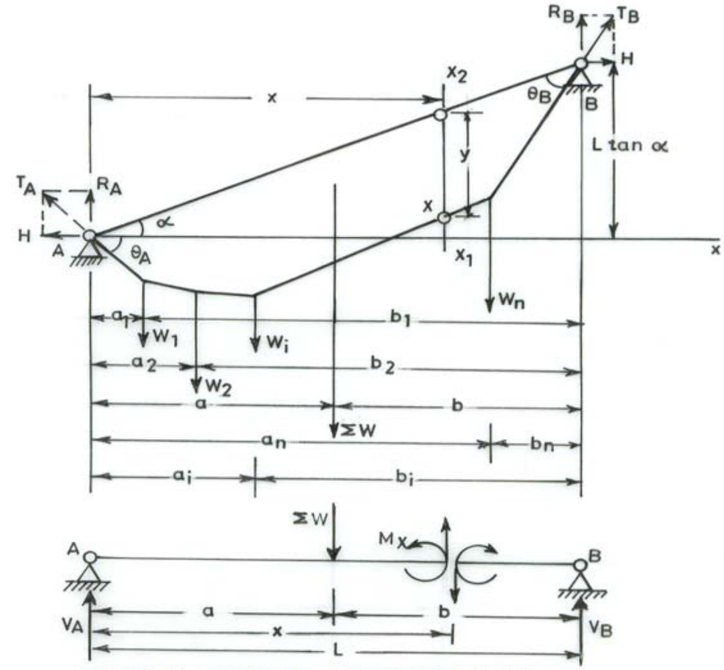


FIG. 10·3. GENERAL CASE OF CABLE.

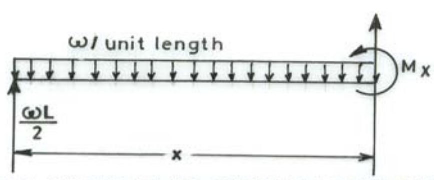


FIG. 10·4. PARTICULAR CASE OF CABLE UNDER UDL.

10.2 General cable theorem

A cable subjected to point loads W_1 to W_n is suspended from supports A and B over a horizontal span L. Line joining supports makes angle $\propto$ with horizontal. Therefore elevation difference between supports is represented by L tan $\propto$ as shown in Fig 10.3.

$$\Sigma W = \sum_{i=1}^{n} W = W_1 + W_2 + ---- + W_i + ----- + W_n$$

$$a = \frac{\Sigma W_i a_i}{\Sigma W}, \quad b = \frac{\Sigma W_i b_i}{\Sigma W}$$

$\sum_B M$ = Counter clockwise moment of vertical downward loads W_1 to W_n about support B.

$$\sum_B M = b\Sigma W$$

$$R_A = \text{Vertical reaction at A} = \frac{\sum_B M}{L} - H \tan \propto$$

$$R_B = (\Sigma W + H \tan \propto - \frac{\sum_B M}{L})$$

Consider a point X on cable at horizontal coordinate x from A and vertical dip y from chord.

$X_1 X_2 = x \tan \propto$, $XX_2 = y$, $XX_1 = (x \tan \propto - y)$

$\sum_X M$ = Counter clockwise moment of all downward loads left of X,

Since cable is assumed to be perfectly flexible the bending moment at any point of cable is zero. Considering moment equilibrium of segment of cable on left of X the relation between H, x and y is obtained which defines general cable theorem.

$$H\,(XX_1) = \sum_X M - R_A\,x$$

$$H\,(x \tan \propto - y) = \sum_X M - \left(\frac{\sum_B M}{L} - H \tan \propto\right) x$$

$$Hy = \left[\frac{x}{L}\sum_B M - \sum_X M\right]$$

Consider a horizontal beam of span L subjected to same vertical loading as cable as shown in Fig.10.3. Let V_A be reaction at A and M_X bending moment at section X at coordinate x.

$$V_A = \frac{b}{L}\sum W = \frac{\sum_B M}{L}$$

$$M_X = V_A x - \sum_X M = \frac{\sum_B M}{L} x - \sum_X M$$

Thus: $Hy = M_X$

The general cable theorem therefore states that at any point on the cable subjected to vertical loads, Hy the product of horizontal component of tension in cable and the vertical dip of that point from cable chord is equal to the bending moment M_X at the same horizontal coordinate in a simply supported beam of same span as cable and subjected to same vertical loading as the cable.

T_A, T_B = Tensions in cable at the supports

θ_A, θ_B = Slopes of cable at supports

$$T_A = \sqrt{H^2 + R_A^2}\,,\ \theta_A = \tan^{-1}\frac{R_A}{H}$$

$$T_B = \sqrt{H^2 + R_B^2}\,,\ \theta_B = \tan^{-1}\frac{R_B}{H}$$

If cable is subjected to vertical downward uniformly distributed load of intensity ω as shown in Fig.10.4, then:

$$Hy = M_X = \frac{\omega L}{2} x - \omega\,\frac{x^2}{2}$$

At mid span $x = \frac{L}{2}$ and y = h the dip of cable.

$$Hh = \frac{\omega L^2}{4} - \frac{\omega L^2}{8} = \frac{\omega L^2}{8}$$

$$H = \frac{\omega L^2}{8h}$$

10.3 Shape of cable

$$Hy = M_X$$

$$\frac{\omega L^2}{8h} y = \frac{\omega L}{2} x - \omega \frac{x^2}{2}$$

$$y = \frac{4h}{L^2} x (L - x)$$

This is the equation of cable curve with respect to cable chord. The cable thus takes the shape of parabola under the action of udl. The same equation is valid when chord is horizontal as shown in Fig.10.5.

10.4 Length of cable with both ends at same level

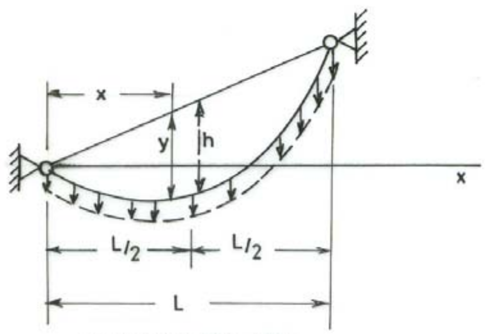
$$S = \int_O^L ds = \int_O^L \left(\sqrt{1 + \left(\frac{dy}{dx} \right)^2} \right) dx$$

$$\frac{dy}{dx} = \frac{4h}{L^2} (L - 2x)$$

$$S = \int_O^L \left[1 + \frac{16h^2}{L^2} (L - 2x)^2 \right]^{\frac{1}{2}} dx$$

This will give:

$$S = L \left[1 + \frac{8}{3} \frac{h^2}{L^2} - \frac{32}{5} \frac{h^4}{L^4} + \frac{256}{7} \frac{h^6}{L^6} - \ldots\ldots \right]$$



CABLE UNDER UDF

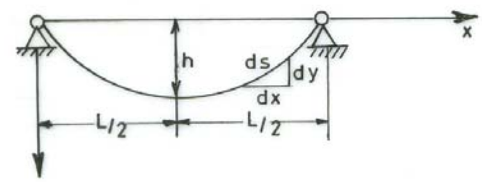


FIG.10·5. CABLE WITH HORIZONTAL CHORD

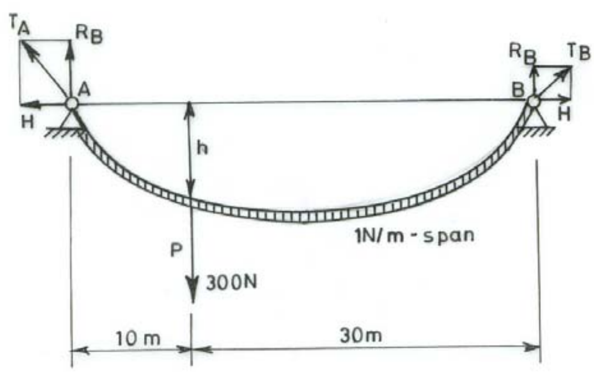


FIG.10·6. NUMERICAL EXAMPLE ON CABLE

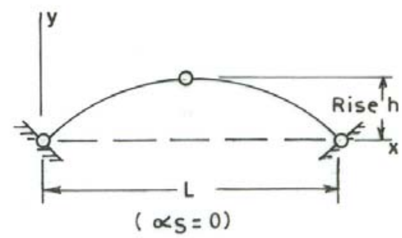


FIG.10·7. THREE HINGED ARCH

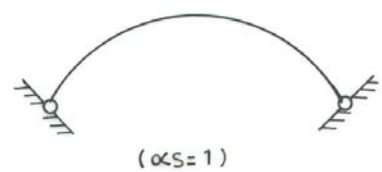


FIG.10·8. TWO HINGED ARCH

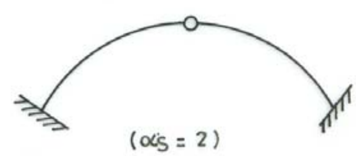


FIG. 10·9. SINGLE HINGED ARCH

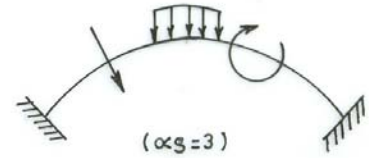


FIG. 10·10. FIXED ARCH

For flat parabolic curves $\frac{h}{L} \leq \frac{1}{10}$, only two terms are retained.

$$S = L\left[1 + \frac{8}{3}\quad\frac{h^2}{L^2}\right]$$

10.5 Example

A flexible cable weighing 1 N/m horizontally is suspended over a span of 40 m as shown in Fig.10.6. It carries a concentrated load of 300 N at point P at horizontal coordinate 10 m from left hand support. Find dip at P so that tension in cable does not exceed 1000 N.

$$R_A = \frac{(300 \text{ x } 30 + 1 \text{ x } 40 \text{ x } 20)}{40} = 245 \text{ N}$$

R_B = (300 + 40 x 1) – 245 = 95 N

R_A + R_B = 340 N = Total vertical load (ok)

Since $R_B < R_A$, maximum tension will occur at A.

$$\sqrt{H^2 + 245^2} = 1000$$

$H^2 = 10^6 - 245^2 = 939975$, H = 969.523 (Rounded to 970)

H = 970 N

Considering segment left of P, the clockwise moment at P is computed and set to zero since cable is flexible.

$M_P = R_A$ x 10 – Hh – 10 x 5 = 2450 – 970 h – 50 = 2400 – 970 h = 0

$$h = \frac{2400}{970} = 2.474 \text{ m.}$$

10.6. Arches

An arch is a curved beam circular or parabolic in form supported at its ends and is subjected to inplane loading. The internal forces developed in the arch are axial force, shear force and bending moment. Depending upon number of hinges the arches are classified as (1) three hinged arch (2) two hinged arch (3) single hinged arch and (4) fixed arch as shown in figures 10.7 to 10.10. A three hinged arch is statically determinate. The remaining three are statically indeterminate to first, second and third degree respectively. Here only determinate three hinged arch will be considered.

Arch under vertical point loads shown in figure 10.11 is a three hinged arch subjected to vertical loads W_1 to W_n. The reactions developed at the supports are shown. It may be noted that moment at the hinge at C in the arch is zero, hence, horizontal component of reaction can be computed from this condition.

$$R_A = \frac{b\Sigma W}{L},\ R_B = \frac{a\Sigma W}{L},\ \Sigma W = W_1 + \ldots.. + W_n$$

$$\sum_C M = R_A \frac{L}{2} - Hh$$

$$H = \frac{\left(\frac{b\Sigma W}{L} - \sum_C M\right)}{h}$$

$\sum_C M$ = Counter clockwise moment about C of all applied vertical loads acting left of C

ΣW = Resultant of all applied vertical loads acting downwards

At any point P on the arch as shown in Fig.10.12, the internal forces F_x,F_y and M_z can easily be computed as explained previously. From F_x and F_y shear force and thrust in the arch can be computed.

θ = Slope of arch axis at P.

V = Shear at P

C = Thrust at P

M = Bending moment at P

$M = M_z$

$V = F_y \cos\theta - F_x \sin\theta$

$C = - F_y \sin\theta - F_x \cos\theta$

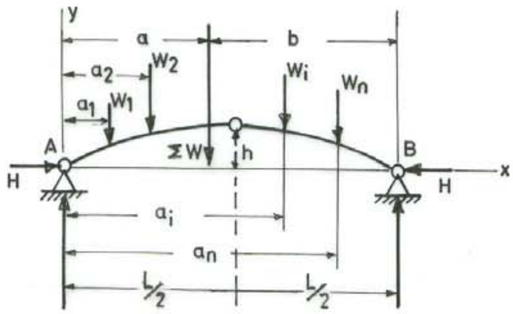


FIG. 10·11. THREE HINGED ARCH UNDER POINT LOADS

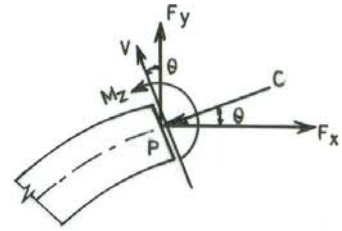


FIG. 10·12. FORCES AT A SECTION OF ARCH

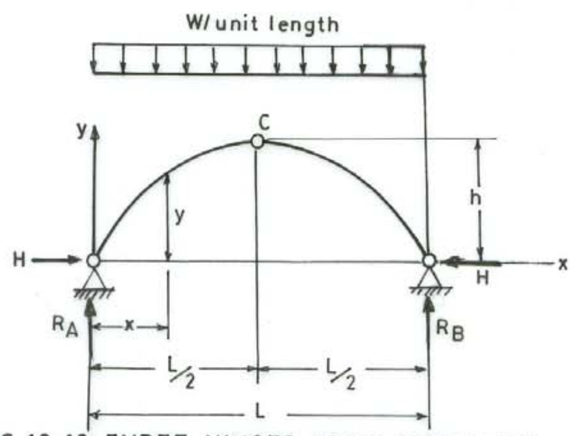


FIG. 10·13. THREE HINGED ARCH UNDER UDL

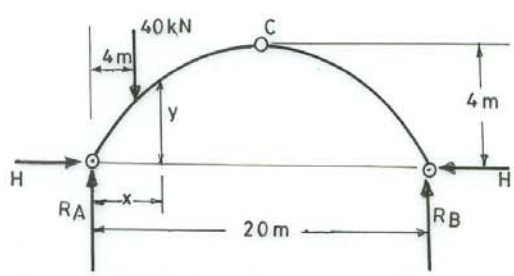


FIG. 10·14. THREE HINGED ARCH UNDER POINT LOAD

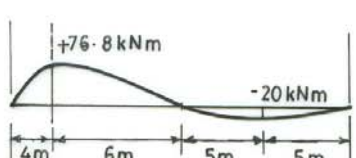


FIG. 10·15. BENDING MOMENT DIAGRAM

10.7. Three hinged parabolic arch under udl

The three hinged arch under udl is shown in Fig.10.13.

The equation of axis of arch is:

$$y = \frac{4h}{L^2}\, x\,(L - x)$$

$$R_A = R_B = \frac{\omega L}{2}$$

By the condition that moment is zero at C:

$$R_A \frac{L}{2} - Hh - \frac{\omega L}{2}\,\frac{L}{4} = 0$$

$$H = \frac{1}{h}\left[-\frac{\omega L^2}{8} + \frac{\omega L^2}{4}\right] = \frac{\omega L^2}{8h}$$

Consider a section P having coordinates (x,y).

$$M_x = \frac{\omega L}{2}\, x - \left(\frac{\omega L^2}{8h}\right) y - \frac{\omega x^2}{2}$$

$$= \frac{\omega L}{2}\, x - \frac{\omega x^2}{2} - \frac{\omega L^2}{8h}\left[\frac{4h}{L^2}\, x\,(L - x)\right]$$

$$= \frac{\omega L}{2}\, x - \frac{\omega x^2}{2} - \frac{\omega L x}{2} + \frac{\omega x^2}{2} = 0$$

The bending moment in parabolic arch under vertical udl is zero.

10.8. Example

A three hinged parabolic arch of 20 m span and 4 m central rise as shown in Fig.10.14 carries a point load of 40 kN at 4 m horizontally from left support. Compute BM, SF and AF at load point. Also determine maximum positive and negative bending moments in the arch and plot the bending moment diagram.

$$y = \frac{4h}{L^2}\ x\ (L - x) = \frac{4 \text{ x } 4}{400}\ x\ (20 - x)$$

$$y = \frac{x}{25}\ (20 - x)$$

$$R_B = \frac{4}{20} \text{ x } 40 = 8 \text{ kN}, R_A = \frac{16}{20} \text{ x } 40 = 32 \text{ kN}$$

$M_C = 0$, $4H = 32 \text{ x } 10 - 40 \text{ x } 6 = 80$, $H = 20$ kN

$0 \leq x \leq 4$ m

$$M_x = 32\ x - 20\ \frac{x}{25}\ (20 - x) = 16\ x + \frac{4}{5}\ x^2$$

$$x = 4, M_x = 16 \text{ x } 4 + \frac{4 \text{ x } 16}{5} = 76.8 \text{ kNm}$$

$4 \text{ m} \leq x \leq 20$ m

$$M_x = 32\ x - 20\ \frac{x}{25}\ (20 - x) - 40\ (x - 4) = 160 - 24\ x + \frac{4}{5}\ x^2$$

$x = 4$, $M_x = 76.8$ kNm (check)

$x = 10$, $M_x = 160 - 240 + 80 = 0$

$$x = 15, M_x = 160 - 24 \text{ x } 15 + \frac{4}{5} \text{ x } 225 = -20 \text{ kNm}$$

$x = 20$, $M_x = 0$ (ok)

The bending moment diagram is parabolic as shown in Fig.10.15.

11. INFLUENCE LINES FOR DETERMINATE STRUCTURES

An influence line is a graph or curve showing the variation of any function such as reaction, bending moment, shearing farce, axial force, torsion moment, stress or stress resultant and displacement at a given section or point of structure, as a unit load acting parallel to a given direction crosses the structure. The influence line gives the value of the function at only one point or section of the structure and at no other point. A separate influence line is to be drawn for the function at any other point.

There are two methods of construction of influence lines for determinate and indeterminate structures.

1) Direct construction of influence lines by analytical method.

2) Construction of influence lines as deflection curves by Muller-Breslau's principle.

11.1. Direct analytical method

In the direct method, first response function and its sign convention are identified. Conventional free body and equilibrium are used to obtain the value of response function for a number of positions of unit load placed along the axis of members of structure. The response function values are plotted as influence line curve. The response function can also be expressed as function of coordinate x measured from a reference point for various segments of structure and then plotted as IL.

11.2. Examples of direct method

Influence lines for simply supported beam

For simply supported beam AB of span L shown in Fig.11.1, the IL diagrams for reactions and bending moment and shear force at section X are plotted as the vertical unit load rolls from A to B along the axis of beam.

Influence lines for support reactions

A vertical unit load at coordinate x from support a is considered as shown in Fig.11.2.

$$R_B = \frac{x}{L},\ R_A = \frac{(L - x)}{L}$$

The above equations are for straight line hence, IL will also be a straight line

$x = 0$, $R_A = 1$, $R_B = 0$

$x = L$, $R_A = 0$, $R_B = 1$

If a horizontal unit force moves along axis of member, the horizontal reaction H at the hinge will be unity. Consequently the IL diagram will be a rectangle with ordinate unity.

$x = 0$ to L, $H = 1$

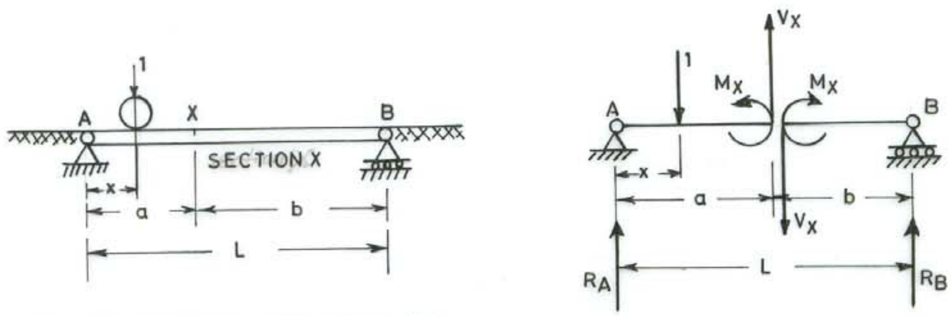


FIG. 11·1. ROLLING UNIT LOAD ON SIMPLY SUPPORTED BEAM

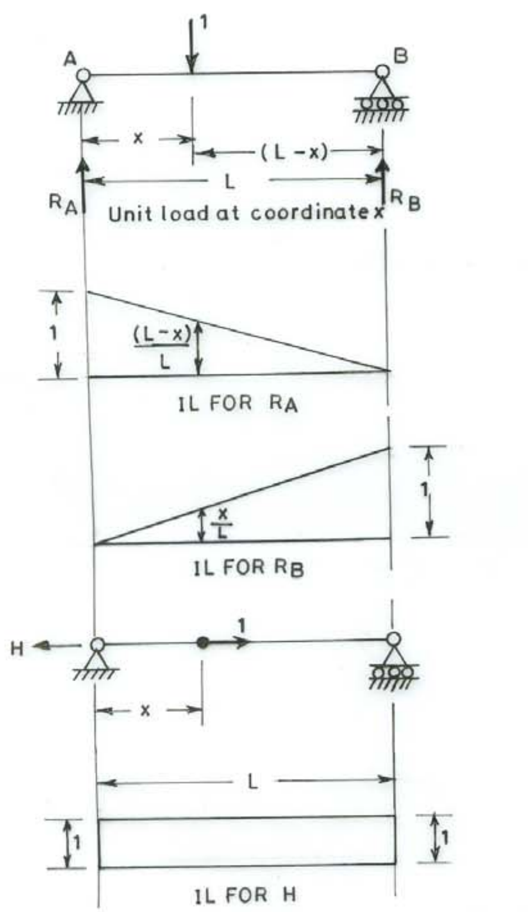


FIG. 11·2 IL FOR SUPPORT REACTIONS

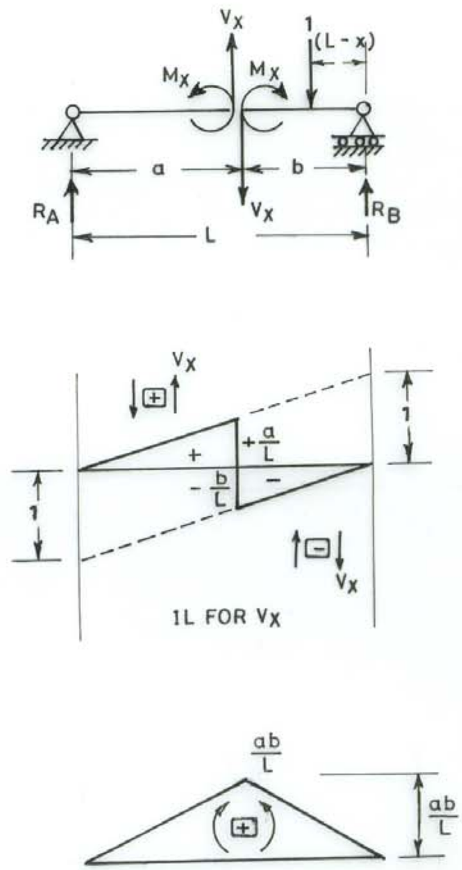


FIG. 11·3. IL FOR INTERNAL FORCES AT A SECTION

The directions identified for R_A and R_B are vertical upwards and direction identified for H is horizontal to left.

<u>Influence lines for BM and SF at a section</u>

The directions of internal forces V_x and M_x at section X are identified as shown in figure 11.3. Unit load is placed at coordinate x.

$0 \leq x \leq a$

$$R_B = \frac{x}{L},\ V_x = \frac{x}{L},\ M_x = \frac{bx}{L}$$

$a \leq x \leq L$

$$R_A = \frac{(L - x)}{L},\ V_x = \frac{(x - L)}{L},\ M_x = \frac{a(L - x)}{L}$$

$x = 0,\ V_x = 0,\ M_x = 0$

$x = a,\ V_x = \frac{a}{L},\ M_x = \frac{ab}{L}$ (load just to left of X)

$x = a,\ V_x = \frac{-b}{L},\ M_x = \frac{ab}{L}$ (load just to right of X)

$x = L,\ V_x = 0,\ M_x = 0$

SF is positive when load is just to leave of section X and it is negative when it is just to the right of section. The BM is positive for all positions of load.

Influence lines for a determinate truss

A four panel truss of span L and height h is shown in figure 11.4. Length of each panel is a. It is required to plot influence lines for forces in members 1,2 and 3 as a unit load moves along bottom chord from A to B.

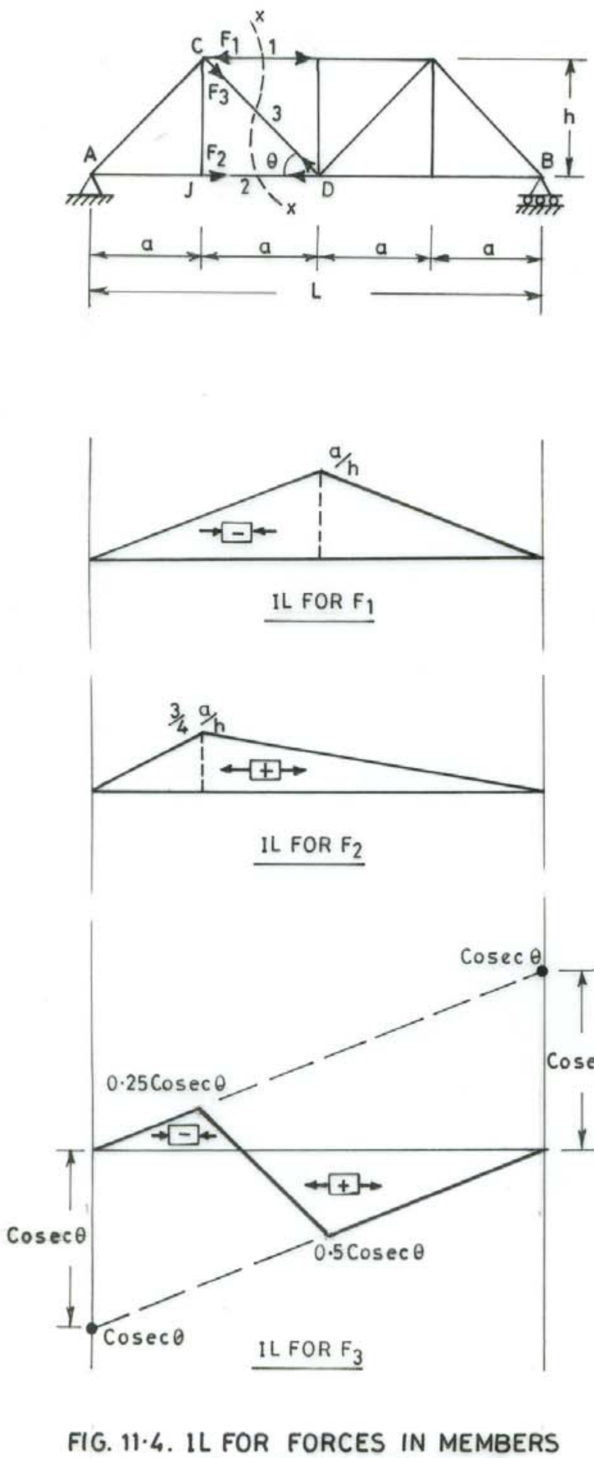


FIG. 11·4. IL FOR FORCES IN MEMBERS OF A DETERMINATE TRUSS

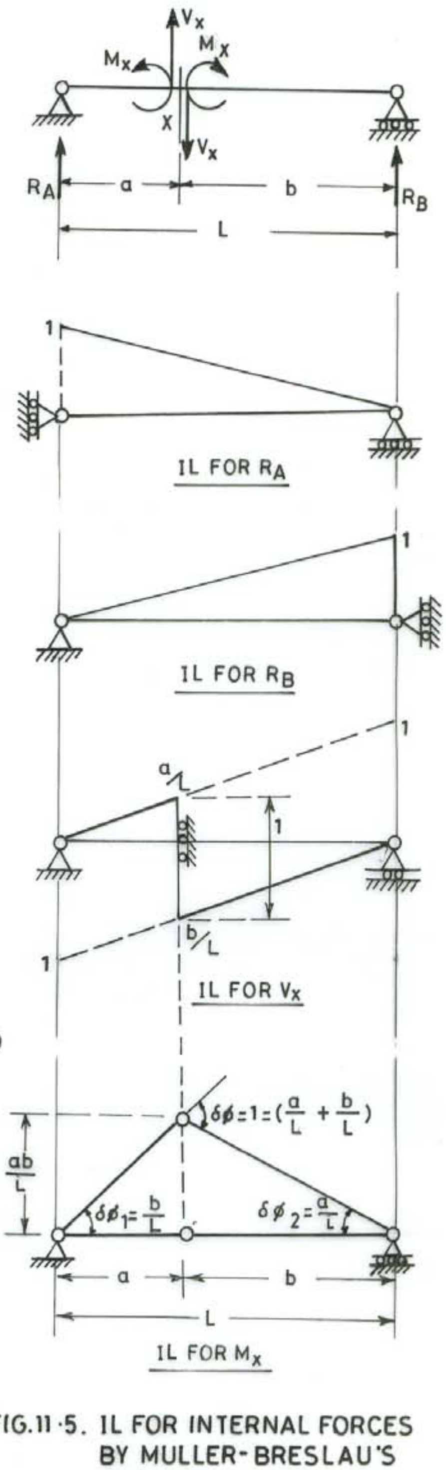


FIG. 11·5. IL FOR INTERNAL FORCES BY MULLER-BRESLAU'S PRINCIPLE

I L for F_1

For any position of load:

$$F_1 = \frac{M_D}{h} \text{ (compressive)}$$

M_D = BM at joint D.

Since height of truss is constant the IL for F_1 is obtained by drawing the IL for moment at D and dividing its ordinates by h. The IL will be triangle with ordinate at D equal to

$$\frac{(2a)(2a)}{hL} = \frac{a}{h}.$$

IL for F_2

Considering equilibrium of left segment about point C:

$$F_2 = \frac{M_C}{h} \text{ (Tensile)}$$

IL for moment at C is a triangle with ordinate $\frac{a(3a)}{4ah} = \frac{3a}{4h}$

IL for F_3

The vertical equilibrium of the parts of the truss on either side of the section xx requires that the vertical component of force F_3 should balance whatever forces may be imposed on these parts that is $\sum V=0$.

Unit load left of joint J

$$F_3 \sin\theta + R_B = 0, \; \sin\theta = \frac{h}{\sqrt{a^2 + h^2}}$$

$$F_3 = -R_B \operatorname{cosec}\theta, \; \operatorname{cosec}\theta = \frac{\sqrt{a^2 + h^2}}{h}$$

The negative sign indicates that the actual force in member 3 in compressive so long as load is to the left of joint J. The IL in this region may therefore be drawn by drawing IL for R_B and multiplying the ordinates by cosec θ.

Unit load to the right of joint D

$F_3 \sin\theta - R_A = 0,\ F_3 = R_A \operatorname{cosec}\theta$

The positive sign indicates that the force in member is tensile so long as the load is right of D. The IL in this region is drawn by plotting the IL for R_A and then multiplying the ordinates by cosec θ.

Unit load between joints J and D

The variation is linear. In fact the IL for diagonal member is proportional to the IL for the shear in panel.

Unit load at J

$R_B = \frac{1}{4},\ F_3 = -\frac{1}{4} \operatorname{cosec}\theta$

Unit load at D

$R_A = \frac{1}{2},\ F_3 = \frac{1}{2} \operatorname{cosec}\theta$

Influence lines by Muller-Breslau's principle

According to this principle if a unit distortion (displacement or discontinuity) corresponding to the desired function or stress resultant is introduced at the given point or section of structure while all other boundary conditions remain unchanged then the resulting elastic line or deflection curve of the structure represents the influence line for the function corresponding to the imposed displacement. An introduction of unit angular change or distortion at a section gives the IL for BM at that section. Similarly, introduction of a unit shear distortion produces deflections equal to IL ordinates for SF at that section where shear distorsion is introduced. The IL for reaction at the support is obtaining by introducing a unit displacement at this support in the direction of required reaction. The distorsion to be introduced must correspond to the type of stress resultant for which IL is sought and it should not be accompanied by any other type of distorsion at the influence section. The influence lines drawn by this method for simply supported beams are shown in figures 11.5.

12.1. Deformations

When a structure is subjected to the action of applied loads each member undergoes deformation due to which the axis of structure is deflected from its original position. The deflections also occur due to temperature variations and misfit of members. The infinitesimal element of length dx of a straight member (ds of curved member) undergoes axial, bending, shearing and torsional deformations as shown in figure 12.1 to 12.7. It is assumed that the material of member obeys Hooke's law. Small displacements are considered so that structure maintains geometry.

Axial deformation

$$\in_x dx = \frac{F_{1x} dx}{EA}$$

E = Modulus of elasticity

A = Area of cross-section of member

F_{1x} = Axial force F_1 along x-axis at coordinate x.

EA = Axial rigidity

$\in_x$ = Strain in x-direction

Axial deformation due to temperature variation ΔT will be

$$\in_x dx = \alpha \Delta T dx$$

α = Coefficient of thermal expansion

Bending deformations

Bending deformations which occur about y & z axes comprise of relative rotations of the sides of the infinitesimal element through an angle $d\theta_y$ and $d\theta_z$ respectively.

Bending about y-axis

$$d\theta_y = \frac{F_{5x} dx}{EI_y} = k_y dx$$

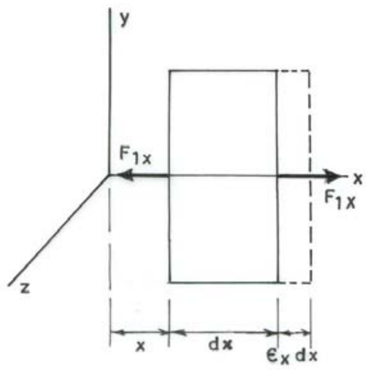


FIG. 12·1. AXIAL DEFORMATION DUE TO AF

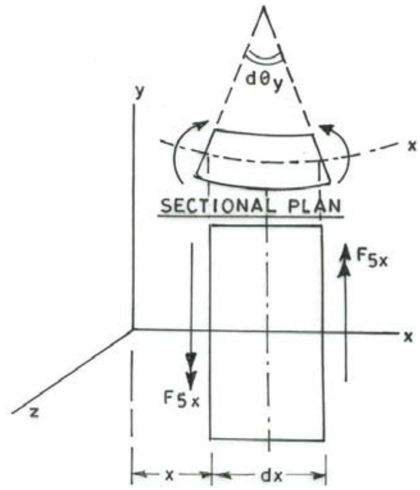


FIG. 12·2. FLEXURAL DEFORMATION IN xz-PLANE DUE TO BM ABOUT y-AXIS

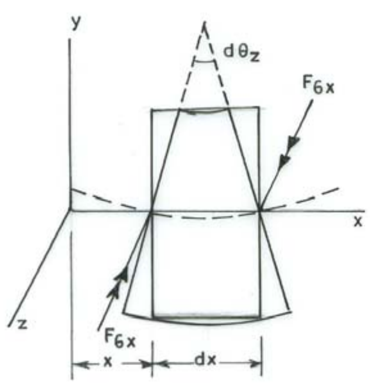


FIG. 12·3. FLEXURAL DEFORMATION IN xy-PLANE DUE TO BM ABOUT z-AXIS

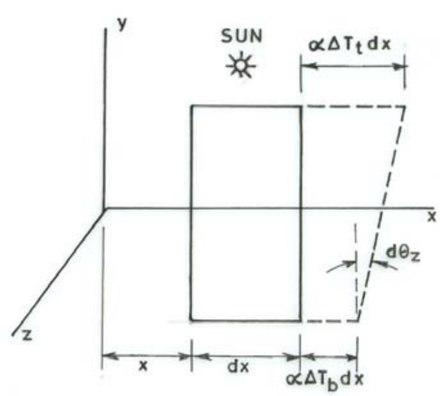


FIG. 12·4. DEFORMATION IN xy-PLANE DUE TO DIFFERENTIAL TEMPERATURE VARIATION ALONG DEPTH

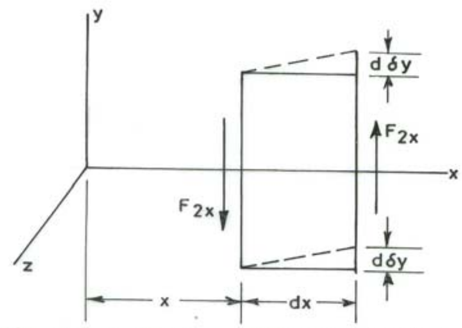


FIG. 12·5. SHEARING DEFORMATION IN xy - PLANE DUE TO SF IN y - DIRECTION

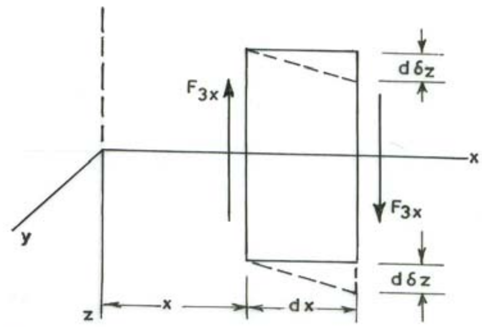


FIG. 12·6. SHEARING DEFORMATION IN xz - PLANE DUE TO SF IN z - DIRECTION

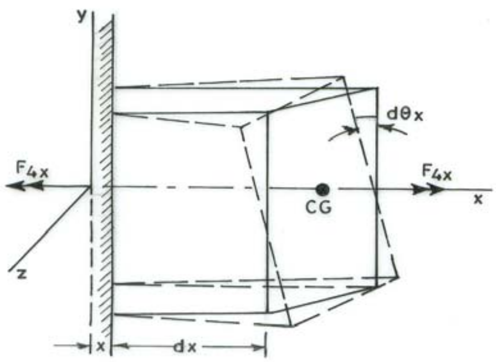


FIG. 12·7. TORSIONAL DEFORMATION DUE TO TM ABOUT x - AXIS

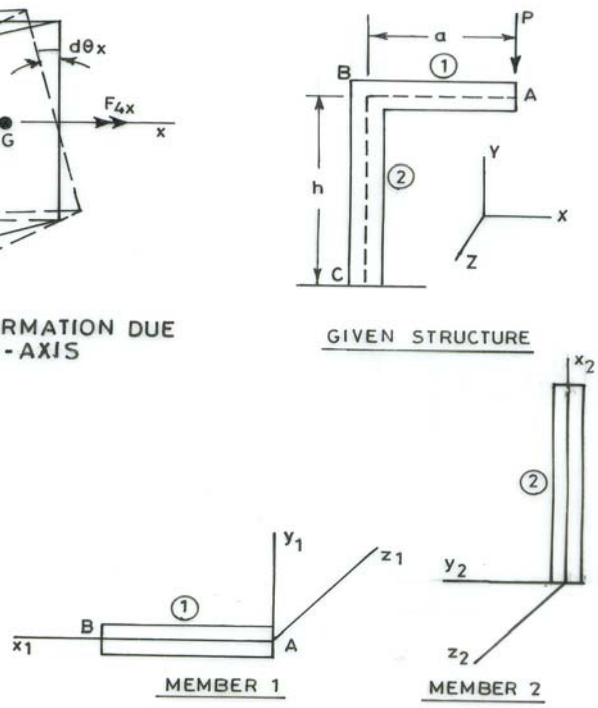


FIG. 12·8. GIVEN STRUCTURE AND ITS MEMBERS 1 AND 2

$d\theta_y$ = Angle change in radians due to bending moment about y-axis

F_{5x} = Bending moment (M_y) on element about y-axis at coordinate x.

I_y = Moment of inertia of cross section about its principal y-axis

EI_y = Flexural rigidity of member with respect to y-axis.

$k_y = \dfrac{F_{5x}}{EI_y}$ = Elastic curvature of axis of member in xz-plane.

Bending about z-axis

$$d\theta_z = \frac{F_{6x}dx}{EI_z} = k_z dx$$

$d\theta_z$ = Angle change in radians due to bending moment about z-axis

F_{6x} = Bending moment (M_z) on element about z-axis at coordinate x.

I_z = Moment of inertia of cross-section about its principal z-axis.

EI_z = Flexural rigidity of member with respect to z-axis.

$k_z = \dfrac{F_{6x}}{EI_z}$ = Elastic curvature of axis of member in xy-plane.

If the element as shown in Fig.12.4. is subjected to linear temperature change from ΔT_t at top to ΔT_b at bottom, the angle change $d\theta_z$ due to this effect will be

$$d\theta_z = \frac{\propto (\Delta T_t - \Delta T_b)dx}{d} = \frac{\propto \Delta(T_t - T_b)dx}{d}$$

d = depth of member

Shearing deformations

The deformations $d\delta_y$ and $d\delta_z$ due to shearing forces consist of displacement $d\delta$ of one side of element with respect to other with respect to y and z directions.

$$d\delta_y = \frac{F_{2x}dx}{GA}\ \mu_y$$

$$d\delta_z = \frac{F_{3x}dx}{GA}\ \mu_z$$

F_{2x} = Shear force in y-direction at coordinate x

F_{3x} = Shear force in z-direction at coordinate x

G = Shear modulus

GA = Shear rigidity

μ_y, μ_z = nondimensional factors depending solely upon the shape and size of cross-section which accounts for the nonuniform distribution of shearing stresses. For rectangular section μ = 1.2 and for circular section μ = 1.11. For I or H sections $\frac{A}{\mu}$ can be taken as web area or in other words μ can be taken as ratio of area of cross-section to web area $\left(\mu = \frac{A}{A_w}\right)$.

Torsional deformation

It is given by angle of twist $d\theta_x$ in radians, which represents the difference in the angles of rotation of its faces about longitudinal axis of the element.

$$d\theta_x = \frac{F_{4x}dx}{GI_x}$$

I_x = Torsion constant or polar moment of inertia of cross section ($I_x = I_y + I_z$).

GI_x = Torsional rigidity

F_{4x} = Torsion moment (M_x).

12.2. Elastic energy of deformation

The elastic or potential energy of member of length L is given by following expression:

$$U^r = \int_o^L \left(\frac{F_{1x}^2}{2EA} + \frac{\mu_y F_{2x}^2}{2GA} + \frac{\mu_z F_{3x}^2}{2GA} + \frac{F_{4x}^2}{2GI_x} + \frac{F_{5x}^2}{2EI_y} + \frac{F_{6x}^2}{2EI_z} \right) dx$$

F_{ex} (e = 1,…,6) = components of vector of internal stress-resultants $\{F_x\}$ at an arbitrary section of member at a coordinate distance x from reference end.

U^r = Elastic strain or potential energy of member number r.

$U = \sum_r U^r$ = Elastic strain or potential energy of all members of structure.

The energy due to shear is neglected being very small compared to that due to other actions.

$$U^r = \int_o^L \left(\frac{F_{1x}^2}{2EA} + \frac{F_{4x}^2}{2GI_x} + \frac{F_{5x}^2}{2EI_y} + \frac{F_{6x}^2}{2EI_z} \right) dx$$

In plane frames subjected to inplane loading primary action is bending moment only hence energy due to axial force is neglected.

Considering only the relevant primary actions the elastic strain energy for various structures is given by following expressions.

<u>Axial force structures (plane and space trusses)</u>

$$U = \sum_r U^r = \sum_r \int_o^L \frac{F_{1x}^2}{2EA} dx$$

<u>Plane frames in xy-plane subjected to in plane loading</u>

$$U = \sum_r U^r = \sum_r \int_o^L \frac{F_{6x}^2 dx}{2EI_z}$$

Plane grids in xz-plane subjected to normal loading.

$$U = \sum_r U^r = \sum_r \int_o^L \left(\frac{F_{4x}^2}{2GI_x} + \frac{F_{6x}^2}{2EI_z} \right) dx$$

Space frames subjected to general loading

$$U = \sum_r U^r = \sum_r \int_o^L \left(\frac{F_{1x}^2}{2EA} + \frac{F_{4x}^2}{2GI_x} + \frac{F_{5x}^2}{2EI_y} + \frac{F_{6x}^2}{2EI_z} \right) dx$$

There are various methods developed for computation of displacements depending upon structural system and nature of loading but basic methods are based on energy principles such as Castigliano's theorem and virtual work.

12.3 Castigliano's theorem

The partial derivative of elastic strain energy U of the structure with respect to any external load P is equal to displacement δ in the structure corresponding to that force. The terms force and displacement are used in the generalized sense that is word force may mean force or moment and word displacement may mean linear or angular displacement. Strain energy U is function of P.

$$\delta = \frac{\partial U}{\partial P}$$

If the deflection is required at a point where there is no load, a load P is placed there and in the expression for partial derivative of elastic energy P is set equal to zero. If the deflection is required in the direction of a particular defined load the load is replaced with P and finally P is set equal to prescribed value.

The deflection calculations are some what simplified if the partial derivatives are worked out before integration.

$$\delta = \frac{\partial U}{\partial P} = \sum_r \left[\frac{\int_L F_{1x} (\partial F_{1x} / \partial P) dx}{EA} + \frac{\int_L F_{4x} (\partial F_{4x} / \partial P)}{GI_x} dx + \frac{\int_L F_{5x} (\partial F_{5x} / \partial P)}{EI_y} dx + \frac{\int_L F_{6x} (\partial F_{6x} / \partial P)}{EI_z} dx \right]$$

The derivatives represent the rate of change of forces F_{ex} with respect to P.

$$f_{1x} = \frac{\partial F_{1x}}{\partial P}, f_{4x} = \frac{\partial F_{4x}}{\partial P}, f_{5x} = \frac{\partial F_{5x}}{\partial P}, f_{6x} = \frac{\partial F_{6x}}{\partial P}$$

These derivatives f are equal to values F_{ex} as caused by a unit load (P=1) and are represented by f_{ex}.

$$\delta = \sum_r \left[\frac{\int_L F_{1x} f_{1x}}{EA} dx + \frac{\int_L F_{4x} f_{4x}}{GI_x} dx + \frac{\int_L F_{5x} f_{5x}}{EI_y} dx + \frac{\int_L F_{6x} f_{6x}}{EI_z} dx \right]$$

12.4. Dummy load method

The equations of dummy load method are derived from principle of virtual work, hence, it is also called virtual work method. It is also called Maxwell-Mohr method.

In this method two systems of loading of same structure are considered.

System 1: Given structure, loading, temperature variations and misfits of parts

System 2: Same structure subjected to unit action corresponding to desired displacement. This action can be unit point load or unit moment or unit pair of opposite forces or moments.

The opposing pair of dummy unit loads is deployed to obtain relative displacement or rotation of two points on the structure.

According to principle of virtual work if the second system is given a small displacement the total work of the forces will be zero.

At a point represented by local coordinate x in a member the internal stress resultants will be f_{ex} (e = 1,….,6) due to dummy unit action. The virtual displacements of the second system are taken as the actual displacements of the first system. Then in accordance with the principle of virtual work.

$$1 x \delta = \sum_r \left[\int_L \frac{F_{1x} f_{1x}}{EA} dx + \int_L \frac{F_{4x} f_{4x}}{GI_x} dx + \int_L \frac{F_{5x} f_{5x}}{EI_y} dx + \int_L \frac{F_{6x} f_{6x}}{EI_z} dx \right]$$

It will be observed that the deflection calculations using Castigliano's theorem are the same as in dummy load method.

12.5. Numerical examples

Example

Determine the horizontal and vertical deflections and the angle of rotation at free end A of cantilever bracket shown in figure 12.8 neglecting axial and shear deformations. The members AB and BC have flexural rigidity EI and axial rigidity EA. Determine additional deflection at A if axial deformations are considered.

The structural system comprises of two members. The members are numbered 1 and 2. The local axes xyz of members are shown. The common axes system for whole structure is XYZ. The bending moment diagrams due to given loading, unit horizontal and vertical and unit couple at A are shown in Fig.12.9. The displacements are computed in system coordinates XYZ.

Vertical deflection at A

$$\delta_Y = -\sum_r \int_L \frac{F_{6x} f_{6x}}{EI} dx$$

$$r = 1: \int_o^a \frac{Px\,x}{EI} dx = \frac{Pa^3}{3EI}$$

$$r = 2: \int_o^h \frac{Pa\,a}{EI} dx = \frac{Pa^2 h}{EI}$$

$$\delta_Y = -\left[\frac{Pa^2 h}{EI} + \frac{Pa^3}{3EI}\right]$$

Horizontal deflection at A

$$\delta_X = +\sum_r \frac{F_{6x} f_{6x}}{EI} dx$$

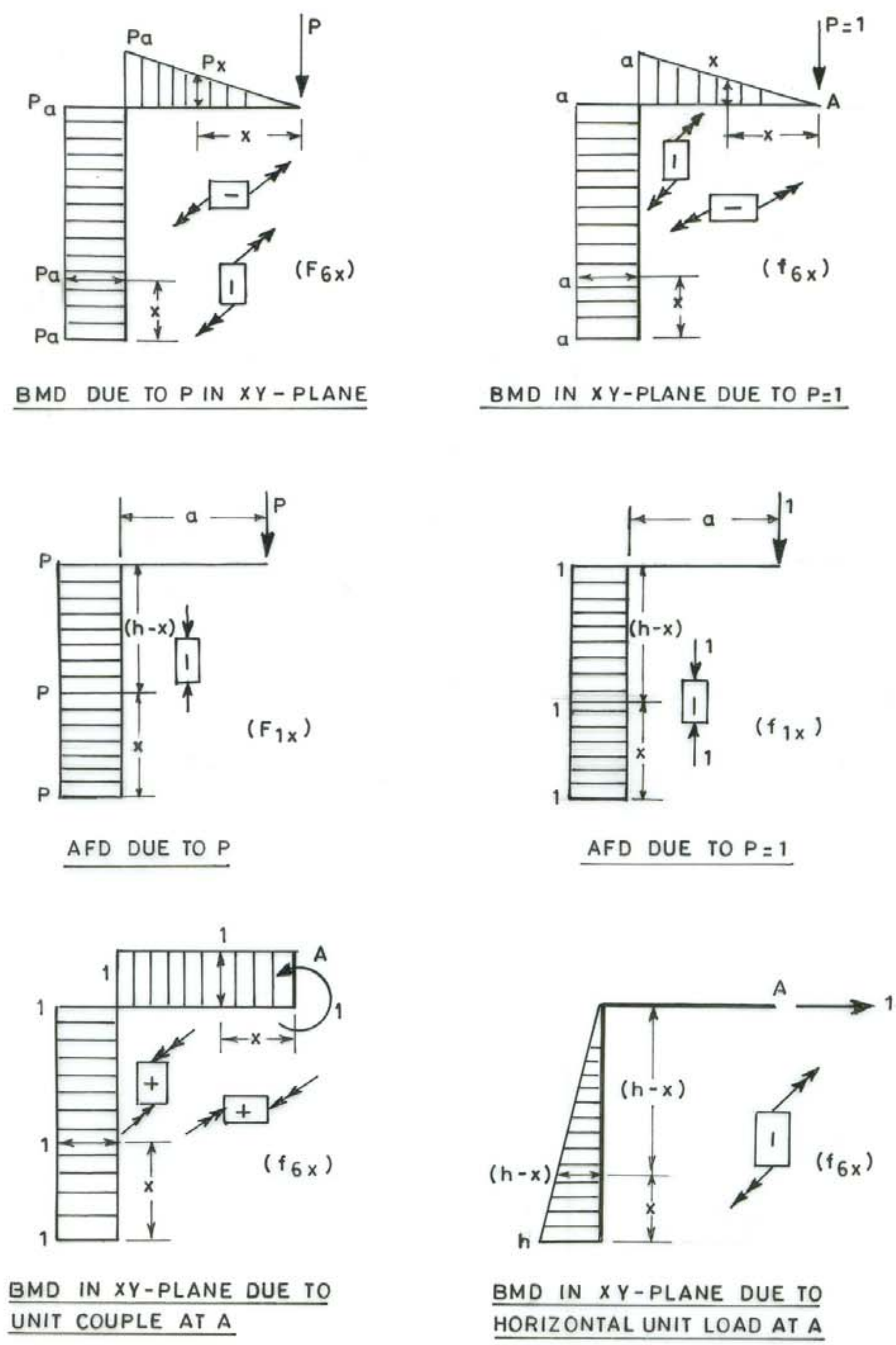


FIG. 12·9. INTERNAL FORCE DIAGRAMS FOR GIVEN AND UNIT LOADS r = 1: $\int_o^a \frac{Px\ x\ 0}{EI} dx = 0$

r = 2: $\int_o^h \frac{Pa(h - x)}{EI} dx = \frac{Pa\ h^2}{EI} - \frac{Pah^2}{2EI} = \frac{Pah^2}{2EI}$

$$\delta_X = \frac{Pah^2}{2EI}$$

Rotation at A

$$\theta_Z = -\sum_r \frac{F_{6x} f_{6x}}{EI} dx$$

r = 1: $\int_o^a \frac{Px\ x\ 1}{EI} dx = \frac{Pa^2}{2EI}$

r = 2: $\int_o^h \frac{Pa\ x\ 1}{EI} dx = \frac{Pah}{EI}$

$$\theta_Z = -\left[\frac{Pa^2}{2EI} + \frac{Pah}{EI}\right]$$

Effect of axial forces

$$\delta_Y = -\sum_r \frac{F_{1x} f_{1x}}{EA} dx$$

r = 2: $\int_o^h \frac{P\ x\ 1}{EA} dx = \frac{Ph}{EA}$

$$\delta_Y = -\frac{ph}{EA}$$

Example

Determine horizontal and vertical displacement of point C and horizontal movement of roller of plane truss shown in figure 12.10.

$A_1 = A_2 = 150 \text{ mm}^2$, $A_3 = 100 \text{ mm}^2$, $E = 200 \text{ kN/mm}^2$

$L_1 = L_2 = 5000$ mm, $L_3 = 8000$ mm.

Computation for given loading

$R_A = R_B = 3$ kN

$\sin\theta = \frac{3}{5}, \cos\theta = \frac{4}{5}$

$F_1 \sin\theta = 3, F_1 = \frac{3 \text{ x } 5}{3} = 5$ kN (C)

$F_2 \sin\theta = 3, F_2 = \frac{3 \text{ x } 5}{3} = 5$ kN (C)

$F_3 = F_2 \cos\theta = 5 \text{ x } \frac{4}{5} = 4$ kN (T)

Unit load at C in y-direction

$F_1 \sin\theta = \frac{1}{2}, F_1 = \frac{5}{6}$ (T)

$F_2 \sin\theta = 1, F_2 = \frac{5}{6}$ (T)

$F_3 = F_1 \cos\theta = \frac{5}{6} \text{ x } \frac{4}{5} = \frac{2}{3}$ (C)

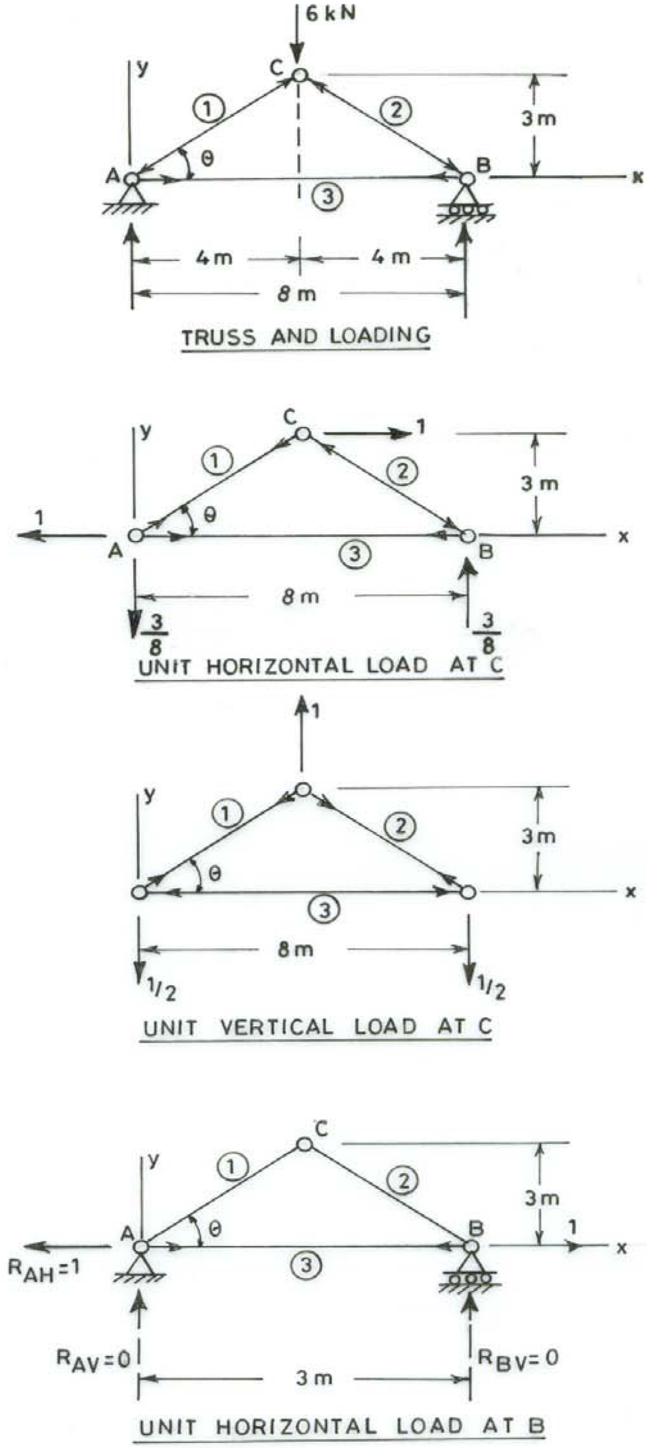


FIG. 12·10. TRUSS UNDER VARIOUS LOADS

Unit load at C in x-direction

$8\ R_B = 3 \text{ x } 1,\ R_B = \frac{3}{8} \uparrow$

$8\ R_A = 3 \text{ x } 1,\ R_A = \frac{3}{8} \downarrow$

Joint A

$F_1 \sin\theta = \frac{3}{8},\ F_1 = \frac{5}{8}$ (T)

$F_3 = 1 - \frac{5}{8} \cos\theta = 1 - \frac{1}{2} = \frac{1}{2}$ (T)

Joint B

$F_2 \sin\theta = \frac{3}{8},\ F_2 = \frac{5}{8}$ (C)

Joint C

$2 \text{ x } \frac{5}{8} \cos\theta = 1$ (check)

Unit load at B in x-direction

$R_{AV} = R_{BV} = 0,\ R_{AH} = 1$ kN

$F_1 = F_2 = 0$

$F_3 = 1$ (T)

Member	L (mm)	A (mm^2)	F_1 (kN)	$\frac{L}{AE}$	f_{1XC} (kN)	f_{1YC} (kN)	f_{1XB} (kN)	F_1f_1 XC	YC	XB
3(AB)	8000	100	+ 4	$\frac{2}{5}$	$+\frac{1}{2}$	$-\frac{2}{3}$	+ 1	+ 2	$-\frac{8}{3}$	+ 4
2 (BC)	5000	150	- 5	$\frac{1}{6}$	$-\frac{5}{8}$	$+\frac{5}{6}$	0	$+\frac{25}{8}$	$-\frac{25}{6}$	0
1 (CA)	5000	150	- 5	$\frac{1}{6}$	$+\frac{5}{8}$	$+\frac{5}{6}$	0	$-\frac{25}{8}$	$-\frac{25}{6}$	0

F_1 = Axial force in member due to given loads

f_{1XC} , f_{1YC}, f_{1XB} = Axial force in member due to unit loads in X and Y directions at point C and B.

$$\delta_{xc} = \sum_{r=1}^{3} \frac{F_1 f_1 L}{AE} = 2 \text{ x } \frac{2}{5} + \frac{25}{8} \text{ x } \frac{1}{6} - \frac{25}{8} \text{ x } \frac{1}{6} = \frac{4}{5} = 0.8 \text{ mm}$$

$$\delta_{yc} = \sum_{r=1}^{3} \frac{F_1 f_1 L}{AE} = -\frac{8}{3} \text{ x } \frac{2}{5} - \frac{25}{6} \text{ x } \frac{1}{6} - \frac{25}{6} \text{ x } \frac{1}{6} = 2.46 \text{ mm}$$

$$\delta_{XB} = \sum_{r=1}^{3} \frac{F_1 f_1 L}{AE} = 4 \text{ x } \frac{2}{5} = \text{o}.8 \text{ mm}$$

Check

Horizontal movement of roller is equal to extension of bar 3.

$$(\Delta L)_3 = \left(\frac{FL}{AE}\right)_3 = \frac{4 \text{ x } 8000}{100 \text{ x } 200} \; 1.6 \text{ mm.}$$

$$(\Delta L)_3 = \delta_{xc} + \delta_{XB} = 0.8 + 0.8 = 1.6 \text{ mm.}$$

Example

Determine displacement in the direction of load P acting at A of a cantilever bracket as shown in figure 12.11.

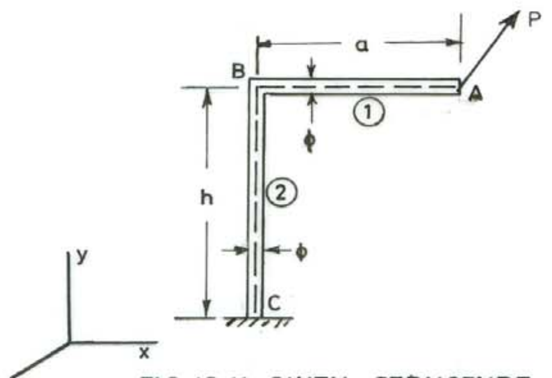
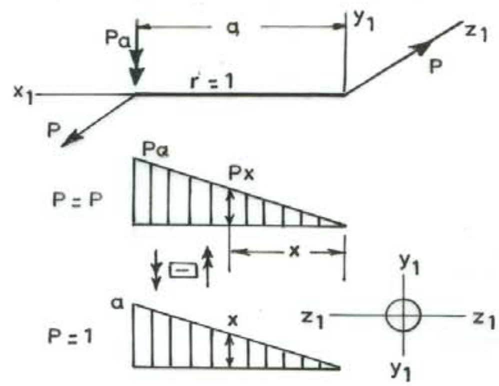
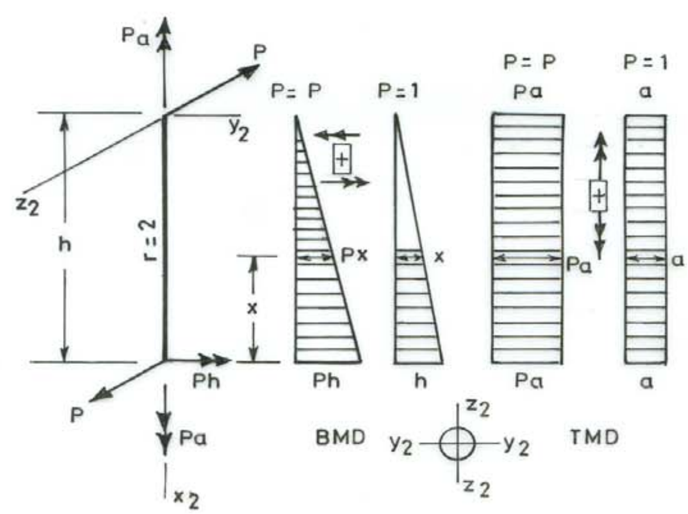


FIG. 12·11. GIVEN STRUCTURE



BMD IN xz PLANE FOR MEMBER 1



BMD AND TMD FOR MEMBER 2

FIG. 12·12. INTERNAL FORCE DIAGRAMS OF MEMBERS FOR GIVEN AND UNIT LOADS

Following numerical data is given. Bars are circular in section.

a = 100 mm, h = 200 mm, ϕ = 100 mm, E = 200 kN/mm^2, G = 70 kN/mm^2, P = 200 kN

The two members are separated as shown in Fig.12.12. Their local axes x_1, y_1, z_1 and x_2, y_2, z_2 are shown. The free bodies of the two members are prepared and BM and TM diagrams are shown for P and P = 1.

$$\delta = \sum_{r=1}^{2}\left[\frac{\int_L F_{4x} f_{4x}}{GI_x}dx + \frac{\int_L F_{5x} f_{5x}}{EI_y}dx\right]$$

For circular section

$I_x = I_y + I_z$

$I_y = I_z = \dfrac{\pi\phi^4}{64} = \dfrac{\pi \text{ x } 100^4}{64} = 4.909 \text{ x } 10^6 \text{ mm}^4$, $I_x = 9.818 \text{ x } 10^6 \text{ mm}^4$

Member no I

The member is subjected to primary action of bending moment about y_1-axis.

δ_{A1} = Deflection of A due to deformation of member 1 that is deflection of A with respect to B if B is clamped.

$$\delta_{A1} = \int_o^a \frac{Px^2}{EI_y}dx = \frac{Pa^3}{3EI_y} = \frac{200 \text{ x } 100^3}{3 \text{ x } 200 \text{ x } 4.909 \text{ x } 10^6} = 0.068 \text{ mm}$$

Member 2

The member no.2 is subjected to primary actions of bending moment about y_2-axis and torsion moment about x_2-axis.

δ_{A2} = Deflection A due to deformation of member 2.

$$\delta_{A2} = \frac{1}{EI_y}\int_o^h Px^2dx + \frac{1}{GI_x}\int_o^h Pa^2dx = \frac{Ph^3}{3EI_y} + \frac{Pa^2h}{GI_x}$$

$$\delta_{A2} = \frac{200 \text{ x } 200^3}{3 \text{ x } 200 \text{ x } 4.909 \text{ x } 10^6} + \frac{200 \text{ x } 100^2 \text{ x } 200}{70 \text{ x } 9.818 \text{ x } 10^6} = 0.543 + 0.582$$

Therefore δ_A = 0.068 + 0.543 + 0.582 = 1.193 mm

Check

θ_{BC} = Angle of twist of B with respect C = $\frac{(Pa)(h)}{GI_x} = \frac{200 \text{ x } 100 \text{ x } 200}{70 \text{ x } 9.818 \text{ x } 10^6}$ = 5.82 x 10^{-3} radians

Displacement of A due to twist = 5.82 x 10^{-3} x 100 = 0.582 mm.

13 NONPRISMATIC MEMBERS

A nonprismatic member has a variable section along its length. For any structural member, the stress resultants do not remain constant throughout its length. Where there is a significant variation of the stress resultants along the length of member, economy can be achieved more efficiently by varying the cross-sectional area of the member, keeping in view the extreme values of the stress resultants in the middle and end sections. Often such variation can be adopted to add to the architectural appearance of the structure. Generally, two types of members are used:

(1) Members with parabolic haunches or parabolic variation of depth.

(2) Members with straight haunches or with linear variation of depth.

The axis of member with haunches or with variable depth is assumed to be the same as for the uniform part usually in the central portion of the member. For a tapered member the uniform part will correspond to the minimum section.

Analysis of such members shown in Fig.13.1 to 13.4 involves the determination of fixed end reactions due to self weight and loads acting on the members and the flexibility and stiffness properties that is force-displacement relationships. For haunched members the fixed end moments, stiffness and flexibility properties are expressed in terms of integrals or bar constants. The bar constants are available in handbooks.

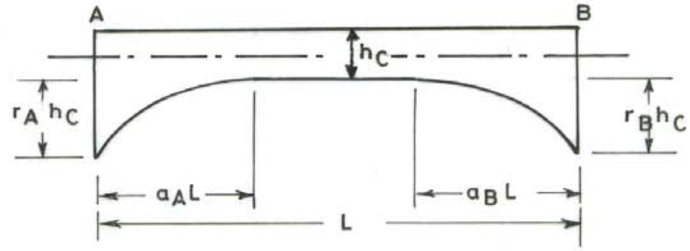


FIG. 13·1. MEMBER WITH PARABOLIC HAUNCHES

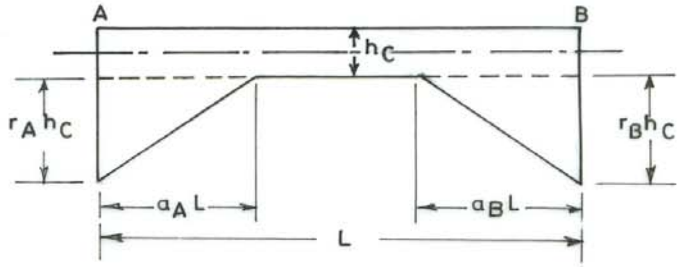


FIG. 13·2. MEMBER WITH STRAIGHT HAUNCHES

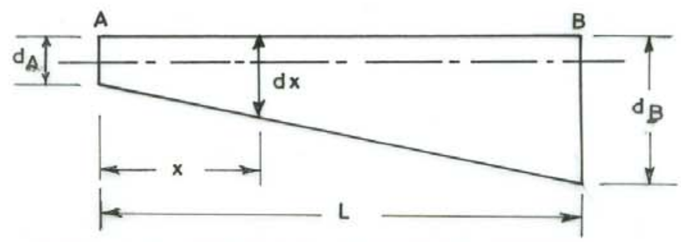


FIG. 13·3. STRAIGHT TAPERED MEMBER

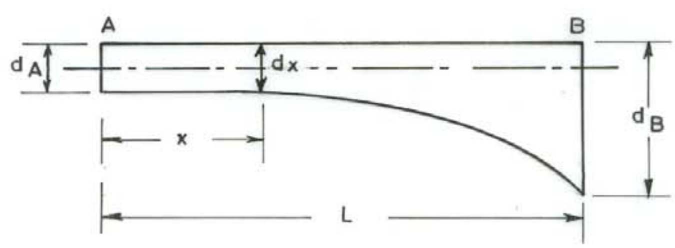


FIG. 13·4. PARABOLICALLY TAPERED MEMBER

13.1 Fixed end reactions

Fixed end reactions for bending in xy-plane.

Concentrated vertical load F and udl of intensity ω will be considered as shown in Fig. 13.5.

(1) Concentrated vertical load F acting at a distance a from left end.

$$\left(M_z\right)_A = F\frac{\int_o^L \frac{xdx}{I_z}\int_a^L \frac{x(x-a)dx}{I_z} - \int_o^L \frac{x^2dx}{I_z}\int_a^L \frac{(x-a)dx}{I_z}}{\int_o^L \frac{dx}{I_z}\int_o^L \frac{x^2dx}{I_z} - \left(\int_o^L \frac{xdx}{I_z}\right)^2}$$

(2) Uniformly distributed vertical load of intensity ω acting on whole span.

$$\left(M_z\right)_A = \frac{\omega}{2}\frac{\int_o^L \frac{xdx}{I_z}\int_o^L \frac{x^3dx}{I_z} - \left(\int_o^L \frac{x^2dx}{I_z}\right)^2}{\int_o^L \frac{dx}{I_z}\int_o^L \frac{x^2dx}{I_z} - \left(\int_o^L \frac{xdx}{I_z}\right)^2}$$

The fixed end moment at the right end B can be determined by performing the integrations in the opposite direction. The vertical reactions at the two ends can be determined from the equilibrium of the free body of the member. The case of bending in xz-plane is treated similarly.

Fixed end reactions for torsion moment

For concentrated torsion moment T acting at intermediate point of member at distance a from left end as shown in Fig.13.6, the fixed end torsional moments are determined from the condition of compatibility at the point of application of the external torsional moment. The angle of twist θ for the two parts namely the left and right cantilevered segments is the same. The left segment is shown in Fig. 13.7 and 13.8. The haunch is shown in Fig.13.9.

The angle of twist between any section on the uniform part of the haunched beam and the fixed end is given by:

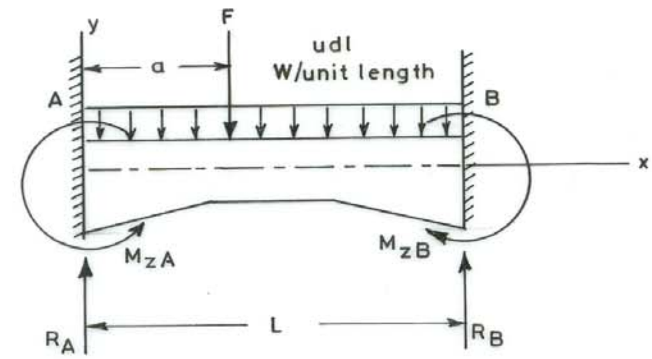


FIG. 13·5. FIXED BEAM OF VARIABLE SECTION UNDER UDL AND VERTICAL POINT LOAD

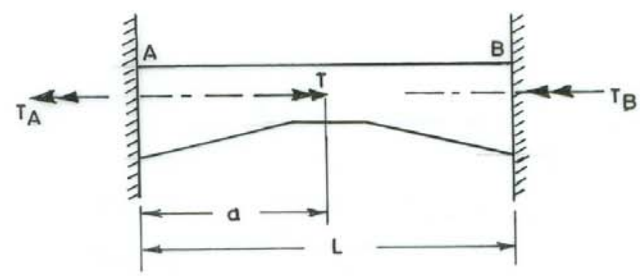


FIG. 13·6. FIXED BEAM OF VARIABLE SECTION UNDER TORSION MOMENT

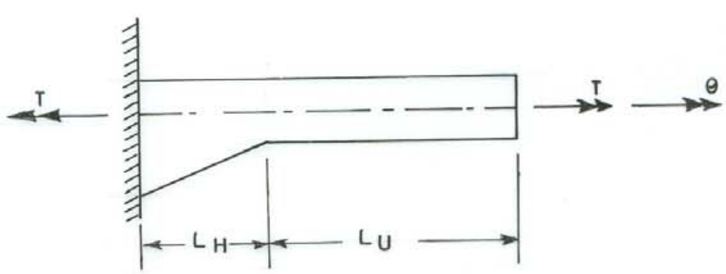


FIG. 13·7. CANTILEVER BEAM OF VARIABLE SECTION UNDER TORSION MOMENT

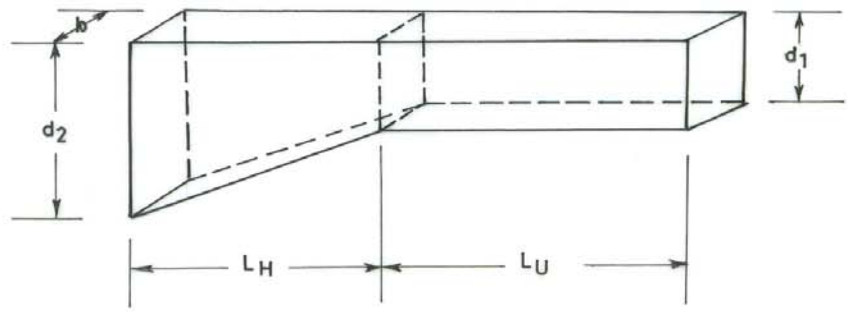


FIG. 13·8. HAUNCHED AND UNIFORM PARTS OF BEAM

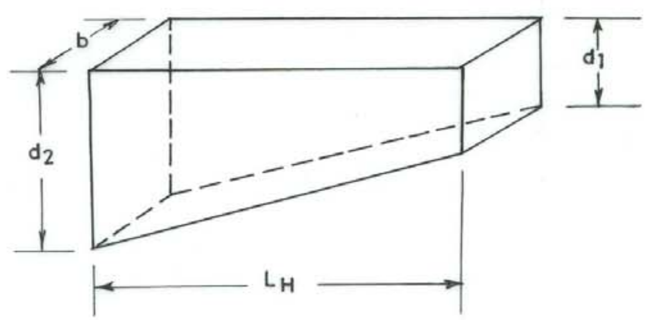


FIG. 13·9. HAUNCH OF A BEAM

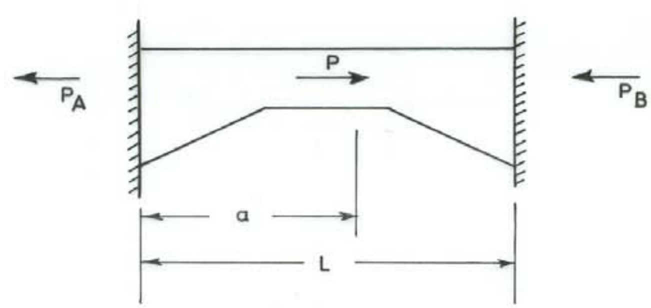


FIG. 13·10. FIXED BEAM OF VARIABLE SECTION UNDER AXIAL FORCE

$$\theta = \frac{T}{G}\left(\frac{L_H}{J_H} + \frac{L_U}{J_U}\right)$$

L_H = Length of haunched part

L_U = Length of uniform part

J_H = Torsion constant of haunched part

J_U = Torsion constant of uniform part

$$J_H = \frac{b^3(d_2 - d_1)}{3\log_e\left(\frac{d_2 - 0.63b}{d_1 - 0.63b}\right)} \text{ for } \frac{d}{b} > 1.2$$

$$J_U = \frac{b^3 d}{3}\left(1 - 0.63\frac{b}{d}\right) \text{ for } \frac{d}{b} > 1.2$$

The torsional stiffness factor K_t for a member with haunched and uniform parts is given by:

$$K_t = \frac{G}{\left(\frac{L_U}{J_U} + \frac{L_H}{J_H}\right)}$$

The fixed end torsional moments T_A and T_B are expressed as:

$$T_A = \frac{T(K_t)_A}{(K_t)_A + (K_t)_B}$$

$$T_B = \frac{T(K_t)_B}{(K_t)_A + (K_t)_B}$$

The fixed end reactions for the case of uniformly distributed torque can be worked out in similar manner.

Fixed end reactions for axial force

The fixed end reactions P_A and P_B for axial force P acting at any point on uniform part of a haunched member as shown in Fig.13.10 are given by:

$$P_A = \frac{P\left(K_F\right)_A}{\left(K_F\right)_A + \left(K_F\right)_B}$$

$$P_A = \frac{P\left(K_F\right)_B}{\left(K_F\right)_A + \left(K_F\right)_B}$$

$$K_F = \frac{E}{\left(\frac{L_U}{A_U} + \frac{L_H}{A_H}\right)}$$

$$A_H = \frac{b\left(d_2 - d_1\right)}{\log_e \frac{d_2}{d_1}}$$

The equivalent uniform area A_{eq} of member of variable section is given in terms of area A_x at coordinate x.

$$\frac{1}{A_{eq}} = \frac{\int_o^L \frac{1}{A_x}\,dx}{L}$$

$$\text{Axial rigidity} = \frac{EA_{eq}}{L}$$

The fixed end reactions for the case of uniformly distributed axial force can be worked out in a similar manner.

Like fixed end reactions the bar constants are also expressed in terms of integrals. Evaluation of integrals from first principles is cumbersome hence, use is made of hand books to save this labour.

13.2 Basic concepts and definitions of bar constants of members with variable section

Ends of member of length L are designated by A and B.

Rotational stiffness factor

The rotational stiffness factor K at one end of member, which is assumed hinged, is defined as the moment required to produce a unit rotation at this end, while the other end is assumed to be fixed as shown in Fig.13.11.

In hand books the stiffness factor K at an end of a haunched member is given as:

$$K = \frac{kEI_{min}}{L}$$

k = Stiffness coefficient

E = Modulus of elasticity

I_{min} = Minimum moment of inertia of member

L = Length of member

Carry over factor

The carry over factor C from the near end to the far end of a bar is defined as the ratio of the moment induced at far end, which is assumed fixed, to the applied moment at the near end which is assumed hinged. The product of the carry over factor and the rotational stiffness factor at one end of member AB is equal to similar product at the other end.

$$C_{AB}\,K_{AB} = C_{BA}\,K_{BA}$$

Modified rotational stiffness factor

The modified rotational stiffness factor K' at one end of the member is defined as the moment required to produce unit rotation at this end which is assumed hinged when the other end is also hinged as shown in Fig.13.12.

$$K'_{AB} = K_{AB}\,(1 - C_{AB}\,C_{BA})$$

$$K'_{BA} = K_{BA}\,(1 - C_{BA}\,C_{AB})$$

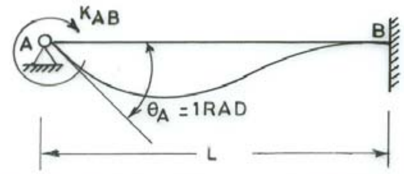


FIG. 13·11. ROTATIONAL STIFFNESS OF MEMBER

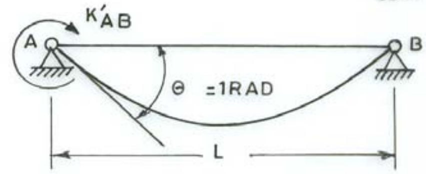


FIG. 13·12. MODIFIED ROTATIONAL STIFFNESS OF MEMBER

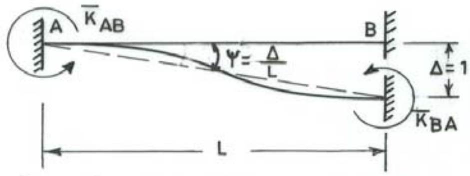


FIG. 13·13. LATERAL STIFFNESS OF MEMBER

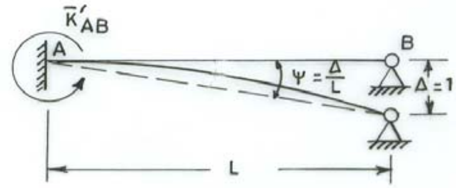


FIG. 13·14. MODIFIED LATERAL STIFFNESS OF MEMBER

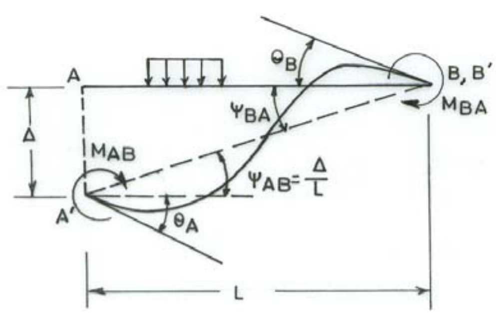


FIG. 14·1. DEFORMATION OF AN ELASTIC BAR

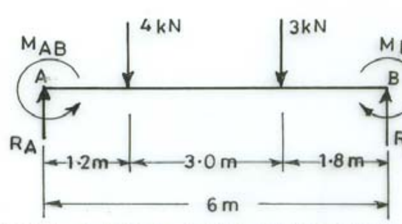


FIG. 14·3. FIXED END REACTIONS

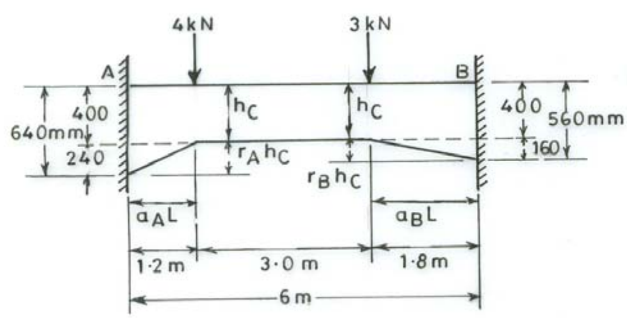


FIG. 14·2. FIXED HAUNCHED BEAM

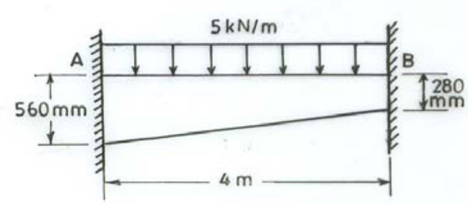


FIG. 14·4. FIXED TAPERED BEAM

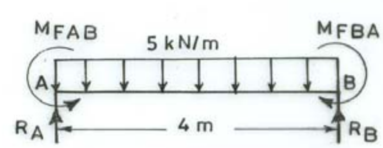


FIG. 14·5. FIXED END REACTIONS

<u>Lateral stiffness factor</u>

<u>The lateral- stiffness factor $\overline{K}$ at one end of the member, which is assumed fixed against rotation, is defined as the moment produced at this end when the other end also fixed against rotation, is displaced laterally through unit distance with respect to the first end, as shown in Fig.13.13.</u>

$$\overline{K}_{AB} = \frac{K_{AB}}{L}(1 + C_{AB})$$

$$\overline{K}_{BA} = \frac{K_{BA}}{L}(1 + C_{BA})$$

<u>Modified lateral stiffness factor</u>

The modified lateral stiffness factor $\overline{K}'$ at one end of a member, which is assumed fixed against rotation is defined as the moment produced at this end when the other end, assumed hinged, is displaced laterally through unit distance with respect to first end as shown in Fig.13.14.

$$\overline{K'}_{AB} = \frac{K_{AB}}{L}(1 - C_{AB}\, C_{BA})$$

$$\overline{K'}_{BA} = \frac{K_{BA}}{L}(1 - C_{BA}\, C_{AB})$$

14 <u>SLOPE DEFLECTION EQUATIONS</u>

<u>Sign convention</u>

Clockwise moments and rotations at the end of bar are treated positive. The lateral displacement between two ends of bar resulting in positive end moments or counter clockwise lateral displacement angle ψ is treated as positive. The shear causing clockwise moment in the bar is treated as positive. The external loads acting vertically downwards are treated positive as shown in Fig.14.1.

The force displacement relationship at the nodes of the elastic bar is defined by following slope deflection equations:

$$M_{AB} = M_{FAB} + K_{AB}\,\theta_A + C_{BA}\,K_{BA}\,\theta_B + \frac{\Delta}{L}\,K_{AB}(1 + C_{AB})$$

$$M_{BA} = M_{FBA} + K_{BA}\,\theta_B + C_{AB}\,K_{AB}\,\theta_A + \frac{\Delta}{L}\,K_{BA}\,(1 + C_{AB})$$

M_{FAB}, M_{FBA} = Fixed end moments at ends A and B.

Designating A as near end and B as far end of member, the two slope deflection equations can be written as:

$$M_{near} = M_{Fnear} + K_{near}\,\theta_{near} + C_{far}\,K_{far}\,\theta_{far} + \frac{\Delta}{L}\,K_{near}\,(1 + C_{near})$$

For a bar of uniform section:

$$C_{near} = C_{far} = \frac{1}{2}$$

$$K_{near} = K_{far} = \frac{4EI}{L}$$

$$M_{near} = M_{Fnear} + \frac{2EI}{L}\left[2\theta_{near} + \theta_{far} + \frac{3\Delta}{L}\right]$$

Example

Find the fixed end moments and reactions for the fixed beam AB loaded as shown in figure 14.2 and due to downward sinking of support B of 10 mm. E = 15 kN/mm^2.

From handbook of bar constants the desired coefficients are taken.

$$a_A = \frac{1.2}{6} = 0.2,\ r_A = \frac{240}{400} = 0.6$$

$$a_B = \frac{1.8}{6} = 0.3,\ r_B = \frac{160}{400} = 0.4$$

<u>For 4 kN load</u>

FEM coeff at A = 0.1371

FEM coeff at B = 0.0327

<u>For 3 kN load</u>

FEM coeff at A = 0.0642

FEM coeff at B = 0.1754

$M_{FAB} = \sum (\text{FEM Coeff})_A \, WL$

$M_{FAB} = 0.137 \times 4 \times 6 + 0.0642 \times 3 \times 6 = 4.446$ kNm

$M_{FBA} = \sum (\text{FEM Coeff})_B \, WL$

$= 0.0327 \times 4 \times 6 + 0.1754 \times 3 \times 6 = 3.942$ kNm

Fixed end moments due to lateral displacement of B with respect to A.

<u>From handbook of bar constants:</u>

$k_{AB} = 6.48$, $k_{BA} = 6.68$, $C_{AB} = 0.622$, $C_{BA} = 0.604$

$$\Delta = 10 \text{ mm},\ L = 6000 \text{ mm},\ I_{min} = \frac{250 \times 400^3}{12} = 1333.333 \times 10^6 \text{ mm}^4$$

$$K_{AB} = \frac{k_{AB} EI_{min}}{L} = \frac{6.48 \times 1333.333 \times 10^6 \times 15}{6000} = 216 \times 10^5 \text{ kN mm}$$

$$M_{FAB} = \frac{\Delta}{L} K_{AB} (1 + C_{AB}) = \frac{10}{6000} \times 216 \times 10^5 (1.622) = 58{,}392 \text{ kN mm}$$

$$K_{BA} = \frac{k_{BA} EI_{min}}{L} = \frac{6.68 \times 15 \times 1333.333 \times 10^6}{6000} = 222 \times 10^5 \text{ kN mm}$$

$$M_{FBA} = \frac{\Delta}{L} K_{BA} (1 + C_{BA}) = \frac{10}{6000} \times 222 \times 10^5 \times 1.604 = 59348 \text{ kN mm}$$

Total fixed end moments due to loads and relative displacement of supports:

$M_{AB} = 4.446 + 58.392 = 62.838$ kN m

$M_{BA} = 3.942 + 59.348 = 55.406$ kN m

Reactions are computed from free body shown in Fig.14.3.

$6R_A = 62.838 – 55.406 + 3 \times 1.8 + 4 \times 4.8$

$R_A = 5.339$ kN

$6R_B = - 62.838 + 55.403 + 4 \times 1.2 + 3 \times 4.2$

$R_B = 1.661$ kN

$R_A + R_B = 7$ kN (check)

Example

A 4 m span fixed tapered beam AB of section 250 x 560 mm at A and 250 x 280 mm at B is subjected to udl of intensity 5 kN/m as shown in Fig.14.4. Using hand book of bar constants determine fixed end moments and hence reactions.

$a_A = 1.0,\ a_B = 0,\ r_A = \dfrac{280}{560} = 0.5,\ r_B = 0$

Fixed end moment coeff at A = 0.1216

Fixed end moment coeff at B = 0.0529

FEM = coeff x w x L^2

$M_{FAB} = 0.1216 \times 5 \times 4 \times 4 = 9.728$ kN m

$M_{FBA} = 0.0529 \times 5 \times 4 \times 4 = 4.232$ kN m

The reactions are computed from free body shown in Fig.14.5.

$R_A = \dfrac{1}{4}\,[9.728 + 5 \times 4 \times 2 – 4.232] = 11.374$ kN

$R_B = \dfrac{1}{4}\,[4.232 + 5 \times 4 \times 2 – 9.728] = 8.626$ kN

$R_A + R_B = 11.374 + 8.626 = 20$ kN (check)

15 MOMENT DISTRIBUTION METHOD

The slope deflection method involves solution of many simultaneous equations. The moment distribution method does not involve as many simultaneous equations. The number of equations reduces considerately and in case of structures with no lateral displacements of nodes of members there are no equations to be solved. Hence, the computational work is much less in this method compared to other methods. This method has an additional advantage as it is based on iterative technique consisting of series of cycles of clamping and unclamping of the joints of structure, each cycle converging on the precise final result. Therefore, the series of cycles can be terminated when the desired degree of accuracy of results has been achieved.

The moment at each end of a member as can be seen from slope-deflection equations comprises of four separate effects.

(1) Fixed end moment (FEM) at the end under consideration due to applied load on the member.

(2) Rotation of member at near end.

(3) Rotation of member at far end.

(4) Rotation of axis or chord of member or the lateral displacement of nodes.

In moment distribution method these effects are considered separately and results are superimposed. Consider a joint A where four members meet rigidly as shown in figure 15.1. The far ends of members are clamped. An artificial clamp is introduced at A. The structure is now kinematically determinate comprising of fixed ended members and the fixed end moments can be computed for the external loads applied to members. The degree of freedom of this structure is one that is rotation at A. Since far ends are clamps they neither rotate nor displace laterally.

If the algebraic sum of fixed end moments at A does not vanish, the resultant moment acting at the joint is termed unbalanced moment. The artificial clamp holds or balances this unbalanced moment.

If now the artificial clamp is removed the joint will rotate under the action of unbalanced moment as shown in Fig.15.2 and end moments will develop in all the members as shown in Fig.15.3. The moments developed at A due to rotation restore the equilibrium of joint and are called distributed moments. The moments developed at far ends are called carry-over-moments. Same sign convention is followed as in slope deflection method.

M = Unbalanced moment at A = $\sum_{A}(\text{FEM})$

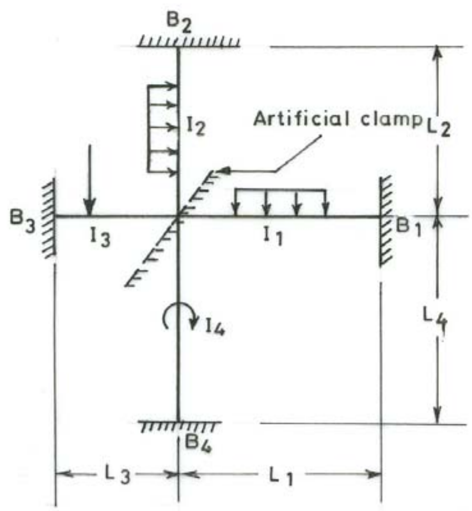


FIG. 15·1. PLANE FRAME WITH ONE DEGREE OF FREEDOM

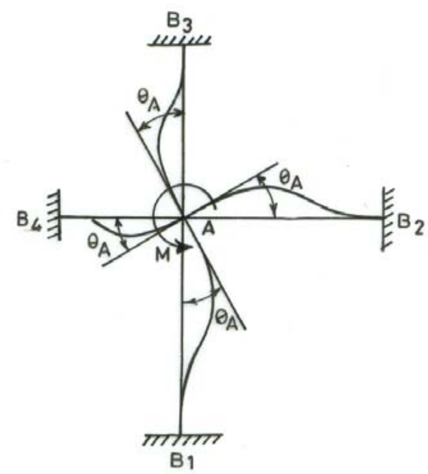


FIG. 15·2. ROTATION AT JOINT A

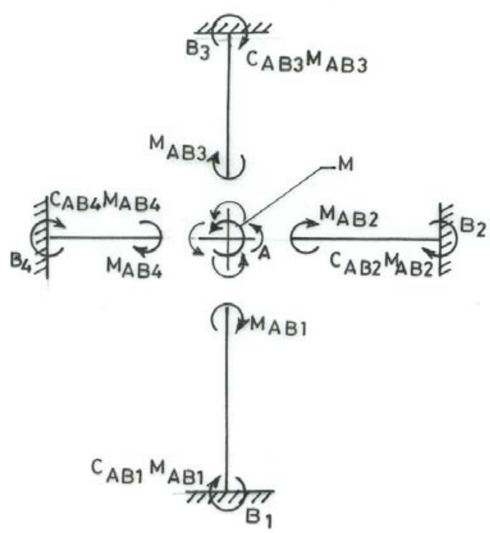


FIG. 15·3. DISTRIBUTION OF UNBALANCED JOINT MOMENT AND CARRY OVER MOMENTS

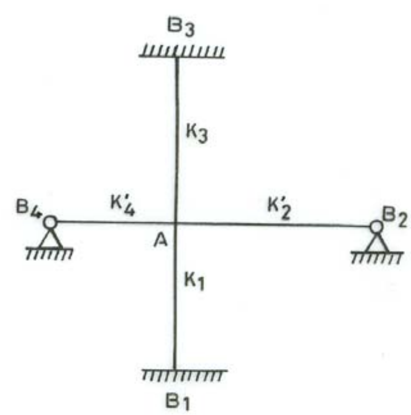


FIG. 15·4. PLANE FRAME WITH THREE DEGREES OF FREEDOM

When clamp at A is removed or relaxed as it is called the structure deforms further and develops distributed member end moments at A and carry over moments at B_1 to B_4.

For moment joint equilibrium at A: $\sum_A M = 0$

$M + M_{AB1} + M_{AB2} + M_{AB3} + M_{AB4} = 0$

$\theta_{B1} = \theta_{B2} = \theta_{B3} = \theta_{B4} = 0$

$M_{AB1} = K_{AB1}\, \theta_A$

$M_{AB2} = K_{AB2}\, \theta_A$

$M_{AB3} = K_{AB3}\, \theta_A$

$M_{AB4} = K_{AB4}\, \theta_A$

$$\theta_A \left[\sum_A K_{AB} \right] + M = 0$$

$$\theta_A = \frac{-M}{\sum_A K_{AB}}$$

$$M_{AB1} = \frac{-K_{AB1}}{\sum_A K_{AB}} M = -(DF)_{AB1}\, M$$

Similarly expression for moments at end A for other members can also be written.

$(DF)_{AB1}$ = Distribution factor at joint A for member AB1.

$$(DF)_{AB1} = \frac{K_{AB1}}{\sum_A K_{AB}}$$

The distributed moment developed at the relaxed end of member under unbalanced moment M at corresponding joint is equal to distribution factor (DF) of this member at this joint times the unbalanced moment with the sign reversed.

The carry over moment at end B_1 will be: $M_{B1A} = C_{AB1}\, M_{AB1}$

The carry-over moment to far end is equal to carry over factor times the corresponding distributed moment and has the same sign.

If far end of a member is hinged then modified stiffness of the member is to be taken in computation of distribution factors as shown in figure 15.4. B_1 and B_3 are clamps and B_2 and B_4 are hinges.

$$\sum_A K_{AB} = K_1 + K_2' + K_3 + K_4'$$

$$(DF)_{AB2} = \frac{K'_{AB2}}{\sum_A K_{AB}}$$

Cantilever portions of structure

The cantilever portions of the structure can be taken care of in two ways

(1) The effect of loads acting on cantilever can be transferred to nearest joint and cantilever part removed as shown in figure 15.5. Nearest joint will be subjected to externally applied vertical load and a moment. This moment is to be taken at this joint as external moment. For the case under consideration the vertical load will be (P+Q) and moment will be (PL+Qa) as shown in figure 15.5. Now the continuous beam ABCD can be analyzed by moment distribution method. D is now a simple support. Distribution factor for DC will be 1. There will be no carry over from C to D. For CD modified stiffness is used for the moment distribution process.

(2) The continuous beam ABCDE is retained as it is and the cantilever part DE is not knocked out. Cantilever moment is entered as FEM for DE. The DF at D will be 1 for DC and zero for DE. Since member is now continuous over support D there will be carry over of moments from C to D and rotational stiffness of CD will not be modified in view of fact that D is not an end simple support as for case when cantilever is knocked out. The moments DE and DC will be of same magnitude and equal to cantilever moment but of opposite sign.

It may be noted that cantilever arm has no restraining effect on the rotation of joint to which it is rigidly connected or in other words its stiffness is zero. Any unbalanced moment is therefore carried or distributed entirely by other members meeting at this joint. In the moment distribution process all the hinged joints should be released first for quick convergence. As regards other joints one should start with the joint which has largest unbalanced moment for rapid convergence. The final results are not affected by the order of relaxation of joints.

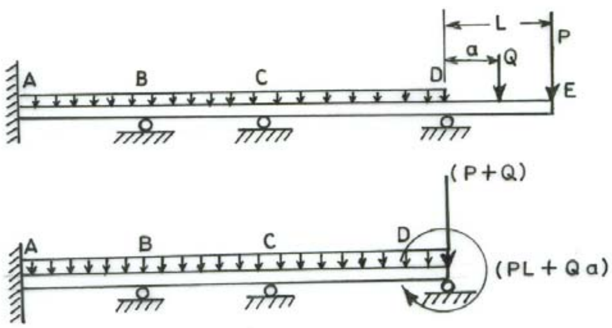


FIG. 15·5. MODIFICATION OF CONTINUOUS BEAM WITH OVERHANG.

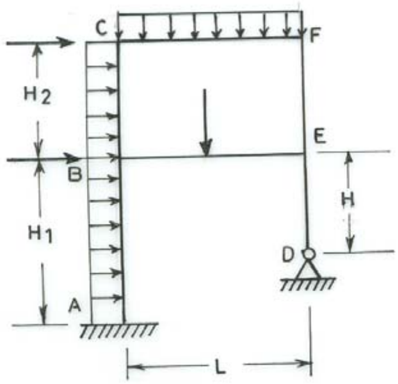


FIG. 15·6. ONE BAY TWO STOREY PLANE FRAME

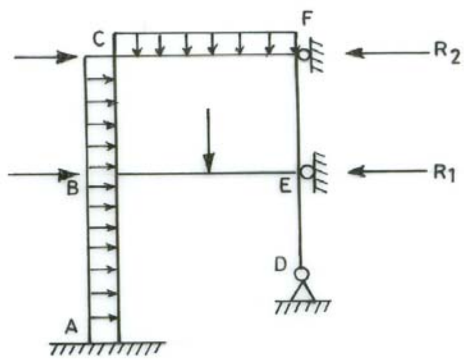


FIG. 15·7. GIVEN STRUCTURE WITH TWO CONSTRAINTS AND LOADS

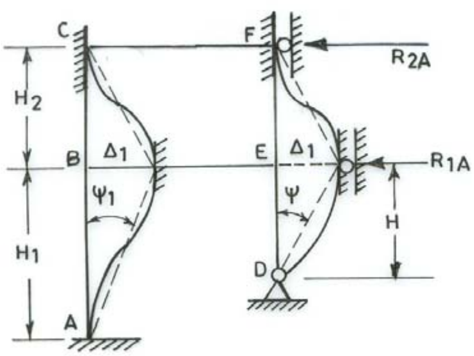


FIG. 15·8. FRAME UNDER LATERAL SWAY Δ1 AT LEVEL BE

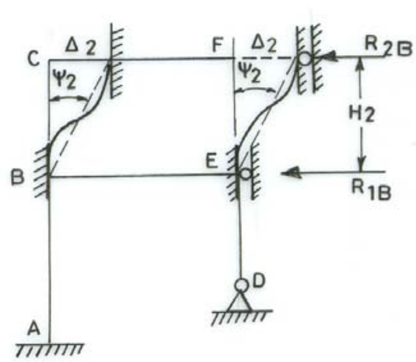


FIG. 15·9. FRAME UNDER LATERAL SWAY Δ2 AT LEVEL CF

In the structures where only rotation of joints is involved no simultaneous equations are required to be solved. Such structures have no side sway. For structures undergoing side sway the number of equations to be framed and solved will be equal to number of unknown lateral displacements Δ or the side sway angles or chord angles ψ.

For problems involving side sway the solution by moment distribution is carried out in two separate parts.

(1) Moment distribution analysis for no sway.

For this artificial constraints are introduced in structure so that there is no sway and reactions are determined at these constraints.

(2) Moment distribution analysis for lateral sway.

Now the structure is subjected to lateral displacements Δ_1, Δ_2, … , Δ_n or ψ angles ψ_1,ψ_2, …, ψ_n corresponding to n constraints introduced and moment distribution analysis is carried out and reactions computed at the n constraints in terms of Δ_1, Δ_2, … , Δ_n. Thus frame is to be analyzed for (n + 1) number of cases. After adding the results of sway and non-sway moment distribution cases, the reactions at the constraints are set equal to zero as they do not exist. Therefore n is corresponding to degree of freedom of structure corresponding to lateral displacements. After lateral displacements are computed, the moments and other desired data can be obtained.

Methodology

The methodology is explained on single bay two storey frame as shown in figure 15.6. Degree of freedom with respect to side sway = 2.

Δ_2 = Side sway at level CF with respect to level at BE.

Δ_1 = Side sway at level BE.

There will be two independent ψ-angles ψ_1 and ψ_2 corresponding to lateral displacements Δ_1 and Δ_2.

This frame will have to be analyzed for (n+1) = (2+1) = 3 cases. The side sway is prevented by introducing two constraints as shown in figure 15.7 and the frame is analyzed by moment distribution for no sway case and reactions R_1 and R_2 determined at the fictitious constraints. Now constraint at E is given unit displacement to right without imparting any rotations to joints B, E, C and F as shown in Fig.15.8. The frame is analyzed and reactions R_{1A} and R_{2A} are determined. Now constraint at F is given unit lateral displacement without rotations at joints B, E, C and F as shown in Fig.15.9. The frame is analyzed and reactions R_{1B} and R_{2B} are determined**.** Since artificial constraints are introduced the total reaction at the two constraints for combined three cases should vanish.

$$R_1 + R_{1A}\,\Delta_1 + R_{1B}\,\Delta_2 = 0$$

$$R_2 + R_{2A}\,\Delta_1 + R_{2B}\,\Delta_2 = 0$$

$$\Psi_1 = \frac{\Delta_1}{H_1},\ \Psi_2 = \frac{\Delta_2}{H_2},\ \Psi = \frac{\Delta_1}{H}$$

The two equations are solved for Δ_1 and Δ_2 and moments obtained in the two sway cases are modified by substituting the values of Δ_1 and Δ_2 respectively. Now sway and non sway moments are added to get final result.

It may be mentioned here that Kani's method which is also based on iteration technique does not involve solution of any simultaneous equations. Kani's method will not be discussed here.

The method of moment distribution will be demonstrated by solving same problems as in case of slope deflection method as shown in figure 15.10.

Example 15.1.

This problem will be solved in two stages.

Stage 1 : For vertical loads

Stage 2 : For concentrated moment at joint B

K_{BA} = 22,200 kNm, K_{BC} = 33,350 kNm

$$\sum_{B} K = K_{BA} + K_{BC} = 55{,}550 \text{ kNm}$$

$$(DF)_{BA} = \frac{22{,}200}{55{,}550} = 0.4,\ (DF)_{BC} = \frac{33{,}350}{55{,}550} = 0.6$$

C_{BA} = 0.604, C_{BC} = 0.294

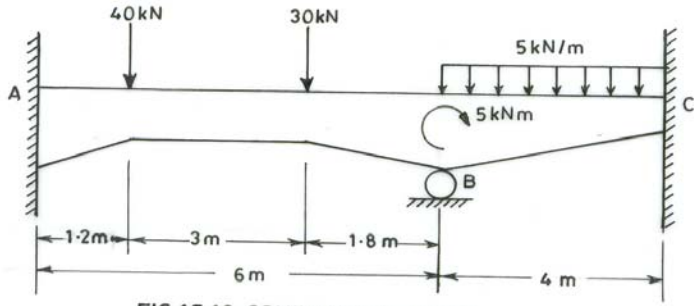


FIG.15-10. CONTINUOUS BEAM

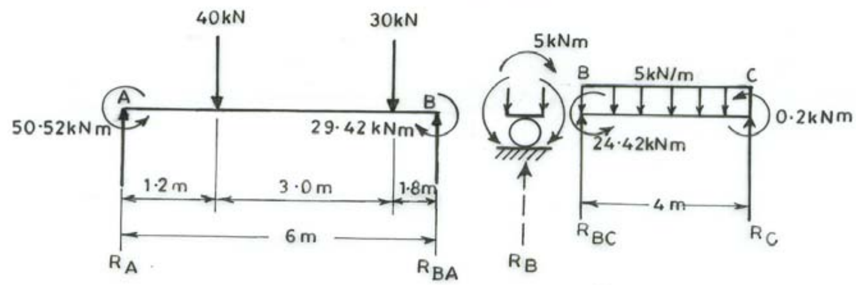


FIG. 15-11. FREE BODY DIAGRAMS OF MEMBERS

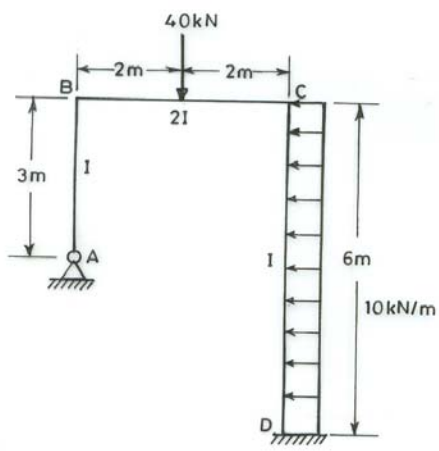


FIG. 15-12. PORTAL FRAME

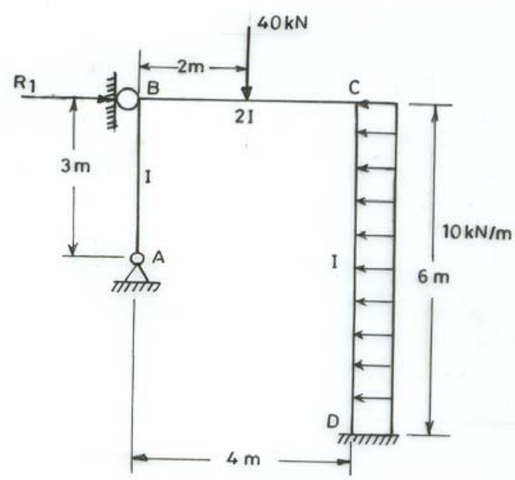


FIG. 15-13. PORTAL FRAME WITH CONSTRAINT AT B FOR NO SWAY

M_{FAB} = - 44.46 kNm, M_{FBA} = + 39.45 kNm

M_{FBC} = - 9.37 kNm, M_{FCB} = + 4.23 kNm

The moment distribution for concentrated moment at joint B is carried out separately. It is entered on top of the joint and then distributed. The moments at the ends of members for the two cases are now added to obtain final moments.

The reactions can be computed from free body diagrams of supports shown in Fig.15.11.

Moment distribution for vertical loads.

JOINT	A	B	B	C
DF & COF		0.604 ← 0.4	0.6 → 0.294	
FEM	-44.46	+ 39.45	- 9.37	+ 4.23
DC		- 12.03	- 18.05	
COC	- 7.27			- 5.31
TOTAL	- 51.73	+ 27.42	- 27.42	- 1.08

Moment distribution for concentrated moment.

-5

JOINT	A	B	B	C
DF & COF		0.604 ← 0.4	0.6 → 0.294	
DC		- 2.00	+ 3.00	
COC	+ 1.21			+ 0.88
TOTAL	+ 1.21	+ 2.00	+ 3.00	+ 0.88

Final moments

		-5		
JOINT	A	B	B	C
DF & COF		0.604 $\leftarrow$ 0.4	0.6 $\rightarrow$ 0.294	
CASE1	- 51.73	+ 27.42	- 27.42	- 1.08
CASE2	+ 1.21	+ 2.00	+ 3.00	+ 0.88
TOTAL	- 50.52	+ 29.42	- 24.42	- 0.20

<u>Beam AB</u>

$R_A = \frac{40 \text{ x } 4.8}{6} + \frac{30 \text{ x } 1.8}{6} + \frac{(50.52 - 29.42)}{6}$ = 32 + 9 + 3.52 = 44.52 kN

$R_{BA} = \frac{40 \text{ x } 1.2}{6} + \frac{30 \text{ x } 4.2}{6} + \frac{29.42 - 50.52}{6}$ = 8 + 21 – 3.52 = 25.48 kN

$R_A + R_{BA}$ = 44.52 + 25.48 = 70 kN (ok)

<u>Beam BC</u>

$R_{BC} = \frac{24.42 + 0.2}{4}$ + 5 x 2 = 6.12 + 10 = 16.12 kN

$R_C = 5 \text{ x } 2 - \frac{24.42 + 0.2}{4}$ = 10 – 6.12 = 3.88 kN

$R_{BC} + R_C$ = 16.12 + 3.88 = 20 kN (ok)

Total reaction at support B:

$R_B = R_{BA} + R_{BC}$ = 25.48 + 16.12 = 41.6 kN

<u>Example 15.2</u>

The portal frame with unequal legs loaded as shown figure 15.12 is analyzed by moment distribution method.

Solution for no sway

A roller constraint is introduced at B so that lateral movement is not permitted as shown in Figure 15.13.

R_1 = Reaction developed at B acting to right.

The dotted line indicates bottom side of members.

$M_{FAB} = M_{FBA} = 0$

$M_{FBC} = -20$ kNm, $M_{FCB} = +20$ kNm

$M_{FCD} = -30$ kNm, $M_{FDC} = +30$ kNm

Distribution factors at A and C

$$K_{BC} = \frac{4E(2I)}{4} = 2EI, \; K_{BA} = \frac{3E(I)}{3} = EI$$

$$\sum_B K = 3\,EI$$

$$(DF)_{BA} = \frac{1}{3}, \; (DF)_{BC} = \frac{2}{3}$$

$$K_{CB} = 2\,EI, \; K_{CD} = \frac{4EI}{6} = \frac{2EI}{3}$$

$$\sum_C K = 2\,EI + \frac{2EI}{3} = \frac{8}{3}\,EI$$

$$(DF)_{CB} = 2 \text{ x } \frac{3}{8} = \frac{3}{4}, \; (DF)_{CD} = \frac{2}{3} \text{ x } \frac{3}{8} = \frac{1}{4}$$

$C_{BA} = 0$, $C_{BC} = C_{CB} = C_{CD} = 0.5$

Fixed-end-moments, distribution factors and carry over factors are entered as shown and the distribution and carry over cycles of moment distribution method are carried out. The final moments for no sway case are obtained.

MOMENT DISTRIBUTION FOR NONSWAY CASE

JOINT	A	B	B	C	C	D
DF & COF		0 ← ⅓	⅔ →0.5	0.5 ← ¾	¼ →0.5	
FEM	0.00	0.00	-20.00	+20.00	- 30.00	+ 30.00
DC	---	+6.67	+13.33	+ 7.50	+ 2.50	---
COC	0.00	---	+ 3.75	+ 6.67	---	+ 1.25
DC	---	- 1.25	- 2.50	- 5.00	- 1.67	---
COC	0.00	---	- 2.50	- 1.25	---	- 0.84
DC	---	+0.83	+ 1.67	+ 0.94	+ 0.31	---
COC	0.00	---	+ 0.47	+ 0.84	---	+ 0.16
DC	0.00	- 0.16	- 0.31	- 0.63	- 0.21	---
COC	0.00	---	- 0.32	- 0.16	---	- 0.11
DC	---	+0.11	+ 0.21	+ 0.12	+ 0.04	---
COC	0.00	---	+ 0.06	+ 0.11	---	+ 0.02
DC	---	- 0.02	- 0.04	- 0.08	- 0.03	---
COC	---	---	- 0.04	- 0.02	---	- 0.02
DC	---	+ 0.01	+ 0.03	+ 0.01	+ 0.01	---
Final moments	0.00	+ 6.19	- 6.19	+ 29.05	- 29.05	+ 30.46

Horizontal shears are computed from free body diagrams of columns shown in Figure 15.14.

$$R_{AH} = \frac{6.19}{3} = 2.063 \text{ kN}$$

$$R_{CH} = 10 \times 3 + \frac{30.46 - 29.05}{10} = 30 + 0.23 = 30.23 \text{ kN}$$

$$R_1 = 60 - 2.063 - 30.235 = 27.702 \text{ kN}$$

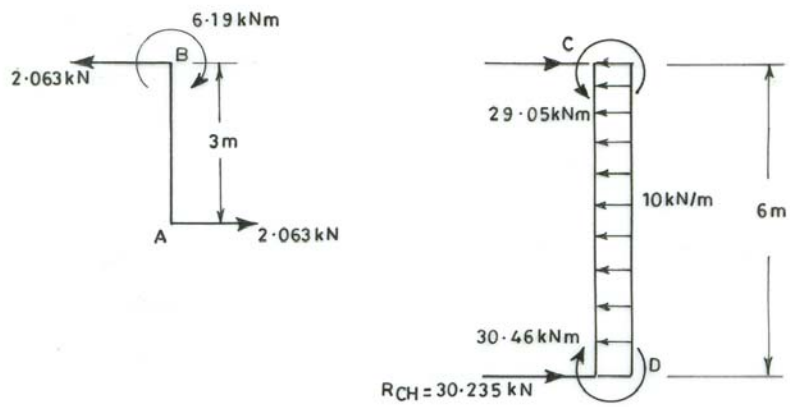


FIG. 15·14. FREE BODY DIAGRAMS OF COLUMNS

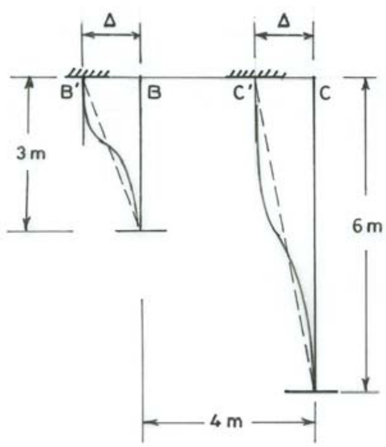


FIG. 15·15. LATERAL SWAY OF COLUMNS

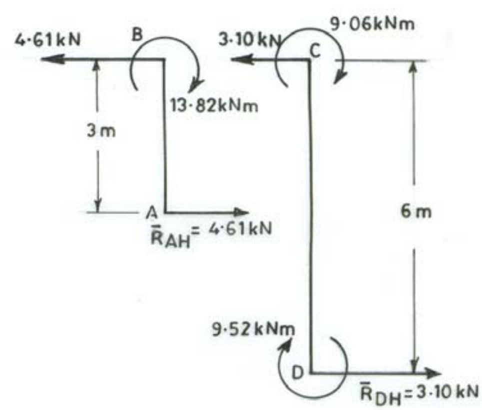


FIG. 15·16. FORCES IN COLUMNS

Solution for sway

A unit lateral displacement is imparted at level BC of the kinematically determinate system to the left without any rotations as shown in Figure 15.15.

$$M_{FAB} = M_{FBA} = \frac{6EI}{3 \times 3} = \frac{2}{3} EI$$

$$M_{FCD} = M_{FDC} = \frac{6EI}{6 \times 6} = \frac{EI}{6}$$

Multiply these FEMs by 60/EI

$$M_{FAB} = M_{FBA} = \frac{2}{3} EI \times \frac{60}{EI} = 40 \text{ kNm}$$

$$M_{FCD} = M_{FDC} = \frac{EI}{6} \times \frac{60}{EI} = 10 \text{ kNm}$$

$$\frac{6EI\Delta}{9} = 40, \Delta = 60/EI$$

$$\frac{6EI\Delta}{36} = 10, \Delta = 60/EI$$

These proportional FEMs are corresponding to $\Delta = \frac{60}{EI}$.

These FEMs are entered for process of moment distribution which is carried out and final proportional moments obtained as shown in scheme of moment distribution.

MOMENT DISTRIBUTION FOR SWAY CASE

JOINT	A	B	B	C	C	D
DF		$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{1}{4}$	
COF		0.5←	→0.5	0.5←	→0.5	
FEM	+ 40	+ 40	---	---	+ 10	+ 10
Release A	- 40	--	---	---	---	---
COC	---	- 20	---	---	---	---
FEM	0	+ 20	---	---	+ 10	+ 10
DC	---	- 6.67	- 13.33	- 7.5	- 2.5	---
COC	0	---	- 3.75	- 6.67	---	- 1.25
DC	---	+1.25	+ 2.50	+ 5.00	+1.67	---
COC	0	---	+2.50	+ 1.25	---	+ 0.84
DC	---	- 0.83	- 1.67	- 0.94	- 0.31	---
COC	0	---	- 0.47	- 0.84	---	- 0.16
DC	---	+ 0.16	+0.31	+0.63	+0.21	---
COC	0	---	+0.32	+0.16	---	+0.11
DC	---	- 0.11	- 0.21	-0.12	- 0.04	---
COC	0	---	- 0.06	-0.11	---	- 0.02
DC	0	+0.02	+0.04	+0.08	+0.03	
Final Moments	0	+13.82	- 13.82	- 9.06	+ 9.06	+ 9.52

From the free body diagrams of columns shown in Figure 15.16 the horizontal shears are obtained.

$$\overline{R}_{AH} = \frac{13.82}{3} = 4.61 \text{ kN}, \quad \overline{R}_{DH} = \frac{(9.06 + 9.52)}{6} = 3.10 \text{ kN}$$

$\overline{R}$ = Reaction at roller constraint for sway case.

$\overline{R}$ = - (4.61 + 3.10) = - 7.71 kN

MF = Modification factor

R + MF x $\overline{R}$ = 0

$$MF = \frac{-R}{\overline{R}} = \frac{-27.702}{-7.71} = 3.593$$

The proportional moments for sway case are corrected by multiplying with modification factor and added to moments obtained for nonsway case to obtain final moments for the given problem.

M_{AB} = 0

M_{BA} = + 6.19 + 13.82 x 3.593 = 55.85 kNm

M_{BC} = - 6.19 – 13.82 x 3.593 = - 55.85 kNm

M_{CB} = + 29.05 – 9.06 x 3.593 = - 3.50 kNm

M_{CD} = - 29.05 + 9.06 x 3.593 = + 3.50 kNm

M_{DC} = + 30.46 + 9.52 x 3.593 = + 64.67 kNm

The results are same as obtained by slope deflection method.

16 ANALYSIS OF CONTINNOUS BEAMS AND PLANE FRAMES CONSISTING OF PRISMATIC AND NON-PRISMATIC MEMBERS.

The continuous beams and plane frames are indeterminate structures. The following two methods will be discussed here for analysis of such structures.

(1) Slope deflection method

(2) Moment distribution method

In these methods the effect of axial deformations is neglected. These are in fact displacement methods. The unknown displacements involved are rotations and lateral displacements at the joints. The joints of the structure are assumed to be rigid, hence, the member end displacements for all the members meeting at the joint are the same. This is known as the condition of compatibility. Besides compatibility, the moment equilibrium of the joints and the lateral shear equilibrium of the storeys of frame should be satisfied. In slope-deflection equations, the member end moments are expressed in terms of member end rotations and relative lateral displacement and loads acting on member. Therefore it is possible to express joint and storey equilibrium equations in terms of unknown displacements of structure by using slope-deflection equations of all members. This gives the system of linear algebraic equations, which is solved by conventional methods in case of slope-deflection method and by an iterative physical relaxation technique in case of moment distribution method. Final results are obtained in the form of member end forces.

Slope-deflection method

The slope-deflection equation for a non-prismatic member AB is as given below. In this equation A and B represent near and far ends.

$$M_{near} = M_{Fnear} + K_{near}\,\theta_{near} + C_{far}\,K_{far}\,\theta_{far} + \frac{\Delta}{L} K_{near}\left(1 + C_{near}\right)$$

For prismatic member:

$$K_{near} = K_{far} = \frac{4EI}{L}$$

$$C_{near} = C_{far} = \frac{1}{2}$$

The slope deflection equation for uniform member takes the following form.

$$M_{near} = M_{Fnear} + \frac{2EI}{L}\left[2\theta_{near} + \theta_{far} + \frac{3\Delta}{L}\right]$$

$$M_{near} = M_{Fnear} + \frac{2EI}{L}\,[2\theta_{near} + \theta_{far} + 3\psi]$$

or $M_{near} = M_{Fnear} + \dfrac{4EI}{L}\,\theta_{near} + \dfrac{2EI}{L}\,\theta_{far} + \dfrac{6EI\Delta}{L^2}$

$$M_{near} = M_{Fnear} + \frac{4EI}{L} \theta_{near} + \frac{2EI}{L} \theta_{far} + \frac{6EI}{L} \psi$$

Joint equilibrium equation

The free body diagram of a rigid joint A of a plane structure is shown in figure 16.1. The joint exerts positive clockwise moments at the ends of members meeting at this joint and is in turn reacted by the same moments in the opposite counter clockwise directions together with an external positive moment M_{extA} acting on it.

An external counter clockwise moment directly acting on the joint is taken positive. The joint moment equilibrium equation is written as follows:

$$\sum M_{near} + M_{ext} = 0$$

$$\sum_{A} M_{AB} + M_{extA} = 0$$

$\sum_{A} M_{AB}$ = The sum of the moments acting at end A of member AB for all the members meeting at this joint.

Shear equilibrium equation

This pertains to the vertical columns of a particular storey in a multistory frame and is derived from the shear equilibrium conditions of that storey. The columns are assumed free of lateral loads directly acting on them. If there are lateral loads directly acting on the columns, the same are replaced by statically equivalent horizontal loads and concentrated moments acting on the nodes of frame and columns assumed free of external loads for their function in the frame.

Inclined columns and columns of different heights can be handled by writing shear equilibrium equation accordingly.

The free body diagram of columns of a particular storey is shown in figure 16.2. The equation of equilibrium for a typical column AB shown in figure 16.3 is written as follows:

$$M_{AB} + M_{BA} + H_{AB} h_{AB} = 0$$

h_{AB} = height of column AB

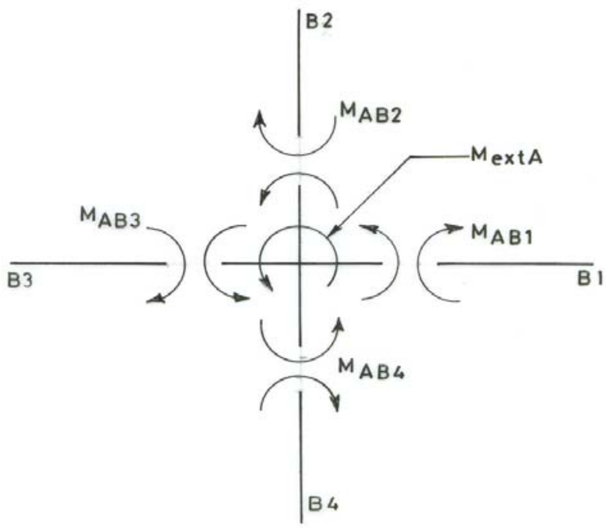


FIG. 16·1. FREE BODY DIAGRAM OF A RIGID JOINT OF FRAME

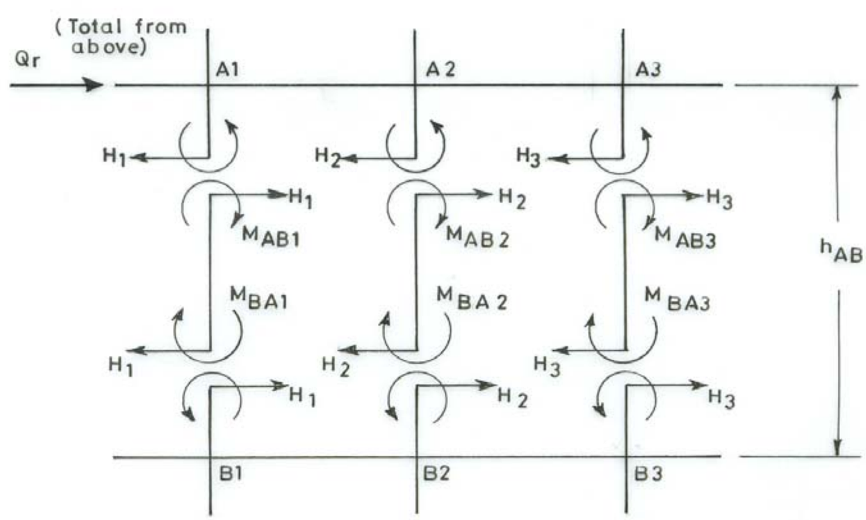


FIG. 16·2. FREE BODY DIAGRAM OF COLUMNS OF r^{TH} STOREY OF FRAME

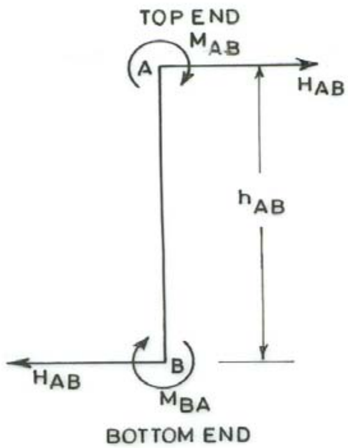


FIG. 16·3. FREE BODY DIAGRAM OF TYPICAL COLUMN OF r^{TH} STOREY

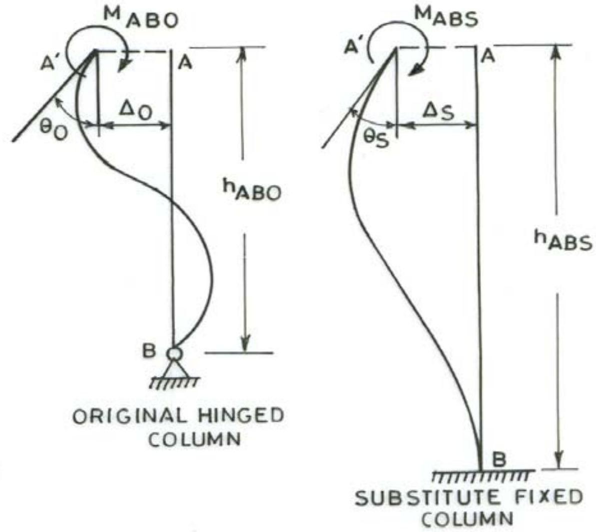


FIG. 16·4. ORIGINAL HINGED AND SUBSTITUTE FIXED COLUMNS

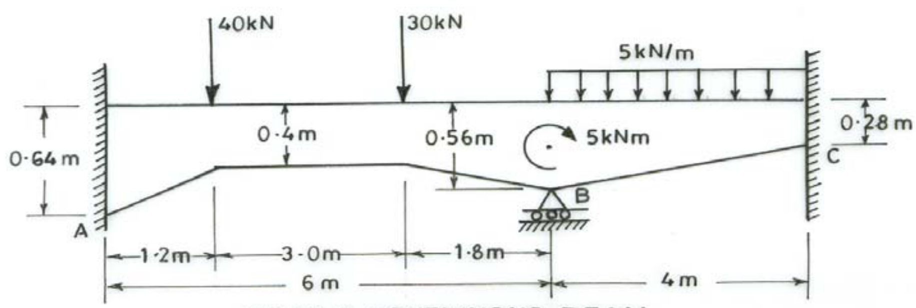


FIG.16·5. CONTINUOUS BEAM

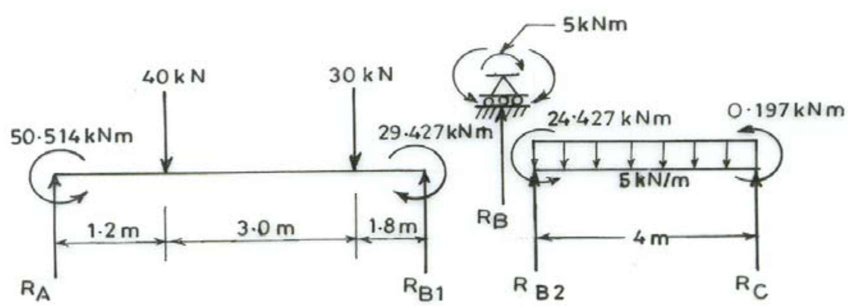


FIG.16·6. FREE BODY DIAGRAMS OF MEMBERS

M_{AB}, M_{BA} = moments at top and bottom taken positive clockwise.

H_{AB} = shear in column taken positive acting to left at bottom.

$$H_{AB} = -\frac{M_{AB} + M_{BA}}{h_{AB}}$$

Let Q_r be the external storey shear for the r^{th} storey of frame, which is taken positive when acting to right. It is equal to sum of all the external horizontal loads acting on the frame above this storey.

$\sum_r H_{AB}$ = Reactive r^{th} storey shear which is sum of base shears of all the columns of this storey.

The shear equilibrium equation for the r^{th} storey is:

$$Q_r - \sum_r H_{AB} = 0$$

$$\sum_r \frac{\left(M_{AB} + M_{BA}\right)}{h_{AB}} + Q_r = 0$$

<u>First storey</u>

The columns of first storey are rigidly connected to the nodes of frame at their upper ends, their lower ends being either clamped or hinged to the foundation while the columns of the other upper storeys are rigidly connected at both the ends. In order that the r^{th} storey shear equilibrium equation is also valid for the first storey, the hinged column need to be replaced with equivalent column fixed at base. For structural similarity of the original hinged column and the substitute fixed column the following conditions should be satisfied as shown in figure 16.4.

$M_{ABO} = M_{ABS}$

$\theta_O = \theta_S$

$\Delta_O = \Delta_S$

These conditions give:

$$K_{ABS} = \frac{3}{4}\ K_{ABO}$$

$$h_{ABS} = \frac{3}{2} h_{ABO}$$

K_{ABO}, K_{ABS} = Rotational stiffnesses of original hinged and substitute fixed columns.

h_{ABO}, h_{ABS} = Heights of original hinged and substitute fixed columns.

The method will be demonstrated on numerical examples.

Numerical example on continuous beam

Shown in figure 16.5 is a two span continuous beam of variable section. The continuous beam is fixed at A and C and is roller supported at B. The degree of freedom is one that is rotation θ_B at B which is unknown.

$\theta_A = \theta_C = 0$

The haunched member AB and the tapered member BC were discussed in previous sections hence the required properties are reproduced below.

Member AB

M_{FAB} = - 44.46 kNm

M_{FBA} = 39.45 kNm

C_{AB} = 0.622

C_{BA} = 0.604

K_{AB} = 21,600 kNm

K_{BA} = 22,200 kNm

Member BC

M_{FBC} = - 9.37 kNm

M_{FCB} = 4.23 kNm

C_{BC} = 0.294

C_{CB} = 0.834

K_{BC} = 33,350 kNm

K_{CB} = 11,780 kNm

Slope-deflection equations:

$M_{AB} = M_{FAB} + K_{AB}\ \theta_A + C_{BA}\ K_{BA}\ \theta_B$

$M_{BA} = M_{FBA} + K_{BA}\ \theta_B + C_{AB}\ K_{AB}\ \theta_A$

$M_{AB} = - 44.46 + 13{,}408.8\ \theta_B$

$M_{BA} = 39.45 + 22{,}200\ \theta_B$

$M_{BC} = M_{FBC} + K_{BC}\ \theta_B + C_{CB}\ K_{CB}\ \theta_C$

$M_{CB} = M_{FCB} + K_{CB}\ \theta_C + C_{BC}\ K_{BC}\ \theta_B$

$M_{BC} = - 9.37 + 33{,}350\ \theta_B$

$M_{CB} = 4.23 + 9804.9\ \theta_B$

Moment equilibrium equation of joint B

$M_{extB} + M_{BA} + M_{BC} = 0$

$- 5 + 39.45 + 22{,}200\ \theta_B + 33{,}350\ \theta_B – 9.37 = 0$

$55{,}550\ \theta_B = - 25.08$

$\theta_B = \dfrac{-25.08}{55{,}550} = - 4.515 \text{ x } 10^{-4}$ radians

$M_{AB} = - 44.46 + 13{,}408.8 \text{ x } (- 4.515 \text{ x } 10^{-4}) = - 50.514$ kNm

$M_{BA} = 39.45 + 22{,}200 \text{ x } (- 4.515 \text{ x } 10^{-4}) = 29.427$ kNm

$M_{BC} = - 9.37 + 33{,}350 \text{ x } (- 4.515 \text{ x } 10^{-4}) = - 24.427$ kNm

$M_{CB} = 4.23 + 9804.9 \text{ x } (- 4.515 \text{ x } 10^{-4}) = - 0.197$ kNm

The reactions are computed as follows using free body diagrams shown in figure 16.6.

Member AB

$$R_A = \frac{40 \text{ x } 4.8}{6} + \frac{30 \text{ x } 1.8}{6} + \frac{(50.514 - 29.427)}{6} = 32 + 9 + 3.15 = 44.51 \text{ kN}$$

$$R_{B1} = \frac{40 \text{ x } 1.2}{6} + \frac{30 \text{ x } 4.2}{6} - \frac{(50.514 - 29.427)}{6} = 8 + 21 - 3.51 = 25.49 \text{ kN}$$

Member BC

$$R_{B2} = \frac{5 \text{ x } 4}{2} + \frac{(24.427 + 0.197)}{4} = 10 + 6.16 = 16.16 \text{ kN}$$

$$R_C = \frac{5 \text{ x } 4}{2} + \frac{(24.427 + 0.197)}{4} = 10 - 6.16 = 3.84 \text{ kN}$$

Roller at B

$R_B = R_{B1} + R_{B2} = 25.49 + 16.16 = 41.65$ kN

$R_A + R_B + R_C = 44.51 + 41.65 + 3.84 = 90$ kN (check)

The final reactions are shown in the figure 16.7.

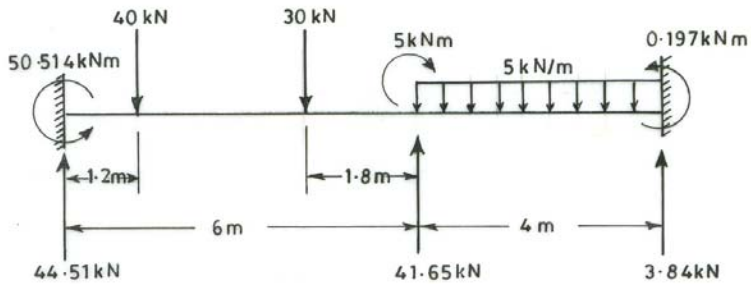


FIG. 16·7. FINAL REACTIONS

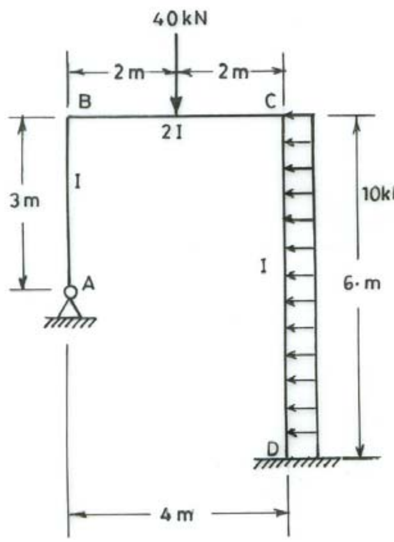


FIG. 16·8. PORTAL FRAME

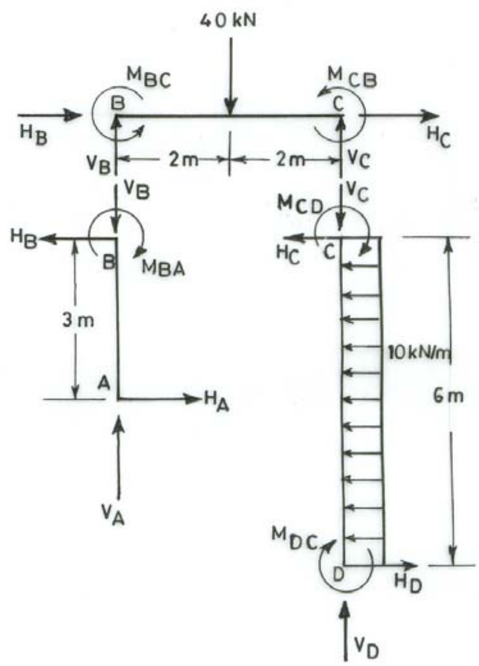


FIG.16·9. FREE BODY DIAGRAMS OF MEMBERS

Numerical example on portal frame

A portal frame loaded as shown in figure 16.8 is analyzed for member end forces by slope-deflection method.

DOF = 4 (θ_A, θ_B, θ_C, Δ)

Rotations are taken positive clockwise at the joints and lateral displacement Δ at the beam level is taken positive to left.

Since A is a hinge : $M_{AB} = 0$

Since D is clamp $\theta_D = 0$

$$M_{FBC} = -\frac{40 \text{ x } 2 \text{ x } 2 \text{ x } 2}{4 \text{ x } 4} = -20 \text{ kNm}$$

$$M_{FCB} = +\frac{40 \text{ x } 4}{8} = +20 \text{ kNm}$$

$$M_{FAB} = M_{FBA} = 0$$

$$M_{FCD} = -\frac{10 \text{ x } 6 \text{ x } 6}{12} = -30 \text{ kNm}$$

$$M_{FDC} = +\frac{10 \text{ x } 6 \text{ x } 6}{12} = +30 \text{ kNm}$$

$$M_{AB} = +\frac{2\,EI}{3}\left(2\theta_A + \theta_B + \frac{3\Delta}{3}\right)$$

$$M_{BA} = +\frac{2\,EI}{3}\left(\theta_A + 2\theta_B + \frac{3\Delta}{3}\right)$$

$$M_{BC} = -20 + \frac{4\,EI}{4}\left(2\theta_B + \theta_C\right)$$

$$M_{CB} = +20 + \frac{4\,EI}{4}\left(\theta_B + 2\theta_C\right)$$

$$M_{CD} = -30 + \frac{2\,EI}{6}\left(2\theta_C + 0 + \frac{3\Delta}{6}\right)$$

M_{DC} = + 30 + $\frac{2\,EI}{6}\left(0 + \theta_C + \frac{3\Delta}{6}\right)$

The unknown displacements θ_A, θ_B, θ_C and Δ can be obtained from three moment equilibrium equations of joints and one shear equilibrium equation.

M_{AB} = 0 ------------------ (1)

M_{BA} + M_{BC} = 0 --------- (2)

M_{CB} + M_{CD} = 0 --------- (3)

H_A + H_D – 60 = 0 ------- (4)

H_A = $\frac{M_{BA}}{3}$

H_D = $\frac{M_{CD} + M_{DC} + 60 \text{ x } 3}{6}$

2 M_{BA} + M_{CD} + M_{DC} = 180 ---- (4)

$\frac{2EI}{3}\left(2\theta_A + \theta_B + \frac{3\Delta}{3}\right) = 0$

θ_A = - $\frac{\theta_B}{2} - \frac{\Delta}{2}$ ------------------ (1)

$\frac{2EI}{3}\left(\theta_A + 2\theta_B + \Delta\right)$- 20 + EI (2$\theta_B$ + θ_C) = 0

2θ_A + 4θ_B + 2Δ + 6θ_B + 3θ_C = $\frac{60}{EI}$

3θ_C + 10θ_B + 2θ_A + 2Δ = $\frac{60}{EI}$ --- (2)

20 + EI (θ_B + 2θ_C) – 30 + $\frac{EI}{3}\left(2\theta_C + \frac{\Delta}{2}\right)$ = 0

$6\theta_B + 12\theta_C + 4\theta_C + \Delta = +\dfrac{60}{EI}$

$6\theta_B + 16\theta_C + \Delta = \dfrac{+60}{EI}$ --------- (3)

$$\frac{4EI}{3}(\theta_A + 2\theta_B + \Delta) - 30 + \frac{EI}{3}\left(2\theta_C + \frac{\Delta}{2}\right) + 30 + \frac{EI}{3}\left(\theta_C + \frac{\Delta}{2}\right) = 180$$

$4\theta_A + 8\theta_B + 3\theta_C + 5\Delta = \dfrac{540}{EI}$ ----- (4)

Substituting value of θ_A from equation (1) in equations (2) and (4):

$3\theta_C + 9\theta_B + \Delta = \dfrac{60}{EI}$ --------- (5)

$6\theta_B + 3\theta_C + 3\Delta = \dfrac{540}{EI}$ ----- (6)

or $2\theta_B + \theta_C + \Delta = \dfrac{180}{EI}$ ----- (6)

Eqn (5) – Eqn (6) gives:

$7\theta_B + 2\theta_C = \dfrac{120}{EI}$ ----------- (7)

Eqn (3) – Eqn (6) gives:

$4\theta_B + 15\theta_C = -\dfrac{120}{EI}$ ----------- (8)

Equations (7) & (8) are solved for θ_B and θ_C.

$(8\theta_C + 105\theta_C) = -\dfrac{480}{EI} + \dfrac{840}{EI} = \dfrac{360}{EI}$

$\theta_C = -\dfrac{360}{97EI} = -\dfrac{3.711}{EI}$

$$\theta_B = \frac{1}{4}\left[\frac{-120}{EI} + \frac{15 \text{ x } 360}{97EI}\right] = \frac{-16.082}{EI}$$

From eqn (6), we get Δ and from eqn (1) θ_A

$$\Delta = \frac{180}{EI} + 2 \text{ x } \frac{16.082}{EI} + \frac{3.711}{EI} = \frac{215.875}{EI}$$

$$\theta_A = -\frac{215.875}{2EI} + \frac{16.082}{2EI} = -\frac{99.897}{EI}$$

Now member end moments are computed from slope deflection equations.

$$M_{AB} = \frac{2}{3} EI\left[2 \text{ x } \frac{-99.897}{EI} - \frac{16.082}{EI} + \frac{215.875}{EI}\right] = 0 \text{ (check)}$$

$$M_{BA} = \frac{2EI}{3}\left[\frac{-99.897}{EI} + \frac{2 \text{ x } (-16.082)}{EI} + \frac{215.875}{EI}\right] = 55.876 \text{ kNm}$$

$$M_{BC} = -20 + EI\left[\frac{2 \text{ x } (-16.082)}{EI} - \frac{3.711}{EI}\right] = -55.875 \text{ kNm}$$

$$M_{CB} = 20 + EI\left[\frac{-16.082}{EI} - \frac{2 \text{ x } 3.711}{EI}\right] = -3.504 \text{ kNm}$$

$$M_{CD} = -30 + \frac{EI}{3}\left[\frac{-2 \text{ x } 3.711}{EI} + \frac{215.875}{2EI}\right] = +3.505 \text{ kNm}$$

$$M_{DC} = 30 + \frac{EI}{3}\left[\frac{-3.711}{EI} + \frac{215.875}{2EI}\right] = 64.742 \text{ kNm}$$

Shears and axial forces at the ends of members are computed from free body diagrams of members shown in Figure 16.9.

<u>Member AB</u>

$$H_A = H_B = \frac{55.876}{3} = 18.625 \text{ kN}$$

Member BC

H_C = - H_B = - 18.625 kN

$$V_B = \frac{40}{2} + \frac{(55.875 + 3.504)}{4} = 34.845 \text{ kN}$$

$$V_C = \frac{40}{2} - \frac{(55.875 + 3.504)}{4} = 5.155 \text{ kN}$$

Member CD

$V_D = V_C$ = 5.155 kN

$$H_D = \frac{10 \text{ x } 6 \text{ x } 3 + 64.742 + 3.505}{6} = 41.375 \text{ kN}$$

Member AB

$V_A = V_B$ = 34.845 kN

$V_A + V_D$ = 40 kN, $H_A + H_D$ = 60 kN (check)

17 ANALYSIS OF INDETERMINATE TRUSSES

External indeterminacy of a truss is computed as follows.

$\propto_E = R - r$

$\propto_E$ = External indeterminacy

R = Total number of reaction components

r = Minimum number of equilibrium equations for truss as a whole for stability.

For plane truss:

$\propto_I = m - (2j - r)$

$\propto_I$ = Internal indeterminacy

m = total number of members

j = total number of joints

$\propto_S$ = Total statical indeterminacy of truss

$\propto_S = \propto_E + \propto_I = (m + r - 2j)$

<u>For space truss:</u>

$\propto_S = (m + R - 3j)$

$\propto_I = m - (3j - r)$

<u>Internally redundant truss to first degree</u>

Shown in figure 17.1 is internally indeterminate truss to first degree. It is made determinate by removing one member as shown in figure 17.2.

F_i = Force in ith member of determinate truss due to given external loads.

m = number of members in determinate truss

L_i = Length of ith member

A_i = area of cross section of ith member

X = Force in member which has been removed from redundant truss due to external loads acting on indeterminate truss as shown in figure 17.3.

k_i = Force in ith member of determinate truss due to unit actions corresponding to member removed as shown in figure 17.4.

$\overline{F_i}$ = Total force in ith member due to external loads and the forces X corresponding to removed member.

$\overline{F_i} = F_i + k_i X$

U = strain energy of determinate truss due to external loads and force X.

$$U = \sum_{i=1}^{m} \frac{(F_i + k_i X)^2 L_i}{2A_i E}$$

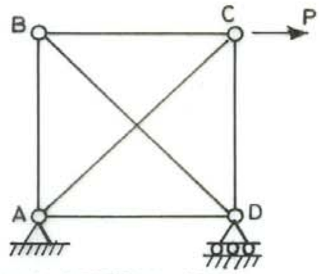


FIG. 17·1. INTERNALLY INDETERMINATE TRUSS

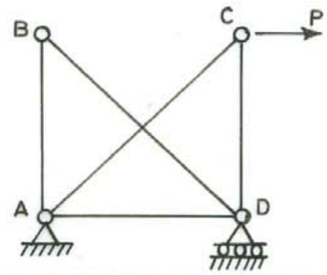


FIG. 17·2. DETERMINATE TRUSS

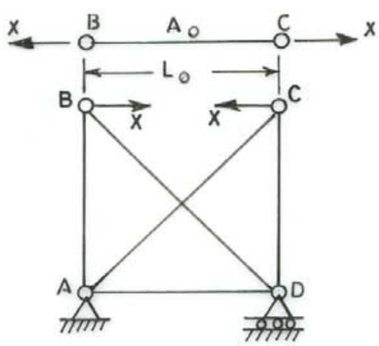


FIG. 17·3. ACTUAL FORCE X (ASSUMED TENSION) CORRESPONDING TO REMOVED MEMBER

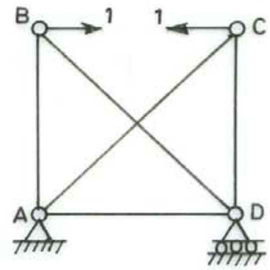


FIG. 17·4. UNIT ACTIONS CORRESPONDING TO REMOVED MEMBER

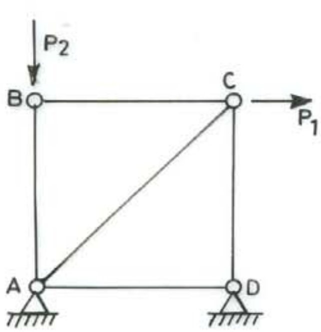


FIG. 17·5. EXTERNALLY INDETERMINATE TRUSS

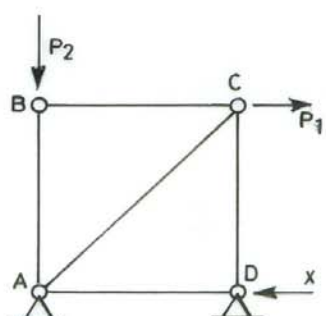


FIG. 17·6. DETERMINATE TRUSS

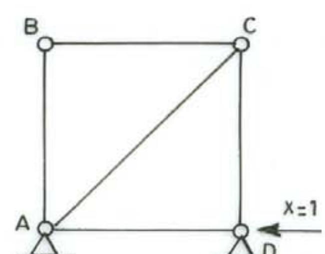


FIG. 17·7. DETERMINATE TRUSS UNDER UNIT REDUNDANT FORCE

$\frac{\partial U}{\partial X}$ = Displacement in the direction of X

ΔL_o = Change in length of removed member (assumed extension).

L_o = Length of removed member.

A_o = Area of cross-section of removed member.

E = modulus of elasticity

$$\Delta L_o = \frac{XL_o}{A_o E}$$

$$\frac{\partial U}{\partial X} = -\Delta L_o$$

$$\sum_{i=1}^{m} \frac{(F_i + k_i X)k_i L_i}{A_i E} = \frac{-XL_o}{A_o E}$$

$$X = -\frac{\sum_{i=1}^{m} F_i k_i L_i / A_i E}{\left(\sum_{i=1}^{m} \frac{k_i^2 L_i}{A_i E} + \frac{L_o}{A_o E}\right)}$$

Externally redundant truss to first degree

Externally indeterminate truss to first degree is shown in figure 17.5. It is made determinate by providing roller at D instead of hinge. Redundant reaction X is introduced at D. Forces in members of determinate truss are computed for external loads and X = 1 separately as shown in figures 17.6. and 17.7.

$$U = \sum_{i=1}^{m} \frac{(F_i + k_i X)^2 L_i}{2A_i E}$$

$$\frac{\partial U}{\partial X} = 0$$

$$\sum_{i=1}^{m} \frac{(F_i + k_i X) k_i L_i}{A_i E} = 0$$

$$X = - \frac{\sum_{i=1}^{m} \frac{F_i k_i L_i}{A_i E}}{\sum_{i=1}^{m} \frac{K_i^2 L_i}{A_i E}}$$

<u>Trusses with n-degrees of redundancy</u>

The truss is made determinate by removing redundant members and reactions. The determinate truss is analyzed for forces in members due to given external loads and due to unit redundant forces X_1, …. X_j, …., X_n.

k_{ij} = Force in member I due to X_j = 1 in determinate truss.

$$\overline{F_i} = F_i + \sum_{j=1}^{n} k_{ij} X_j$$

The strain energy of the truss including strain energy of redundant members is given by:

$$U = \sum_{i=1}^{m} \frac{\left(F_i + \sum_{j=1}^{n} k_{ij} X_j\right)^2 L_i}{2A_i E} + \frac{X_j^2 L_j}{2A_j E}$$

L_j = Length of member corresponding to redundant member.

For strain energy to be minimum.

$$\frac{\partial U}{\partial X_1} = \frac{\partial U}{\partial X_2} = \ldots.. \frac{\partial U}{\partial X_j} = \ldots\ldots\ldots = \frac{\partial U}{\partial X_n} = 0$$

For unknown forces corresponding to redundant members:

$$\frac{\partial U}{\partial X_j} = \sum_{i=1}^{m} \frac{\left(F + \sum_{j=1}^{n} k_{ij} X_j\right) k_i L_i}{A_i E} + \frac{X_j L_j}{A_j E} = 0$$

For unknown forces corresponding to reactions L_j is set equal to zero. The partial derivatives with respect to all the n-unknowns give a system of simultaneous equations, which is solved for unknowns and the forces in indeterminate truss.

Lack of fit in members of indeterminate trusses

No forces are generated in members of determinate truss due to lack of fit but in case of indeterminate truss the forces are generated in members when the members with lack of fit are forced into position. Consider an indeterminate truss ABCD in which member BC is shorter than its exact length by amount δ as shown in figure 17.8. The member will come in tension when fitted.

X = force in member due to lack of fit.

System is made determinate by removing members having lack of fit. Unit forces X = 1 are applied as shown in figure 17.9.

$$\overline{F_i} = k_i X$$

$$U = \sum_{i=1}^{m} \frac{(k_i X)^2 L_i}{A_i E}$$

$$\delta = \frac{\partial U}{\partial X} = \text{movement in the direction of force}$$

$$\sum_{i=1}^{m} \frac{k_i^2 X L_i}{A_i E} = \frac{-XL_o}{A_o E} + \delta$$

L_o = Length of member having lack of fit

A_o = Area of member having lack of fit

$$X = \frac{\delta}{\left(\sum_{i=1}^{m} \frac{k_i^2 L_i}{A_i E} + \frac{L_o}{A_o E} \right)}$$

If member with lack of fit is longer than L_o, δ will be taken negative.

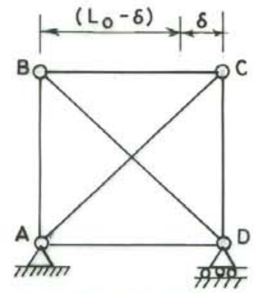


FIG.17·8. TRUSS WITH LACK OF FIT δ IN MEMBER BC

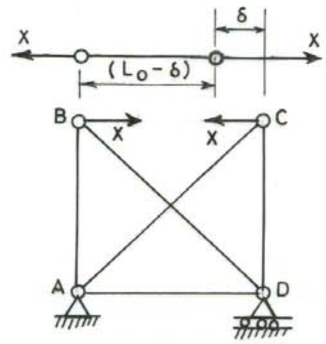


FIG.17·9. DETERMINATE TRUSS BY REMOVAL OF MEMBER WITH LACK OF FIT

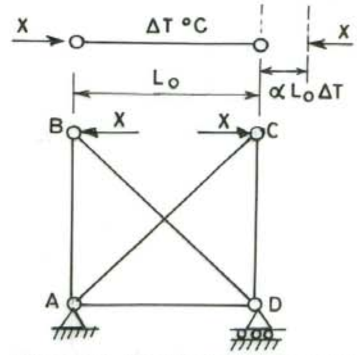


FIG.17·10. DETERMINATE TRUSS BY REMOVAL OF MEMBER SUBJECTED TO TEMPERATURE RISE

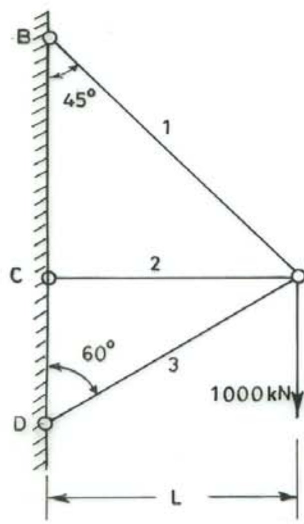


FIG.17·11. INDETERMINATE CANTILEVER TRUSS.

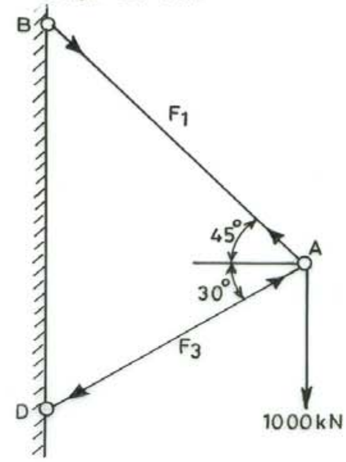


FIG.17·12. DETERMINATE CANTILEVER TRUSS

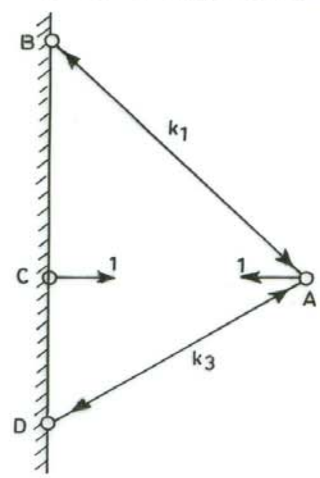


FIG.17·13. DETERMINATE TRUSS UNDER UNIT REDUNDANT FORCES

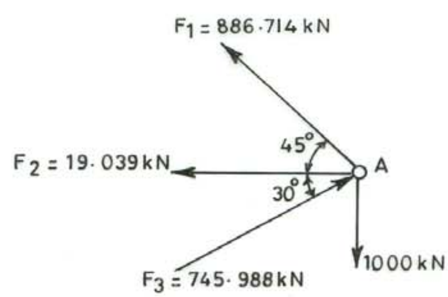


FIG.17·14. FINAL FORCES IN MEMBERS

Temperature forces in redundant trusses

The member BC is subjected to rise in temperature of $\Delta T^{o}C$.

$\delta L_o = \propto L_o \, \Delta T$

$\propto$ = Coefficient of thermal expansion

The truss is made determinate by removing the member which is subjected to temperature change. Unit loads are applied at the joints of the member as shown in figure 17.10. and forces in various members are computed.

Force in redundant member BC = kX

$$U = \sum_{i=1}^{m} \frac{(k_i X)^2 L_i}{2A_i E}$$

$$\frac{\partial U}{\partial X} = \text{movement in the direction of force X}$$

$$\sum_{i=1}^{m} \frac{k_i^2 X L_i}{A_i E} = \delta L_o - \frac{XL_o}{A_o E} = L_o \propto \Delta T - \frac{XL_o}{A_o E}$$

$$X = \frac{L_o \propto \Delta T}{\left(\sum_{i=1}^{m} \frac{k_i^2 L_i}{A_i E} + \frac{L_o}{A_o E} \right)}$$

If member BC is subjected to fall in temperature ΔT will be taken negative.

The method discussed above is applicable to plane as well as space trusses and pertains to force method.

Numerical examples

Example 17.1.

For the cantilever truss shown in figure 17.11 find the forces in members. The axial rigidity AE of all the members is same.

There are three members meeting at joint A and two equations of equilibrium, hence, static indeterminacy is 1. Let AC be redundant member. It is removed to obtain determinate system as shown in figure 17.12.

Joint A

$F_1 \text{ Sin } 45^o + F_3 \text{ Sin } 30^o = 1000$

$F_1 \text{ Cos } 45^o = F_1 \text{ Cos } 30^o$

$F_1 = \dfrac{\sqrt{3}\sqrt{2}}{2} \quad F_3 = \sqrt{\dfrac{3}{2}}\ F_3$

$\left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}\right) F_3 = 1000$

$F_3 = \dfrac{2000}{\left(\sqrt{3}+1\right)} = 732.051 \text{ (comp)}$

$F_1 = 896.576$ (Tension)

We apply unit actions corresponding to unknown tensile force in AC as shown in figure 17.13.

Joint A

$k_1 \text{ Sin } 45^o = k_3 \text{ Sin } 30^o$

$k_3 = \sqrt{2}\ k_1$

$k_1 \text{ Cos } 45^o = k_3 \text{ Cos } 30^o = 1$

$\dfrac{k_1}{\sqrt{2}} + \dfrac{\sqrt{3}}{2}\sqrt{2}\ k_1 = 1$

$k_1 \left(1 + \sqrt{3}\right) = \sqrt{2}$

$$k_1 = \frac{\sqrt{2}}{\left(1+\sqrt{3}\right)} = \frac{1.4142}{2.7321} = 0.518 \text{ (comp)}$$

$$k_3 = \frac{2}{\left(1+\sqrt{3}\right)} = 0.732 \text{ (comp)}$$

$$X = -\frac{\sum_{i=1,3} \frac{F_i k_i L_i}{A_i E}}{\left(\sum_{i=1,3} \frac{k_i^2 L_i}{A_i E} + \frac{L_o}{A_o E}\right)} = -\frac{F_1 k_1 L_1 + F_3 k_3 L_3}{k_1^2 L_1 + k_3^2 L_3 + L_o}$$

$$X = -\frac{896.576 \text{ x } (-0.518)\sqrt{2}\,L + (-732.051)(-0.732)\frac{2}{\sqrt{3}}\,L}{(-0.518)^2\sqrt{2}\,L + (-0.732)^2\frac{2}{\sqrt{3}}\,L + L}$$

$$X = -\frac{(-656.798 + 618.759)}{(0.379 + 0.619 + 1)} = \frac{38.039}{1.998} = 19.039 \text{ kN}$$

$\overline{F_1}$ = F_1 + k_1 X = 896.576 – 0.518 x 19.039 = 886.714 kN

$\overline{F_2}$ = X = 19.039 kN

$\overline{F_3}$ = F_3 + k_3 X = - 732.051 – 0.732 x 19.039 = - 745.988 kN

The final forces in members are shown in figure 17.14.

<u>Check</u>

<u>x – direction</u>

$$-19.039 + 745.988 \text{ x } \frac{\sqrt{3}}{2} - 886.714 \text{ x } \frac{1}{\sqrt{2}} = -19.039 + 646.045 - 627.001 = 0.004 \text{ (ok)}$$

y-direction

$$886.714 \times \frac{1}{\sqrt{2}} + 745.988 \times \frac{1}{2} - 1000 = 627.001 + 372.994 - 1000 = 0 \text{ (ok)}$$

It can be seen that there are small computational errors as the forces are not summing up to zero exactly. However, residuals are small and neglected.

Example 17.2.

For internally indeterminate truss shown in figure 17.15 determine forces in all members.

$E = 200$ kN/mm^2

Size of vertical members = 30 x 20 mm

Size of horizontal members = 30 x 30 mm

Size of diagonal members = 20 x 20 mm

Member BC is removed to make the truss determinate as shown in figure 17.16. Forces in members F are now determined.

$R_{AV} = R_D = 7.5$ kN

$R_{AH} = 10$ kN

Joint C

$F_4 \cos\theta = 10$

$$F_4 = 10 \times \frac{5}{4} = \frac{50}{4} = 12.5 \text{ kN (Tension)}$$

$$F_2 = F_4 \sin\theta = \frac{50}{4} \times \frac{3}{5} = \frac{15}{2} = 7.5 \text{ kN (comp)}$$

Joint B

$F_1 = F_5 = 0$

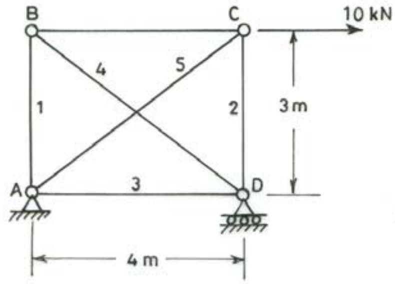


FIG. 17.15. INDETERMINATE TRUSS

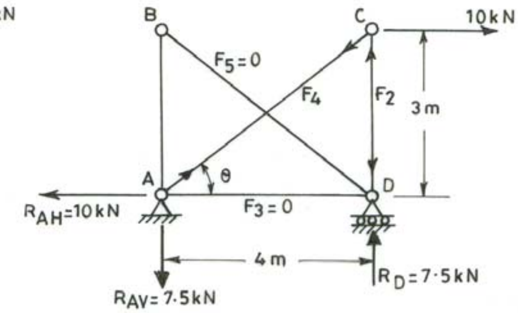


FIG. 17.16. DETERMINATE TRUSS

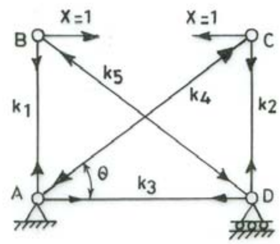


FIG. 17·17. UNIT FORCES ALONG REMOVED MEMBER

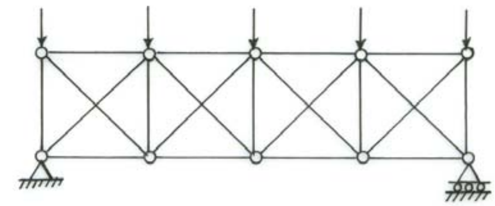


FIG. 18·1. PARALLEL CHORD INDETERMINATE TRUSS

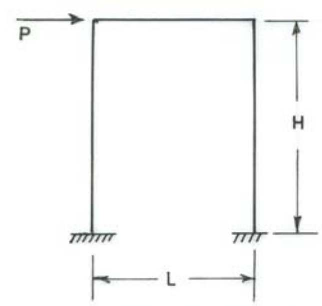


FIG. 18·2. INDETERMINATE PORTAL FRAME

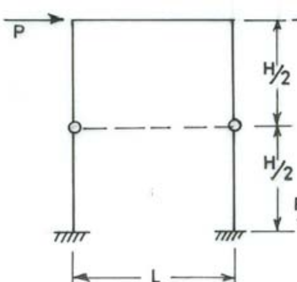


FIG. 18·3. PORTAL FRAME HINGED AT MID HEIGHT OF COLUMNS

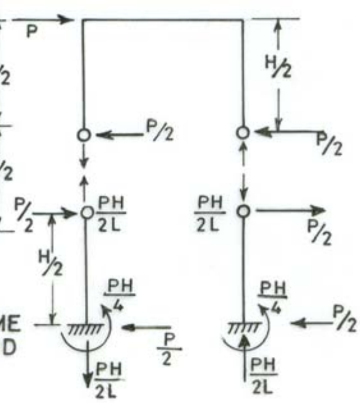


FIG. 18·4. DETERMINATE SYSTEM FOR APPROXIMATE ANALYSIS

Joint A

$F_3 = 10 - F_4 \text{ Cos } \theta = 10 - \frac{50}{4} \text{ x } \frac{4}{5} = 0$

Now unit actions are applied at B and C as shown in figure 17.17 and forces are determined in members.

Joint B

$k_5 \text{ Cos } \theta = 1, k_5 = \frac{5}{4} = 1.25 \text{ kN (comp)}$

$k_1 = k_5 \text{ Sin } \theta = \frac{5}{4} \text{ x } \frac{3}{5} = \frac{3}{4} = 0.75 \text{ kN (Tension)}$

Joint A

$k_4 \text{ Sin } \theta = k_1 = \frac{3}{4}$

$k_4 = \frac{5}{3} \text{ x } \frac{3}{4} = \frac{5}{4} \text{ kN (comp)}$

$k_3 = k_4 \text{ Cos } \theta = \frac{5}{4} \text{ x } \frac{4}{5} = 1 \text{ kN (Tension)}$

Joint C

$K_2 = k_4 \text{ Sin } \theta = \frac{5}{4} \text{ x } \frac{3}{5} = \frac{3}{4} = 0.75 \text{ kN (Tension)}$

$$X = - \frac{\sum \frac{FkL}{AE}}{\sum \frac{k^2L}{AE} + \frac{L_o}{A_oE}} = \frac{-\sum \frac{FkL}{A}}{\sum \frac{k^2L}{A} + \frac{L_o}{A_o}}$$

Computations are carried out in tabular form

Member	Force F (kN)	Force k (kN)	Length (mm)	Area (mm^2)	$\frac{FkL}{A}$	$\frac{k^2L}{A}$	Final Force F (F + k X)
1	0	0.75	3000	600	0	2.8125	3.128
2	-7.5	0.75	3000	600	- 28.125	2.8125	- 4.373
3	0	1.00	4000	900	0	4.4444	4.170
4	12.5	-1.25	5000	400	-195.312	19.5312	7.288
5	0	-1.25	5000	400	0	19.5312	- 5.213
				$\sum$	-223.437	49.1318	

$$\frac{L_o}{A_o} = \frac{4000}{900} = 4.44444$$

$$X = - \frac{-223.437}{(49.1318 + 4.4444)} = + 4.17 \text{ kN}$$

18 APPROXIMATE METHODS OF ANALYSIS OF STATICALLY INDETERMINATE STRUCTURES

Even the so called exact analysis of structures is approximate in the broad sense because every analysis is based on assumptions with respect to geometry and material behavior of structure. However, the approximate analysis for statically indeterminate structures is based on assumptions, which render the indeterminate system to a determinate system, which can be analyzed exactly by use of equations of equilibrium. The advantage is that analysis of determinate systems does not depend on elastic properties of members where as analysis of statically indeterminate systems depends on elastic properties of members. Approximate analysis is also required for assumption of initial elastic properties of members in order to carry out design of indeterminate structure. Approximate methods are also adopted under situation of configuration complexity of structure for which exact method of analysis is not available. Where exact analysis requires prohibitive time and cost it should be abandoned and approximate method adopted which generally gives conservative design. The first step in approximate analysis of statically indeterminate structure is to determine its statical indeterminacy and then introduce equal number of releases to make it determinate.

The following types of statically indeterminate structures are generally analyzed by approximate methods.

1. Parallel chord trusses with two diagonals in each panel.
2. Portals, trussed portals and mill bents.
3. Towers with straight legs.
4. Building frames.

The approximate analysis will be explained on following examples.

1. Parallel chord indeterminate truss shown in figure 18.1.

   The degree of static indeterminacy is 4 as four additional diagonal members have been introduced in just rigid or determinate simple truss. The assumption that will be made here is that in each panel the shear is equally divided between the two diagonals or in other words the numerical values of force in diagonal members of same panel are equal. This amounts to four independent assumptions or reduction of static indeterminacy by 4, hence, truss can be analyzed as determinate truss.

2. Fixed portal under horizontal load.

   For the portal shown in figure 18.2 the degree of static indeterminacy is 3. Hence, three assumptions are required. The points of contra flexure occur at mid height of columns. Hence, two hinges will be introduced at mid height as shown in Figure 18.3. The third assumption will be made as equal resistance to shear by the columns. Now analysis is possible by equations of equilibrium and is self explanatory from figure 18.4.

3. Trussed portal and mill bents.

   Here also, the hinges are introduced at mid height of columns and it is assumed that columns carry same horizontal shear as shown in figures 18.5 and 18.6.

   Vertical reaction at hinges = $\dfrac{P\left(\dfrac{H}{2}+h\right)}{L}$

   Other results are self explanatory.

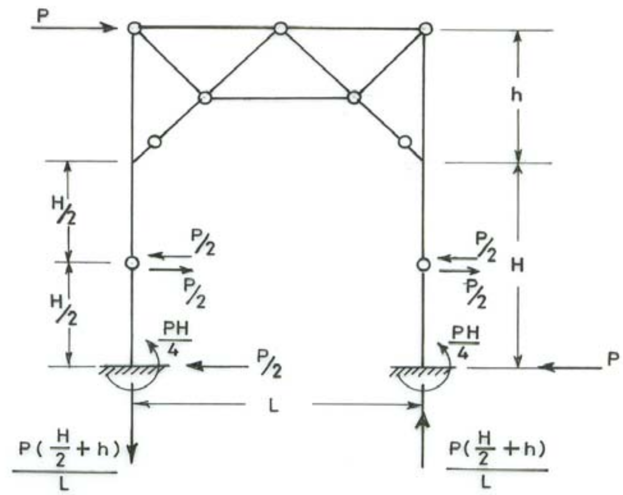


FIG. 18·5. DETERMINATE SYSTEM FOR APPROXIMATE ANALYSIS

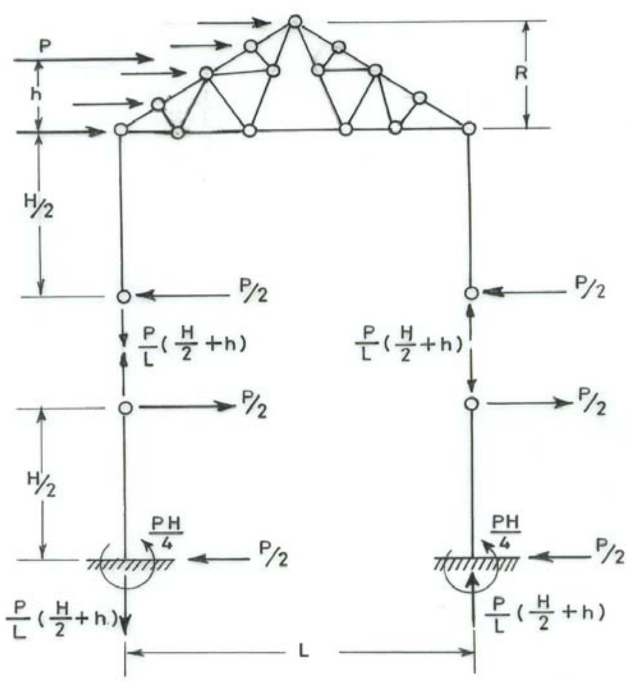


FIG. 18·6. DETERMINATE SYSTEM FOR APPROXIMATE ANALYSIS

4. Tower with straight legs shown in figure 18.7.

   Tower is a space truss. For approximate analysis it is necessary that each face of tower should be a plane. Each face is separated out as a plane truss as shown in figure 18.8 and equivalent inplane loads are shown. The indeterminate plane truss is replaced by two determinate plane trusses as shown figures 18.9 and 18.10 and the loads are apportioned on these trusses. Final result of forces in the members is obtained by superposing the results of individual trusses.

5. Plane multistory indeterminate frames as shown in figure 18.11.

   The analysis is carried out separately for vertical and lateral loads. For lateral loads following methods are used.

   (1) Portal method

   (2) Cantilever method

   (3) Factor method

Analysis for vertical loads

The analysis for vertical loads is carried out on following assumptions

(a) The axial force in the girder is zero

(b) A point of inflection occurs in the girder at one-tenth point measured along span from each end. The hinges are introduced at these points as shown in figure 18.12.

(c) The girder is analyzed for positive BM as a simply supported beam of span 0.8L and the vertical reactions are transferred to cantilever parts of span 0.1L as shown in Figure 18.13. The negative moments at the ends of beam are computed as the negative moments at the cantilever clamps as shown in figure 18.14. The vertical reactions of clamps are transferred to the columns as axial loads.

(d) At the beam column junction the clamp bending moment is distributed between the columns above and below in proportion to their stiffness. A typical girder AB is considered as shown in figure 18.15.

Maximum positive BM at centre = $\frac{\omega(0.8\,L)^2}{8} = 0.08\,\omega L^2$.

Maximum reaction at hinge = $0.4\omega L$.

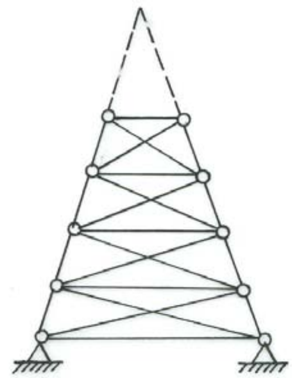


FIG.18·7. INDETERMINATE SPACE TRUSS TOWER

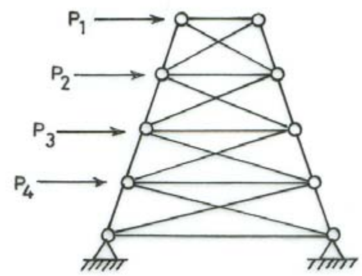
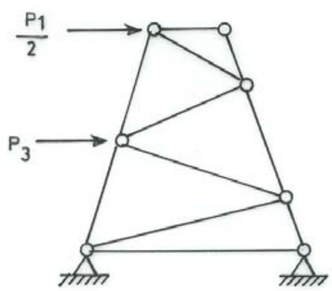


FIG. 18·8. INDETERMINATE PLANE TRUSS FOR APPROXIMATE ANALYSIS



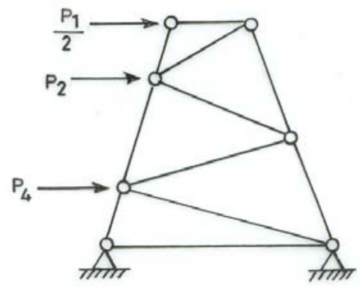


FIG. 18·9 & 18·10. DETERMINATE PLANE TRUSSES WITH APPORTIONED LOADS FOR APPROXIMATE ANALYSIS

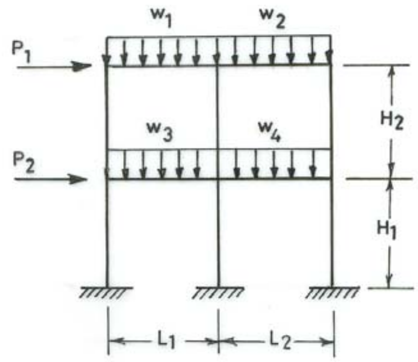


FIG.18·11. INDETERMINATE PLANE FRAME

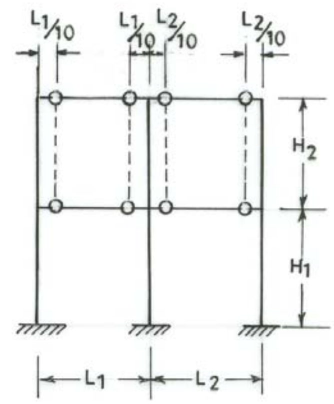


FIG.18·12. PLANE FRAME WITH HINGES IN BEAMS FOR APPROXIMATE ANALYSIS FOR VERTICAL LOADS

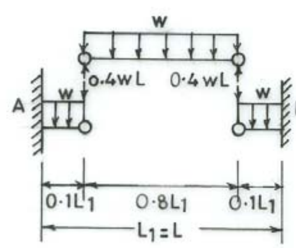


FIG.18·13. FREE BODY DIAGRAM OF GIRDER

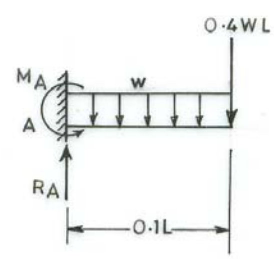


FIG.18·14. FREE BODY DIAGRAM OF CANTILEVER CLAMP

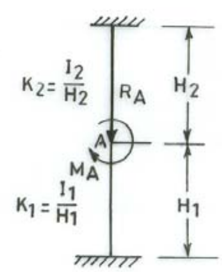


FIG.18·15. FREE BODY DIAGRAM OF COLUMNS AT A

Maximum negative bending moment is computed as cantilever clamp moment.

Maximum negative BM at end $M_A = \omega(0.1L)\,\frac{(0.1L)}{2} + 0.4\omega L\,(0.1L) = 0.045\,\omega L^2$

Reaction at end A $R_A = \omega(0.1L) + 0.\,4\omega L = 0.5\,\omega L$

At A axial load transmitted to column is $R_A = 0.5\,\omega L$.

The moment M_A is distributed to upper and lower columns proportional to their stiffnesses.

$$M_{AU} = M_A\,\frac{K_2}{(K_1 + K_2)}$$

$$M_{AL} = M_A\,\frac{K_1}{(K_1 + K_2)}$$

If at the joint their is beam on the other side also, its analysis can also be carried out similarly and forces in column superposed.

Analysis for horizontal loads

Portal Method

In portal method following assumptions are made.

(a) The point of contra flexure in each girder is located at its mid span point.

(b) The point of contra flexure in a column is located at mid height of each column.

(c) The total horizontal external shear on each storey is distributed between the column of that storey so that each interior column carries twice as much shear as the exterior column.

Shown in figure 18.16 is a plane frame with columns C_1 to C_6 and girders g_1 to g_4. It is subjected to lateral loads P and Q. The hinges are assumed at mid points of all the members and the horizontal shears are distributed in columns in accordance with the rules as shown in figure 18.17.

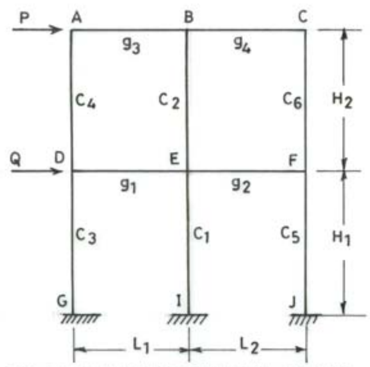


FIG. 18·16. INDETERMINATE FRAME UNDER HORIZONTAL LOADS

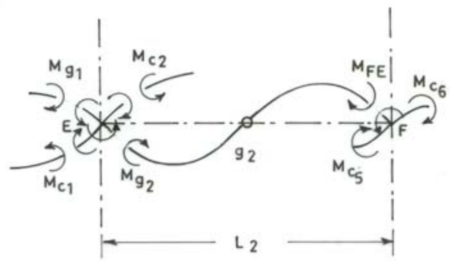


FIG. 18·18. FREE BODY DIAGRAM AT NODES OF GIRDER

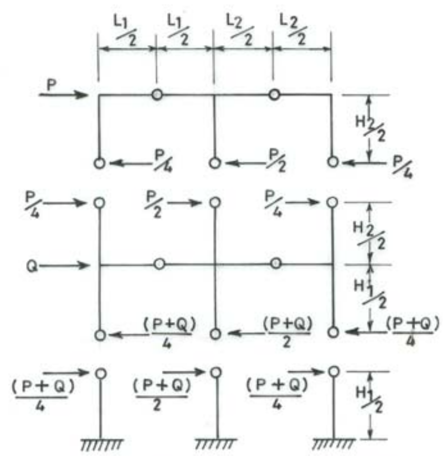


FIG. 18·17. FREE BODY DIAGRAMS FOR APPROXIMATE ANALYSIS FOR HORIZONTAL LOADS

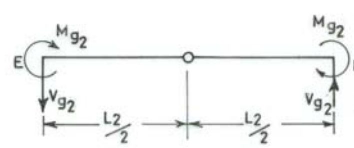


FIG. 18·19. FREE BODY DIAGRAM OF GIRDER

BM in column

BM in a column is equal to shear in column multiplied by half its height.

$$\text{BM in } C_1 = \left(\frac{P+Q}{2}\right)\frac{H_1}{2} = \frac{(P+Q)H_1}{4}$$

$$\text{BM in } C_2 = \frac{(PH_2)}{4}$$

BM in girder

The girder and column moments act in apposite directions at their junction in accordance with their elastic line or deflection curve as shown in figure 18.18.

At joint E:

$M_{C1} + M_{C2} = M_{g1} + M_{g2}$

Since column end moments are now known the beam end moments can be computed. At end F of g_2:

$M_{FE} = M_{C5} + M_{C6}$

Since moment at centre of girder is zero, $M_{g2} = M_{FE}$.

Hence, $M_{g1} = M_{C1} + M_{C2} - M_{g2}$.

Thus proceeding across the girder of first level the girder moments are determined. The girder moments at other level can also be determined in the same manner.

SF in girder

Considering free body of g_2, as shown in figure 18.19 the shear at ends is obtained

$$V_{g2} = \frac{2M_{g2}}{L_2}$$

Likewise shear is determined in all girders.

AF in columns

Axial forces in columns can be obtained by summing up from the top of the column, the shears transferred to the column by the girders connected to this column.

AF in girder

Axial forces in girders are obtained by summing up the horizontal forces from one end of the girder to the other end.

Cantilever method

The following assumptions are made.

1) As in portal method the points of contra flexure are assumed at mid points of all beams and columns as shown in figure 18.20.

2) The intensity of axial stress in each column of a storey is proportional to the horizontal distance of that column from the centre of gravity of all columns of that storey. The steps of this method are demonstrated on a three by two plane frame as shown in figure 18.20. It is assumed that all columns have same area of cross-section A. The distance of CG of columns x in horizontal direction from A is computed as follows as shown in figure 18.21.

$$x\,(4A) = AL_1 + A(L_1 + L_2) + A(L_1 + L_2 + L_3)$$

$$x = \frac{(3L_1 + 2L_2 + L_3)}{4}$$

F_{AE} = axial force in AE (T)

$$F_{BF} = \text{Axial force in BF} = \frac{(x - L_1)}{x} F_{AE} \text{ (T)}$$

$$F_{CG} = \text{Axial force in CG} = \frac{(L_1 + L_2 - x)}{x} F_{AE} \text{ (C)}$$

$$F_{DH} = \text{Axial force in DH} = \frac{(L_1 + L_2 + L_3 - x)}{x} F_{AE} \text{ (C)}$$

Taking a free body above the horizontal section through mid points of columns of first storey the expression for moment about mid point of column DH is set equal to zero.

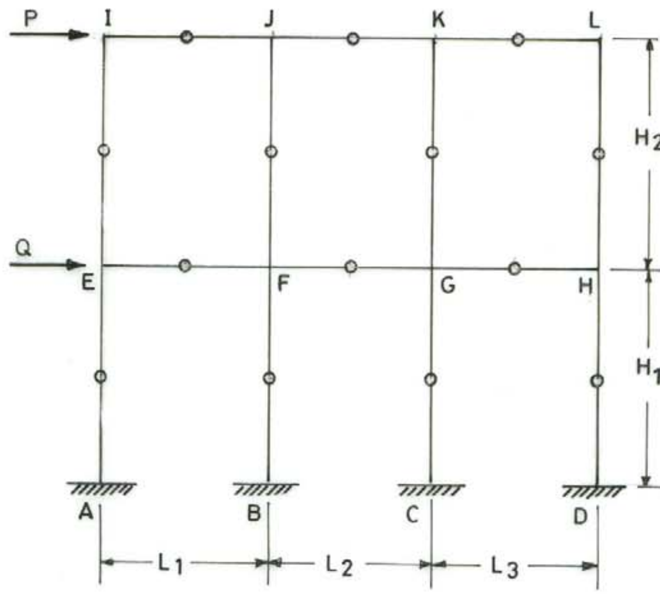


FIG. 18·20. INDETERMINATE PLANE FRAME WITH HINGES AT MID POINTS OF ALL MEMBERS

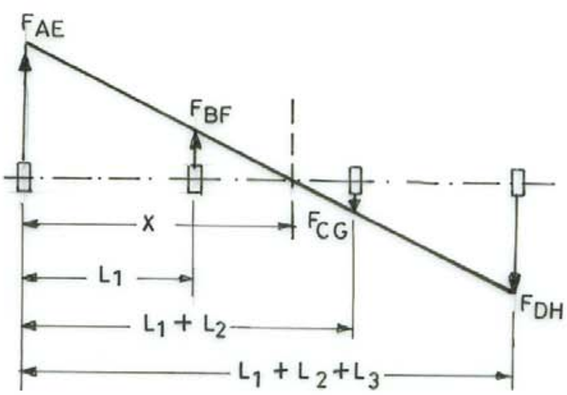


FIG. 18·21. FORCES IN COLUMNS

$$P\left(H_2+\frac{H_1}{2}\right)+Q\left(\frac{H_1}{2}\right)-F_{AE}\,(L_1+L_2+L_3)-\frac{F_{AE}}{x}(x-L_1)\,(L_2+L_3)$$

$$+\frac{F_{AE}}{x}(L_1+L_2-x)\,L_3=0$$

This gives the value of F_{AE}, hence, forces in columns of first storey are computed. The forces in columns of second storey are computed in the same manner by taking moment about mid point of column HL of the free body above the horizontal section through mid points of columns of second storey.

Girder shears

The girder shears can be obtained from the column axial forces at the various joints as shown in figure 18.22.

Joint I

$V_{IJ} = F_{IE}$

Joint E

$V_{EF} = F_{EA} - V_{IJ} = F_{EA} - F_{IE}$

Girder moments

The moment at each end of girder equals the shear in that girder multiplied by half the length of that girder as shown in figure 18.23. For example:

$$M_{EF} = V_{EF}\,\frac{L_1}{2}$$

Column moments

Column moments are determined by beginning at the top of each column stack and working progressively down to base as shown in figure 18.24.

Joint J

$$M_{JF} = V_{IJ}\,\frac{L_1}{2} + V_{JK}\,\frac{L_2}{2}$$

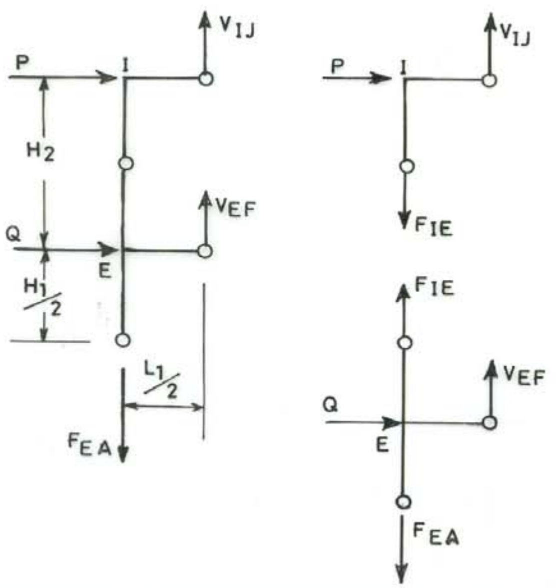


FIG. 18·22. FREE BODY DIAGRAMS FOR COMPUTATION OF SHEARS IN GIRDERS

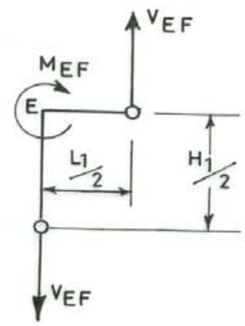


FIG. 18·23. FREE BODY DIAGRAM FOR COMPUTATION OF GIRDER MOMENT

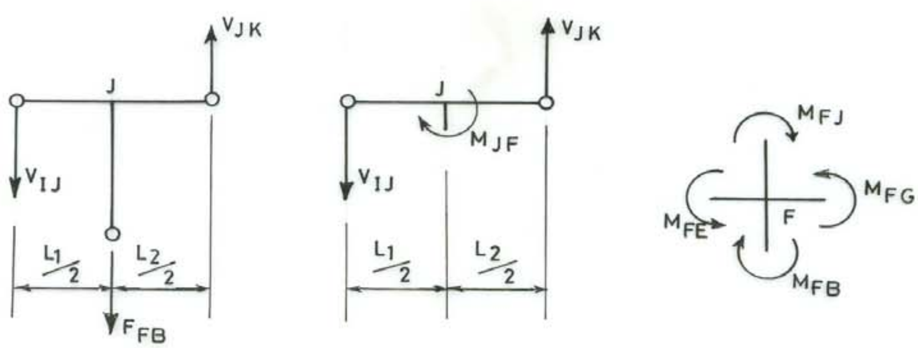


FIG. 18·24. FREE BODY DIAGRAMS FOR COMPUTATION OF COLUMN MOMENTS

Since there is a point of contra flexure at the center of FJ, M_{FJ} will be equal to M_{JF}.

$M_{FJ} = M_{JF}$

$M_{FB} + M_{FJ} = M_{FG} + M_{FE}$

$M_{FB} = M_{FG} + M_{FE} - M_{FJ}$

Because of hinge in middle of column:

$M_{BF} = M_{FB}$

Proceeding in this manner moments, shears and axial forces in all members are determined.

<u>The factor method</u>

This method is more accurate compared to portal and cantilever methods. The assumptions in factor method are based on elastic action of structure hence results of this method correspond to approximate slope-deflection analysis. The method is not discussed here.

***** <u>CONCLUDING REMARKS</u> *****

Readers are welcome to point out any errors noticed in the lecture course material so that corrections can be incorporated. Any suggestions are welcome by the author.