A FILL IN THE BLANKS

1. If the expression $\dfrac{\left[\sin\left(\dfrac{x}{2}\right)+\cos\left(\dfrac{x}{2}\right)-i\tan(x)\right]}{\left[1+2i\sin\left(\dfrac{x}{2}\right)\right]}$ is real, then the set of all possible values of x is... **(IIT 1987; 2M)**

2. For any two complex numbers z_1, z_2 and any real numbers a and b. $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = \ldots$ **(IIT 1988; 2M)**

3. If a and b are real numbers between 0 and 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle, then $a = \ldots$ and $b = \ldots$ **(IIT 1990)**

4. $ABCD$ is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy $BD = 2AC$. If the points D and M represent the complex numbers $1+i$ and $2-i$ respectively, then A represents the complex number ...or... **(IIT 1993; 2M)**

5. Suppose z_1, z_2, z_3 are the vertices of an equilateral triangle inscribed in the circle $|z| = 2$. If $z_1 = 1 + i\sqrt{3}$, then $z_2 = \ldots$, $z_3 = \ldots$ **(IIT 1994; 2M)**

6. The value of the expression
$1\cdot(2-\omega)(2-\omega^2)+2(3-\omega)(3-\omega^2)+\ldots+(n-1)\cdot(n-\omega)(n-\omega^2)$, where ω is an imaginary cube root of unity, is... **(IIT 1996; 2M)**

B TRUE / FALSE

1. For complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we write $z_1 \cap z_2$, if $x_1 \le x_2$ and $y_1 \le y_2$. Then for all complex numbers z with $1 \cap z$, we have $\dfrac{1-z}{1+z} \cap 0$ **(IIT 1981; 2M)**

2. If the complex numbers, z_1, z_2 and z_3 represent the vertices of an equilateral triangle such that $|z_1| = |z_2| = |z_3|$, then $z_1 + z_2 + z_3 = 0$ **(IIT 1984; 1M)**

3. The cube roots of unity when represented on argand diagram form the vertices of an equilateral triangle. **(IIT 1988; 1M)**

C OBJECTIVE QUESTIONS

➥ ***Only one option is correct :***

1. The smallest positive integer n for which $\left(\dfrac{1+i}{1-i}\right)^n = 1$, is : **(IIT 1980)**
(a) 8
(b) 16
(c) 12
(d) none of these

2. The complex numbers $z = x + iy$ which satisfy the equation $\left|\dfrac{z-5i}{z+5i}\right| = 1$, lie on : **(IIT 1981; 2M)**
(a) the x-axis
(b) the straight line $y = 5$
(c) a circle passing through the origin
(d) none of these

3. If $z = \left(\dfrac{\sqrt{3}}{2}+\dfrac{i}{2}\right)^5 + \left(\dfrac{\sqrt{3}}{2}-\dfrac{i}{2}\right)^5$, then : **(IIT 1982; 2M)**
(a) Re $(z) = 0$ (b) Im $(z) = 0$
(c) Re $(z) > 0$, Im $(z) > 0$ (d) Re $(z) > 0$, Im $(z) < 0$

4. The inequality $|z-4| < |z-2|$ represents the region given by : **(IIT 1982; 2M)**
(a) Re $(z) \ge 0$ (b) Re $(z) < 0$
(c) Re $(z) > 0$ (d) none of these

5. If $z = x + iy$ and $w = (1 - iz)/(z - i)$, then $|w| = 1$ implies that , in the complex plane : **(IIT 1983; 1M)**
(a) z lies on the imaginary axis
(b) z lies on the real axis
(c) z lies on the unit circle
(d) none of these

6. The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order, if and only if : **(IIT 1983; 1M)**

(a) $z_1 + z_4 = z_2 + z_3$ (b) $z_1 + z_3 = z_2 + z_4$
(c) $z_1 + z_2 = z_3 + z_4$ (d) none of these

7. If a, b, c and u, v, w are complex numbers representing the vertices of two triangle such that $c = (1-r)a + rb$ and $w = (1-r)u + rv$, where r is a complex number, then the two triangles : **(IIT 1985; 2M)**

(a) have the same area (b) are similar
(c) are congruent (d) none of these

8. The value of $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right)$ is : **(IIT 1987; 2M)**

(a) -1 (b) 0
(c) $-i$ (d) i
(e) none of these

9. If z_1 and z_2 are two non zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then arg z_1 − arg z_2 is equal to : **(IIT 1987; 2M)**

(a) $-\pi$ (b) $-\frac{\pi}{2}$
(c) 0 (d) $\frac{\pi}{2}$
(e) π

10. The complex numbers $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other, for : **(IIT 1988; 2M)**

(a) $x = n\pi$ (b) $x = 0$
(c) $x = (n + 1/2)\pi$ (d) no value of x

11. If $\omega\,(\neq 1)$ is a cube root of unity and $(1+\omega)^7 = A + B\omega$, then A and B are respectively : **(IIT 1995)**

(a) 0, 1 (b) 1, 1
(c) 1, 0 (d) −1, 1

12. Let z and w be two non zero complex numbers such that $|z| = |w|$ and arg z + arg $w = \pi$, then z equals : **(IIT 1995)**

(a) w (b) $-w$
(c) $\overline{w}$ (d) $-\overline{w}$

13. Let z and w be two complex numbers such that $|z| \le 1$, $|w| \le 1$ and $|z + iw| = |z - i\overline{w}| = 2$, then z equals : **(IIT 1995)**

(a) 1 or i (b) i or $-i$
(c) 1 or −1 (d) i or −1

14. For positive integers n_1, n_2 the value of expression $(1+i)^{n_1} + (1+i^3)^{n_1} + (1+i^5)^{n_2} + (1+i^7)^{n_2}$, here $i = \sqrt{-1}$ is a real number, if and only if : **(IIT 1996)**

(a) $n_1 = n_2 + 1$ (b) $n_1 = n_2 - 1$
(c) $n_1 = n_2$ (d) $n_1 > 0,\ n_2 > 0$

15. If ω is an imaginary cube root of unity, then $(1 + \omega - \omega^2)^7$ is equal to : **(IIT 1998; 2M)**

(a) 128ω (b) -128ω
(c) $128\omega^2$ (d) $-128\omega^2$

16. The value of sum $\sum_{n=1}^{13}(i^n + i^{n+1})$, where $i = \sqrt{-1}$ equals : **(IIT 1998; 2M)**

(a) i (b) $i - 1$
(c) $-i$ (d) 0

17. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then : **(IIT 1998; 2M)**

(a) $x = 3,\ y = 1$ (b) $x = 1,\ y = 1$
(c) $x = 0,\ y = 3$ (d) $x = 0,\ y = 0$

18. If $i = \sqrt{-1}$, then $4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365}$ is equal to : **(IIT 1999; 2M)**

(a) $1 - i\sqrt{3}$ (b) $-1 + i\sqrt{3}$
(c) $i\sqrt{3}$ (d) $-i\sqrt{3}$

19. If arg $(z) < 0$, then arg $(-z)$ − arg (z) = : **(IIT 2000)**

(a) π (b) $-\pi$
(c) $-\pi/2$ (d) $\pi/2$

20. If z_1, z_2 and z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$, then $|z_1 + z_2 + z_3|$ is : **(IIT 2000)**

(a) equal to 1 (b) less than 1
(c) greater than 3 (d) equal to 3

21. Let z_1 and z_2 be n^{th} roots of unity which subtend a right angle at the origin, then n must be of the form : **(IIT 2001)**

(a) $4k + 1$ (b) $4k + 2$
(c) $4k + 3$ (d) $4k$

22. The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1 - z_2}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$ are the vertices of a triangle which is : **(IIT 2001)**

(a) of area zero
(b) right-angled isosceles
(c) equilateral
(d) obtuse-angled isosceles

23. Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, then value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$ is : **(IIT 2002)**

(a) 3ω (b) $3\omega(\omega - 1)$
(c) $3\omega^2$ (d) $3\omega(1 - \omega)$

24. For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, the minimum value of $|z_1 - z_2|$ is : **(IIT 2002)**

(a) 0 (b) 2
(c) 7 (d) 17

25. If $|z|=1$ and $w=\dfrac{z-1}{z+1}$ (where $z \neq -1$), then Re (w) is : **(IIT 2003)**

(a) 0 (b) $\dfrac{1}{|z+1|^2}$

(c) $\left|\dfrac{1}{z+1}\right| \cdot \dfrac{1}{|z+1|^2}$ (d) $\dfrac{\sqrt{2}}{|z+1|^2}$

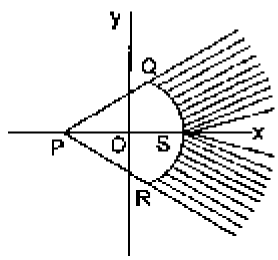
26. If $\omega(\neq 1)$ be a cube root of unity and $(1+\omega^2)^n=(1+\omega^4)^n$, then the least positive value of n is **(IIT 2004)**

(a) 2 (b) 3
(c) 5 (d) 6

27. The minimum value of $|a+b\omega+c\omega^2|$, where a, b and c are all not equal integers and $\omega\ (\neq 1)$ is a cube root of unity, is : **(IIT 2005)**

(a) $\sqrt{3}$ (b) $\dfrac{1}{2}$
(c) 1 (d) 0

28. The shaded region, where $P=(-1,0)$, $Q=(-1+\sqrt{2}, \sqrt{2})$ $R=(-1+\sqrt{2}, -\sqrt{2})$ $S=(1,0)$ is represented by : **(IIT 2005)**



(a) $|z+1|>2, |arg(z+1)|<\dfrac{\pi}{4}$

(b) $|z+1|<2, |arg(z+1)|<\dfrac{\pi}{2}$

(c) $|z+1|>2, |arg(z+1)|>\dfrac{\pi}{4}$

(d) $|z-1|<2, |arg(z+1)|>\dfrac{\pi}{2}$

29. If $w=\alpha+i\beta$, where $\beta \neq 0$ and $z \neq 1$, satisfies the condition that $\left(\dfrac{w-\overline{w}z}{1-z}\right)$ is purely real, then the set of values of z is : **(IIT 2006)**

(a) $|z|=1, z \neq 2$ (b) $|z|=1$ and $z \neq 1$
(c) $z=\overline{z}$ (d) none of these

30. A man walks a distance of 3 units from the origin towards the north-east (N 45° E) direction. From there, he walks a distance of 4 units towards the north-west (N 45° W) direction to reach a point P. Then the position of P in the Argand plane is : **(IIT 2007)**

(a) $3e^{i\pi/4}+4i$ (b) $(3-4i)e^{i\pi/4}$
(c) $(4+3i)e^{i\pi/4}$ (d) $(3+4i)e^{i\pi/4}$

31. If $|z|=1$ and $z \neq \pm 1$, then all the values of $\dfrac{z}{1-z^2}$ lie on : **(IIT 2007)**

(a) a line not passing through the origin
(b) $|z|=\sqrt{2}$
(c) the x-axis
(d) the y-axis

D OBJECTIVE QUESTIONS

➡ ***More than one options are correct :***

1. If $z_1=a+ib$ and $z_2=c+id$ are complex numbers such that $|z_1|=|z_2|=1$ and Re $(z_1\overline{z}_2)=0$, then the pair of complex numbers $w_1=a+ic$ and $w_2=b+id$ satisfies : **(IIT 1985; 2M)**

(a) $|w_1|=1$ (b) $|w_2|=1$
(c) Re $(w_1\overline{w}_2)=0$ (d) none of these

2. Let z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1|=|z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $\dfrac{z_1+z_2}{z_1-z_2}$ may be . **(IIT 1986; 2M)**

(a) zero (b) real and positive
(c) real and negative (d) purely imaginary
(e) none of these

E SUBJECTIVE QUESTIONS

1. It is given that n is an odd integer greater than 3, but n is not a multiple of 3. Prove that x^3+x^2+x is a factor of $(x+1)^n-x^n-1$: **(IIT 1980)**

2. Find the real values of x and y for which the following equation is satisfied :
$$\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i.$$ **(IIT 1980)**

3. Let the complex numbers z_1, z_2 and z_3 be the vertices of an equilateral triangle. Let z_0 be the circumcentre of the triangle. Then prove that : **(IIT 1981; 4M)**
$$z_1^2+z_2^2+z_3^2=3z_0^2.$$

4. A relation R on the set of complex numbers is defined by $z_1\ R\ z_2$, if and only if $\dfrac{z_1-z_2}{z_1+z_2}$ is real. **(IIT 1982)**

Show that R is an equivalence relation.

5. Prove that the complex numbers z_1, z_2 and the origin form an equilateral triangle only if $z_1^2+z_2^2-z_1z_2=0$. **(IIT 1983)**

6. If $1, a_1, a_2, \ldots, a_{n-1}$ are the n roots of unity, then show that **(IIT 1984)**
$$(1-a_1)(1-a_2)(1-a_3)\ldots(1-a_{n-1})=n$$

7. Show that the area of the triangle on the argand diagram formed by the complex number z, iz and $z+iz$ is $\frac{1}{2}|z|^2$ (IIT 1986, $2\frac{1}{2}$ M)

8. Complex numbers z_1, z_2, z_3 are the vertices A, B, C respectively of an isosceles right angled triangle with right angle at C. show that

$$(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$$ (IIT 1986; $2\frac{1}{2}$ M)

9. Let $z_1 = 10+6i$ and $z_2 = 4+6i$. If z is any complex number such that the argument of $(z-z_1)/(z-z_2)$ is $\pi/4$, then prove that $|z-7-9i| = 3\sqrt{2}$. (IIT 1991; 4M)

10. If $iz^3 + z^2 - z + i = 0$, then show that $|z| = 1$. (IIT 1995; 5M)

11. $|z| \le 1, |w| \le 1$, show that

$$|z-w|^2 \le (|z|-|w|)^2 + (\arg z - \arg w)^2$$ (IIT 1995; 5M)

12. Find all non-zero complex numbers z satisfying $\bar{z} = iz^2$. (IIT 1996; 2M)

13. Let z_1 and z_2 be roots of the equation $z^2 + pz + q = 0$. where the coefficients p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = \alpha \ne 0$ and $OA = OB$, where O is the origin prove that $p^2 = 4q\cos^2\left(\frac{\alpha}{2}\right)$ (IIT 1997; 5M)

14. Let $\bar{b}z + b\bar{z} = c$, $b \ne 0$, be a line in the complex plane, where $\bar{b}$ is the complex conjugate of b. If a point z_1 is the reflextion of the point z_2 through the line, then show that $c = \bar{z}_1 b + z_2\bar{b}$. (IIT 1997C; 5M)

15. For complex numbers z and w, prove that $|z|^2 w - |w|^2 z = z - w$, if and only if $z = w$ or $z\bar{w} = 1$. (IIT 1999; 10M)

16. Let a complex number $\alpha, \alpha \ne 1$, be a root of the equation $z^{p+q} - z^p - z^q + 1 = 0$ (IIT 2002; 5M)

Where p, q are distinct primes. Show that either

$$1+\alpha+\alpha^2+\dots+\alpha^{p-1} = 0$$

or

$$1+\alpha+\alpha^2+\dots+\alpha^{q-1} = 0$$

but not both together.

17. If z_1 and z_2 are two complex numbers such that $|z_1| < 1 < |z_2|$, then prove that

$$\left|\frac{1-z_1\bar{z}_2}{z_1 - z_2}\right| < 1.$$ (IIT 2003; 2M)

18. Prove that there exists no complex number z such that $|z| < \frac{1}{3}$ and $\sum_{r=1}^{n} a_r z^r = 1$, where $|a_r| < 2$. (IIT 2003; 2M)

19. Find the centre and radius of the circle formed by all the points represented by $z = x+iy$ satisfying the relation $\left|\frac{z-\alpha}{z-\beta}\right| = k\ (k \ne 1)$, where α and β are constant complex numbers given by $\alpha = \alpha_1 + i\alpha_2, \beta = \beta_1 + i\beta_2$. (IIT 2004; 2M)

20. If one of the vertices of the square circumscribing the circle $|z-1| = \sqrt{2}$ is $2+\sqrt{3}i$. Find the other vertices of square. (IIT 2005)

ANSWERS

A Fill in the Blanks

1. $x = 2n\pi + 2\alpha, \alpha = \tan^{-1}k$, where $k \in (1, 2)$ or $x = 2n\pi$
2. $(a^2+b^2)(|z_1|^2 + |z_2|^2)$
3. $a = b = 2 \pm \sqrt{3}$
4. $3 - \frac{i}{2}$ or $1 - \frac{3i}{2}$
5. $z_2 = -2, z_3 = 1 - i\sqrt{3}$
6. $\frac{1}{4}n(n-1)(n^2+3n+4)$

B True / False

1. True 2. True 3. True

C Objective Questions (Only one option)

1. (d)	2. (a)	3. (b)	4. (d)	5. (b)	6. (b)	7. (b)
8. (d)	9. (c)	10. (d)	11. (b)	12. (d)	13. (c)	14. (d)
15. (d)	16. (b)	17. (d)	18. (c)	19. (a)	20. (a)	21. (d)
22. (c)	23. (b)	24. (b)	25. (a)	26. (b)	27. (c)	28. (a)
29. (b)	30. (d)	31. (d)				

D Objective Questions (More than one option)

1. (a, b, c) 2. (a, d)

E Subjective Questions

2. $x = 3$ and $y = -1$

12. $z = i, \pm\frac{\sqrt{3}}{2} - \frac{i}{2}$

19. Centre $= \frac{\alpha - k^2\beta}{1-k^2}$, Radius $= \left|\frac{k(\alpha-\beta)}{1-k^2}\right|$

20. $z_2 = -\sqrt{3}\,i$, $z_3 = (1-\sqrt{3}) + i$ and $z_4 = (1+\sqrt{3}) - i$

A FILL IN THE BLANKS

1. If

$$\frac{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)-i\tan x}{1+2i\sin\frac{x}{2}}\in R$$

$$\Rightarrow \frac{\left\{\sin\frac{x}{2}+\cos\frac{x}{2}-i\tan x\right\}\left\{1-2i\sin\frac{x}{2}\right\}}{1+4\sin^2\frac{x}{2}}$$

It will be real, if imaginary part is zero

$\therefore \quad -2\sin\frac{x}{2}\left\{\sin\frac{x}{2}+\cos\frac{x}{2}\right\}-\tan x=0$

or $2\sin\frac{x}{2}\left\{\sin\frac{x}{2}+\cos\frac{x}{2}\right\}\cos x+2\sin\frac{x}{2}\cos\frac{x}{2}=0$

or $\sin\frac{x}{2}\left[\left\{\sin\frac{x}{2}+\cos\frac{x}{2}\right\}\left\{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right\}+\cos\frac{x}{2}\right]=0$

$\therefore \quad \sin\frac{x}{2}=0 \Rightarrow x=2n\pi \quad \ldots(1)$

or $\left\{\sin\frac{x}{2}+\cos\frac{x}{2}\right\}\left\{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right\}+\cos\frac{x}{2}=0$

dividing by $\cos^3\frac{x}{2}$

$$\left(\tan\frac{x}{2}+1\right)\left(1-\tan^2\frac{x}{2}\right)+\left(1+\tan^2\frac{x}{2}\right)=0$$

$\Rightarrow \tan^3\frac{x}{2} - \tan\frac{x}{2}-2=0$

Let $\quad \tan\frac{x}{2}=t$

$$f(t)=t^3-t-2$$

then, $f(1)=-2<0$ and $f(2)=4>0$

Thus $f(t)$ changes sign from negative to positive in $(1,2)$

$\therefore$ let $t=k$ be the root for which

$$f(k)=0 \quad \text{and} \quad k\in(1,2)$$

$\therefore \quad t=k \quad$ or $\quad \tan\frac{x}{2}=k=\tan\alpha$

Hence, $\quad \frac{x}{2}=n\pi+\alpha$

$\Rightarrow \begin{cases} x=2n\pi+2\alpha,\ \alpha=\tan^{-1}k, \text{ where } k\in(1,2) \\ \text{or } \ x=2n\pi \end{cases}$

2. $|az_1-bz_2|^2+|bz_1+az_2|^2$

$=\{a^2|z_1|^2+b^2|z_2|^2-2ab\,\mathrm{Re}\,(z_1\bar{z}_2)\}$

$+\{b^2|z_1|^2+a^2|z_2|^2+2ab\,\mathrm{Re}\,(z_1\bar{z}_2)\}$

$=(a^2+b^2)(|z_1|^2+|z_2|^2)$

3. Since z_1, z_2 and z_3 forms an equilateral Δ

$\Rightarrow z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$

or $(a+i)^2+(1+ib)^2+(0)^2=(a+i)(1+ib)+0+0$

$\Rightarrow a^2-1+2ai+1-b^2+2ib=a+i(ab+1)-b$

$\Rightarrow \quad (a^2-b^2)+2i(a+b)=(a-b)+i(ab+1),$

on comparing $a^2-b^2=a-b$ and $2(a+b)=ab+1$

$\Rightarrow \quad (a-b)(a+b-1)=0 \quad$ and $\quad 2(a+b)=ab+1$

$\Rightarrow \quad (a=b$ or $a+b=1) \quad$ and $\quad 2(a+b)=ab+1$

If $a=b \Rightarrow 2(2a)=a^2+1$

or $\quad a^2-4a+1=0$

$\Rightarrow \quad a=\frac{4\pm\sqrt{16-4}}{2}=2\pm\sqrt{3}$

If $a+b=1$,

$\Rightarrow \quad 2=a(1-a)+1$

$\Rightarrow \quad a^2-a+1=0$

$\Rightarrow a=\frac{1\pm\sqrt{1-4}}{2}$, but a and $b\in R$

$\therefore$ only solution when $a=b$

$\Rightarrow \quad a=b=2\pm\sqrt{3}$

4. $D=(1+i)$, $M=(2-i)$ (given)

and diagonals of a rhombus bisect each other.

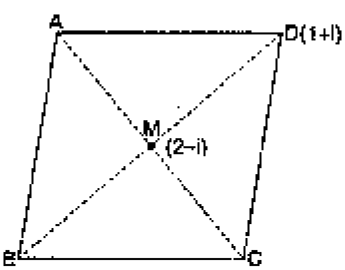
Let $B\equiv(a+ib)$, therefore

$$\frac{a+1}{2}=2,\ \frac{b+1}{2}=-1$$

$\Rightarrow \quad a+1=4,\ b+1=-2$

$\Rightarrow \quad a=3,\ b=-3$

$\Rightarrow \quad B\equiv(3-3i)$



Again $DM=\sqrt{(2-1)^2+(-1-1)^2}=\sqrt{1+4}=\sqrt{5}$

but $\quad BD=2DM \Rightarrow BD=2\sqrt{5}$

and $\quad 2AC=BD \Rightarrow 2AC=2\sqrt{5} \Rightarrow AC=\sqrt{5}$

and $\quad AC=2AM \Rightarrow \sqrt{5}=2AM \Rightarrow AM=\frac{\sqrt{5}}{2}$.

Now, let coordinate of A be $(x+iy)$

but in a rhombus $AD=AB$, therefore we have $AD^2=AB^2$

$$\Rightarrow \quad (x-1)^2+(y-1)^2=(x-3)^2+(y+3)^2$$

$$\Rightarrow \quad x^2+1-2x+y^2+1-2y=x^2+9-6x+y^2+9+6y$$

$$\Rightarrow \quad 4x-8y=16$$

$$\Rightarrow \quad x-2y=4$$

$$\Rightarrow \quad x=2y+4 \qquad \ldots\text{(i)}$$

Again $\quad AM=\dfrac{\sqrt{5}}{2}$

$$\Rightarrow \quad AM^2=\frac{5}{4}$$

$$\Rightarrow \quad (x-2)^2+(y+1)^2=\frac{5}{4}$$

From eq. (i) putting the value of x

$$\Rightarrow \quad (2y+4-2)^2+(y+1)^2=\frac{5}{4}$$

$$\Rightarrow \quad (2y+2)^2+(y+1)^2=5/4$$

$$\Rightarrow \quad 4y^2+4+8y+y^2+1+2y=5/4$$

$$\Rightarrow \quad 5y^2+10y+5=5/4$$

$$\Rightarrow \quad 20y^2+40y+20=5$$

$$\Rightarrow \quad 20y^2+40y+15=0$$

$$\Rightarrow \quad 4y^2+8y+3=0$$

$$\Rightarrow \quad 4y^2+6y+2y+3=0$$

$$\Rightarrow \quad 2y(2y+3)+1(2y+3)=0$$

$$\Rightarrow \quad (2y+1)(2y+3)=0$$

$$\Rightarrow \quad 2y+1=0,\ 2y+3=0$$

$$\Rightarrow \quad y=-1/2,\ y=-3/2$$

Putting these values in eq. (i)

$$x=2(-1/2)+4,\ x=2(-3/2)+4$$

$$\Rightarrow \quad x=-1+4,\ x=-3+4$$

$$\Rightarrow \quad x=3,\ x=1$$

Hence, A is either $3-\dfrac{i}{2}$ or $1-\dfrac{3i}{2}$

Alter

As M is the centre of Rhombus,

$\therefore$ By rotating D about M through an angle of $\pm\pi/2$, we get possible position of A.

$$\Rightarrow \quad \frac{z_3-z_2}{z_1-z_2}=\frac{|z_3-z_2|}{|z_1-z_2|}e^{\pm i\pi/2}$$

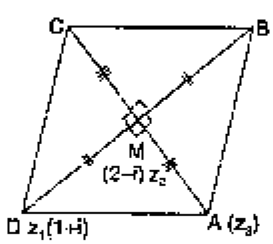
$$\Rightarrow \quad \frac{z_3-(2-i)}{-1+2i}=\frac{1}{2}(\pm i)$$

$$\Rightarrow \quad z_3=(2-i)\pm\frac{1}{2}i(2i-1)$$

$$=(2-i)\pm\frac{1}{2}(-2-i)$$

$$=\frac{(4-2i-2-i)}{2},\ \frac{4-2i+2+i}{2}$$

$$=1-\frac{3}{2}i,\ 3-\frac{i}{2}$$



$\therefore$ A is either $\left(1-\dfrac{3}{2}i\right)$ or $\left(3-\dfrac{i}{2}\right)$.

5. $z_1=1+i\sqrt{3}=r(\cos\theta+i\sin\theta)$ (let)

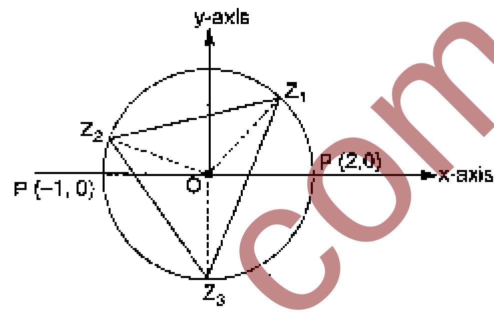
$$\Rightarrow r\cos\theta=1,\ r\sin\theta=\sqrt{3}$$

$$\Rightarrow \quad r=2 \quad \text{and} \quad \theta=\pi/3$$

so $\quad z_1=2(\cos\pi/3+(\sin\pi/3))$

since $\quad |z_2|=|z_3|=2$ (given)

Now the triangle z_1, z_2 and z_3 being an equilateral and the sides z_1z_2 and z_1z_3 make an angle $2\pi/3$ at the centre.



therefore $\quad \angle POz_2=\dfrac{\pi}{3}+\dfrac{2\pi}{3}=\pi$

and $\quad \angle POz_3=\dfrac{\pi}{3}+\dfrac{2\pi}{3}+\dfrac{2\pi}{3}=\dfrac{5\pi}{3}$

therefore $\quad z_2=2(\cos\pi+i\sin\pi)=2(-1+0)=-2$

and $\quad z_3=2\left(\cos\dfrac{5\pi}{3}+i\sin\dfrac{5\pi}{3}\right)=2\left(\dfrac{1}{2}-i\dfrac{\sqrt{3}}{2}\right)=1-i\sqrt{3}$

Alter

Whenever vertices of an equilateral Δ having centroid is given its vertices are of the from z, $z\omega$, $z\omega^2$.

$\therefore$ If one of the vertex is $z_1=1+i\sqrt{3}$,

the other two are $(z_1\omega)$, $(z_1\omega^2)$

$$\Rightarrow \quad (1+i\sqrt{3})\frac{(-1+i\sqrt{3})}{2},\ (1+i\sqrt{3})\frac{(-1-i\sqrt{3})}{2}$$

$$\Rightarrow \quad \frac{-(1+3)}{2},\ -\frac{(1+i^2(\sqrt{3})^2+2i\sqrt{3})}{2}$$

$$\Rightarrow \quad -2,\ -\frac{(-2+2i\sqrt{3})}{2}=1-i\sqrt{3}$$

$$\therefore \quad z_2=-2 \quad \text{and} \quad z_3=1-i\sqrt{3}.$$

6. $\quad T_r=r[(r+1)-\omega][(r+1)-\omega^2]$

$$=r[(r+1)^2-(\omega+\omega^2)(r+1)+\omega^3]$$

$$=r[(r+1)^2+(r+1)+1]$$

$$=r[r^2+1+2r+r+1+1]$$

$$=r[r^2+3r+3]$$

$$=r^3+3r^2+3r$$

Therefore, the sum of the given series

$$=\sum_{r=1}^{(n-1)}(r^3+3r^2+3r)$$

$$= \left|\frac{(n-1)(n)}{2}\right|^2 - 3\left[\frac{(n-1)\,n(2n-1)}{6}\right] + 3\,\frac{(n-1)(n)}{2}$$

$$= (n-1)\,(n)\left[\frac{(n-1)(n)}{4} + \frac{(2n-1)}{2} + \frac{3}{2}\right]$$

B TRUE / FALSE

1. Let $z = x + iy$

$\Rightarrow$ $1 \cap z$ gives $1 \cap x + iy$

or $1 \le x$ and $0 \le y$ ...(1)

also $\dfrac{1-z}{1+z} \cap 0$

$$\frac{1-x-iy}{1-x+iy} \cap 0$$

$$\frac{(1-x-iy)\,(1+x-iy)}{(1+x+iy)\,(1+x-iy)} \cap 0+0i$$

$$\Rightarrow \frac{1-x^2-y^2}{(1+x)^2+y^2} - \frac{2iy}{(1+x)^2+y^2} \cap 0+0i$$

$$\Rightarrow \quad x^2+y^2 \ge 1 \quad \text{and} \quad -2y \le 0$$

C OBJECTIVE (ONLY ONE OPTION)

1. As, $\left(\dfrac{1+i}{1-i}\right)^n = 1 \Rightarrow \left(\dfrac{1+i}{1-i} \times \dfrac{1+i}{1+i}\right)^n = 1$

$\Rightarrow$ $\left(\dfrac{2i}{2}\right)^n - 1 \Rightarrow i^n = 1$

The smallest (+ve) integer n for which $i^n = 1$ is 4.

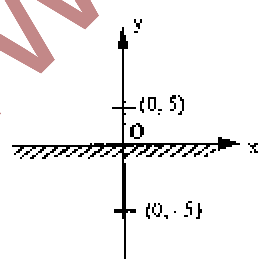
$\therefore$ $n = 4$

Hence, (d) is the correct answer.

2. $\left|\dfrac{z-5i}{z+5i}\right| = 1 \Rightarrow |z-5i| - |z+5i|$

{using, definition $|z - z_1| = |z - z_2|$ gives perpendicular bisector of z_1 and z_2}

$\therefore$ Perpendicular bisector of (0, 5) and (0, – 5)



$\Rightarrow$ x-axis.

Hence, (a) is the correct answer.

$$= \frac{1}{4}(n-1)\,(n)\,[(n-1)n + 2(2n-1) + 6]$$

$$= \frac{1}{4}(n-1)\,n[n^2 - n + 4n - 2 + 6]$$

$$= \frac{1}{4}(n-1)\,n[n^2 + 3n + 4]$$

or $x^2 + y^2 \ge 1$ and $y \ge 0$ which is true by (1).

2. If z_1, z_2, z_3 are vertices of equilateral Δ and $|z_1| = |z_2| = |z_3|$

$\Rightarrow$ z_1, z_2, z_3 lie on a circle with centre at origin.

$\Rightarrow$ Circum centre = Centroid $\Rightarrow 0 = \dfrac{z_1 + z_2 + z_3}{3}$

$\therefore$ $z_1 + z_2 + z_3 = 0$

3. Since, cube root of unity are 1, ω, ω^2 given by,

$$A\,(1, 0),\ B\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), C\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

$\Rightarrow AB = BC = CA = \sqrt{3}$. Thus cube roots of unity form an equilateral Δ.

3. As $z = \left(\dfrac{\sqrt{3}}{2} + \dfrac{i}{2}\right)^5 + \left(\dfrac{\sqrt{3}}{2} - \dfrac{i}{2}\right)^5$

{we know, $\omega = \dfrac{-1 + i\sqrt{3}}{2}$ and $\omega^2 = \dfrac{-1 - i\sqrt{3}}{2}$}

$\therefore$ $\dfrac{\sqrt{3}+i}{2} = -i\left(\dfrac{-1+i\sqrt{3}}{2}\right) = -\omega i$

and $\dfrac{\sqrt{3}-i}{2} = i\left(\dfrac{-1-i\sqrt{3}}{2}\right) = \omega^2 i$

$\Rightarrow$ $z = (-i\omega)^5 + (i\omega^2)^5 = -i\omega^2 + i\omega$

$= i(\omega - \omega^2)$

$= i(i\sqrt{3}) = -\sqrt{3}$

$\Rightarrow$ Re$(z) < 0$, and Im$(z) = 0$

Hence, (b) is correct answer.

Alter

As, $z + \bar{z} = 2\,\text{Re}(z)$

$\therefore$ If $z = \left(\dfrac{\sqrt{3}}{2} + \dfrac{i}{2}\right)^5 + \left(\dfrac{\sqrt{3}}{2} - \dfrac{i}{2}\right)^5$

$\Rightarrow$ z is purely real.

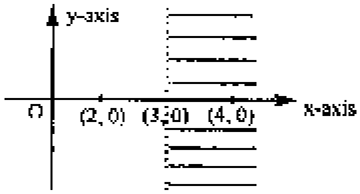
or Im $(z) = 0$.

4. $|z-4|<|z-2|$

as, $|z-z_1|>|z-z_2|$ represents the region on right side of perpendicular bisector of z_1 and z_2

$\therefore \quad |z-2|>|z-4|$

$\Rightarrow \quad \text{Re}(z)>3 \quad \text{and} \quad \text{Im}(z)\in R$



Hence, (d) is the correct answer.

5. $|w|=1 \Rightarrow |w|=\left|\frac{1-iz}{z-i}\right|$ gives, $|z-i|=|1-iz|$

$\Rightarrow |z-i|=|z+i|$

as, $\quad |1-iz|=|-i||z+i|=|z+i|$

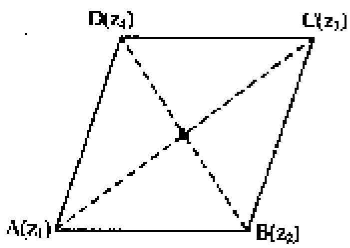
perpendicular bisector of (0, 1) and (0, −1) *i.e.*, x-axis

Thus, z lies on real axis.

Hence, (b) is the correct answer.

6. As, z_1, z_2, z_3, z_4 are vertices of parallelogram.

$\therefore$ mid point of AC = mid point of BD



$$\Rightarrow \quad \frac{z_1+z_3}{2}=\frac{z_2+z_4}{2}$$

$$\Rightarrow \quad z_1+z_3=z_2+z_4$$

Hence, (b) is the correct answer.

7. Since a, b, c and u, v, w are vertices of two triangles

$$\Rightarrow \begin{vmatrix} a & u & 1\\ b & v & 1\\ c & w & 1\end{vmatrix}, \text{ where } c=(1-r)a+rb$$

and $\quad w=(1-r)u+rv \quad \ldots(i)$

applying $R_3 \to R_3-\{(1-r)R_1+rR_2\}$ we get

$$=\begin{vmatrix} a & u & 1\\ b & v & 1\\ c-(1-r)a-rb & w-(1-r)u-rv & 1-(1-r)-r\end{vmatrix}$$

$$=\begin{vmatrix} a & u & 1\\ b & v & 1\\ 0 & 0 & 0\end{vmatrix} \qquad \text{[using (i)]}$$

$=0$

Thus, two triangles are similar.

Hence, (b) is the correct answer.

8. Here, $\sum_{k=1}^{6}\left(\sin\frac{2k\pi}{7}-i\cos\frac{2k\pi}{7}\right)$

$$=\sum_{k=1}^{6}-i\left(\cos\frac{2k\pi}{7}+i\sin\frac{2k\pi}{7}\right)$$

$$=-i\left\{\sum_{k=1}^{6}e^{\frac{i2k\pi}{7}}\right\}$$

$$=-i\{e^{i2\pi/7}+e^{i4\pi/7}+e^{i6\pi/7}+e^{i8\pi/7}+e^{i10\pi/7}+e^{i12\pi/7}\}$$

$$=-i\left\{\frac{e^{i2\pi/7}(1-e^{i12\pi/7})}{1-e^{i2\pi/7}}\right\}$$

$$=-i\left\{\frac{e^{i2\pi/7}-e^{i14\pi/7}}{1-e^{i2\pi/7}}\right\}, \left(\text{as } e^{\frac{i14\pi}{7}}=1\right)$$

$$=-i\left\{\frac{e^{i2\pi/7}-1}{1-e^{i2\pi/7}}\right\}=i$$

Hence (d) is the correct answer.

9. As, $|z_1+z_2|=|z_1|+|z_2|$, squaring both sides

$\Rightarrow |z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\arg z_1-\arg z_2)$

$\quad =|z_1|^2+|z_2|^2+2|z_1||z_2|$

$\Rightarrow \quad 2|z_1||z_2|\cos(\arg z_1-\arg z_2)=2|z_1||z_2|$

$\Rightarrow \quad \cos(\arg z_1-\arg z_2)=1$

$\Rightarrow \quad \arg z_1-\arg z_2=0$

Hence, (c) is correct answers.

10. As $(\sin x+i\cos 2x)=\cos x-i\sin 2x$

$\Rightarrow \quad \sin x=\cos x$ and $\cos 2x=\sin 2x$

$\Rightarrow \quad \tan x=1$ and $\tan 2x=1$

$\Rightarrow \quad x=\pi/4$ and $x=\pi/8$ which is not possible at same time thus no solution.

Hence (d) is the correct answer.

11. $(1+\omega)^7=(1+\omega)(1+\omega)^6$

$\quad =(1+\omega)(-\omega^2)^6$

$\quad =1+\omega$

$\Rightarrow \quad A+B\omega=1+\omega \;\Rightarrow\; A=1, B=1$

12. We have to find z in terms of w under given condition

Let $\quad w=re^{i\theta}, \quad \therefore \bar{w}=re^{-i\theta}$

$\therefore \quad z=re^{i(\pi-\theta)}=re^{i\pi}\cdot e^{-i\theta}$

$\quad =-re^{-i\theta}=-\bar{w}$

13. We have $\quad |z+iw|=|z-i\bar{w}|=2$

$\Rightarrow \quad |z-(-iw)|=|z-(i\bar{w})|=2$

$\Rightarrow \quad |z-(-iw)|=|z-(-\overline{iw})|.$

$\therefore$ z lies on the perpendicular bisector of the line joining $-iw$ and $-\overline{iw}$. Since $-\overline{iw}$ is the mirror image of $-iw$ in the x-axis, the locus of z is the x-axis.

Let $z = x + iy$ and $y = 0$.

Now $|z| \le 1 \Rightarrow x^2 + 0^2 \le 1 \Rightarrow -1 \le x \le 1$.

$\therefore$ z may take values given in (c).

14. $(1+i)^{n_1} + (1-i)^{n_1} + (1+i)^{n_2} + (1-i)^{n_2}$

$$= [{}^{n_1}C_0 + {}^{n_1}C_1 i + {}^{n_1}C_2 i^2 + {}^{n_1}C_3 i^3 + \dots]$$
$$+ [{}^{n_1}C_0 - {}^{n_1}C_1 i + {}^{n_1}C_2 i^2 - {}^{n_1}C_3 i^3 + \dots]$$
$$+ [{}^{n_2}C_0 + {}^{n_2}C_1 i + {}^{n_2}C_2 i^2 + {}^{n_2}C_3 i^3 + \dots]$$
$$+ [{}^{n_2}C_0 - {}^{n_2}C_1 i + {}^{n_2}C_2 i^2 - {}^{n_2}C_3 i^3 + \dots]$$
$$= 2[{}^{n_1}C_0 + {}^{n_1}C_2 i^2 + {}^{n_1}C_4 i^4 + \dots]$$
$$+ 2[{}^{n_2}C_0 + {}^{n_2}C_2 i^2 + {}^{n_2}C_4 i^4 + \dots]$$
$$= 2[{}^{n_1}C_0 - {}^{n_1}C_2 + {}^{n_1}C_4 + \dots] + 2[{}^{n_2}C_0 - {}^{n_2}C_2 + {}^{n_2}C_4 - \dots]$$

This is a real number irrespective of the values of n_1 and n_2.

Alter

$\{(1+i)^{n_1} + (1-i)^{n_1}\} + \{(1-i)^{n_2} + (1-i)^{n_2}\}$

$\Rightarrow$ a real number for all n_1 and $n_2 \in R$.

{as, $z + \bar{z} = 2\text{Re}(z)$

$\Rightarrow (1+i)^{n_1} + (1-i)^{n_1}$ is real number for all $n \in R$}

$\therefore$ (d) is the best option.

15. $(1+\omega-\omega^2)^7 = (-\omega^2-\omega^2)^7$

$$= (-2\omega^2)^7 = (-2)^7(\omega^2)^7 = -128.\omega^{14} = -128\omega^2$$

Therefore, (d) is the ans.

16. $\sum_{n=1}^{13}(i^n + i^{n+1}) = \sum_{n=1}^{13} i^n (1+i) = (1+i)\sum_{n=1}^{13} i^n$

$$= (1+i)(i + i^2 + i^3 + \dots + i^{13}) = (1+i)\left\{\frac{i(1-i)}{1-i}\right\}$$

$= (1+i)\,i = -1 + i$, Therefore, (b) is the ans.

Alter

As sum of any four consecutive powers of iota is zero.

$$\therefore \quad \sum_{n=1}^{13}(i^n + i^{n+1}) = (i + i^2 + \dots + i^{13}) + (i^2 + i^3 + \dots + i^{14})$$
$$= i + i^2$$
$$= i - 1$$

17. $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$ (given)

$$\Rightarrow \quad -3i\begin{vmatrix} 6i & 1 & 1 \\ 4 & -1 & -1 \\ 20 & i & i \end{vmatrix} = 0$$

[$\because C_2$ and C_3 are identical].

$\Rightarrow x + iy = 0 \Rightarrow x = 0,\ y = 0$. Therefore, (d) is the ans.

18. If in a complex number $a + ib$, the ratio $a : b$ is $1 : \sqrt{3}$ is then always try to convert that complex number in ω.

Here $\quad \omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

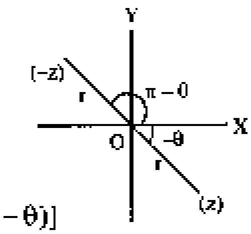
Therefore, $4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365}$

$$= 4 + 5\omega^{334} + 3\omega^{365}$$
$$= 4 + 5.(\omega^3)^{111}.\omega + 3.(\omega^3)^{121}.\omega^2$$
$$= 4 + 5\omega + 3\omega^2 \qquad (\because \omega^3 = 1)$$
$$= 1 + 3 + 2\omega + 3\omega + 3\omega^2$$
$$= 1 + 2\omega + 3(1 + \omega + \omega^2) = 1 + 2\omega + 3 \times 0$$
$$(\because 1 + \omega + \omega^2 = 0)$$
$$= 1 + (-1 + \sqrt{3}i) = \sqrt{3}i.$$

Therefore, (c) is the ans.

19. Arg $(z) < 0$ (given)

$\Rightarrow \quad \arg(z) = -\theta$



Now, $\quad z = r\cos(-\theta) + i\sin(-\theta)$

$$= r(\cos\theta - i\sin\theta)$$

again $\quad -z = -r[\cos\theta - i\sin\theta]$

$$= r[\cos(\pi-\theta) + i\sin(\pi-\theta)]$$

$\therefore \quad \arg(-z) = \pi - \theta$

Thus, $\arg(-z) - \arg(z) = \pi - \theta - (-\theta) = \pi$

Therefore, (a) is the ans.

Alter

$$\arg(-z) - \arg(z) = \arg\left(\frac{-z}{z}\right) = \arg(-1) = \pi$$

20. $|z_1| = |z_2| = |z_3| = 1$ (given)

Now, $\quad |z_1| = 1$

$\Rightarrow \quad |z_1|^2 = 1$

$\Rightarrow \quad z_1\bar{z}_1 = 1,$

Similarly $z_2\bar{z}_2 = 1$, $z_3\bar{z}_3 = 1$

Now, $\quad \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$

$\Rightarrow \quad |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 1$

$\Rightarrow \quad |\overline{z_1 + z_2 + z_3}| = 1$

$\Rightarrow \quad |z_1 + z_2 + z_3| = 1$

Therefore, (a) is the ans.

21. $\arg\frac{z_1}{z_2} = \frac{\pi}{2} \Rightarrow \frac{z_1}{z_2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

$$(\because |z_2| = |z_1| = 1)$$

$$\therefore \quad \frac{z_1^n}{z_2^n} = (i)^n$$

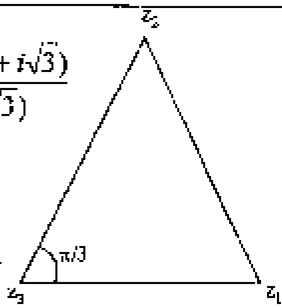
Hence, $\quad i^n = 1 \quad \Rightarrow \quad n = 4k$

Therefore, (d) is the answer.

22. $\dfrac{z_1 - z_3}{z_2 - z_3} = \dfrac{1 - i\sqrt{3}}{2} = \dfrac{(1 - i\sqrt{3})(1 + i\sqrt{3})}{2(1 + i\sqrt{3})}$

$= \dfrac{1 - i^2 3}{2(1 + i\sqrt{3})}$

$= \dfrac{4}{2(1 + i\sqrt{3})} = \dfrac{2}{(1 + i\sqrt{3})}$



$\Rightarrow \dfrac{z_2 - z_3}{z_1 - z_3} = \dfrac{1 + i\sqrt{3}}{2} = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$

$\Rightarrow \left|\dfrac{z_2 - z_3}{z_1 - z_3}\right| = 1$ and $\arg\left(\dfrac{z_2 - z_3}{z_1 - z_3}\right) = \dfrac{\pi}{3}$

Hence the Δ is equilateral. Therefore, (c) is the answer.

23. Operate $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$ the given determinant reduce to

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2\ \ \omega^2 & \omega^2 - 1 \\ 0 & \omega^2 - 1 & \omega - 1 \end{vmatrix} \qquad (\because \omega^4 = \omega)$$

$= (-2 - \omega^2)(\omega - 1) - (\omega^2 - 1)^2$

$= -(-3\omega^2 + 3\omega)$

$= 3\omega(\omega - 1)$

24. We know,

$|z_1 - z_2| = |z_1 - (z_2 - 3 - 4i) - (3 + 4i)|$

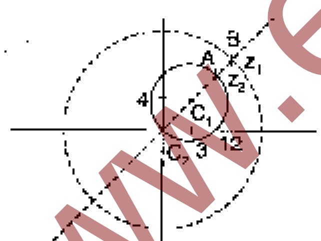
$\geq |z_1| - |z_2 - 3 - 4i| - |3 + 4i|$

$\geq 12 - 5 - 5$ (using $|z_1 - z_2| \geq |z_1| - |z_2|$)

$\therefore \quad |z_1 - z_2| \geq 2$

Alter

Clearly from the figure $|z_1 - z_2|$ is minimum when z_1, z_2 lie along the diameter.



$\therefore \quad |z_1 - z_2| \geq C_2B - C_2A$

$\geq 12 - 10 = 2$

25. Since, $|z| = 1$ and $w = \dfrac{z - 1}{z + 1}$

$z - 1 = wz + w$ or $z = \dfrac{1 + w}{1 - w}$

$\Rightarrow \quad |z| = \dfrac{|1 + w|}{|1 - w|}$

$\Rightarrow \quad |1 - w| = |1 + w|$ {as, $|z| = 1$}

Squaring both sides, we get

$1 + |w|^2 - 2|w|\operatorname{Re}(w) = 1 + |w|^2 + 2|w|\operatorname{Re}(w)$

{using; $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2|z_1||z_2|\operatorname{Re}(z_1\bar{z}_2)$}

$\Rightarrow 4|w|\operatorname{Re}|w| = 0$ or $\operatorname{Re}(w) = 0$

26. Here $(1 + \omega^2)^n = (1 + \omega^4)^n$

$\Rightarrow \quad (-\omega)^n = (-\omega^2)^n$ {as $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$}

$\Rightarrow \quad \omega^n = 1$

$\Rightarrow n = 3$ is least positive value of n.

27. Let $x = |a + b\omega + c\omega^2|$

$\Rightarrow \quad x^2 = |a + b\omega + c\omega^2|^2$

$= (a^2 + b^2 + c^2 - ab - bc - ca)$

or $\quad x^2 = \dfrac{1}{2}\{(a - b)^2 - (b - c)^2 + (c - a)^2\}$ ...(i)

Since, a, b, c are all integers but not all simultaneously equal

$\Rightarrow$ If $a = b$ then $a \neq c$ and $b \neq c$

As, difference of integers = integer.

$\Rightarrow (b - c)^2 \geq 1$ {as minimum difference of two consecutive integers is (± 1)} also $(c - a)^2 \geq 1$

and we have taken $a = b \Rightarrow (a - b)^2 = 0$

thus, from equation (i),

$x^2 = \dfrac{1}{2}\{(a - b)^2 + (b - c)^2 + (c - a)^2\} \geq \dfrac{1}{2}\{0 + 1 + 1\}$

$\Rightarrow \quad x^2 \geq 1$

or minimum value of $|x| = 1$

28. As $|PQ| = |PS| = |PR| = 2$

∴ Shaded part represents the external part of circle having centre $(-1, 0)$ and radius 2.

As we know equation of circle having centre z_0 and radius r, is $|z - z_0| = r$

$\therefore \quad |z - (-1 - 0i)| > 2$

$\Rightarrow \quad |z + 1| > 2$ ...(i)

also argument of $z + 1$ with respect to positive direction of x-axis is $\pi/4$.

$\therefore \quad \arg(z + 1) \leq \dfrac{\pi}{4}$

and argument of $z - 1$ in anticlockwise direction is $-\pi/4$

$\therefore \quad -\pi/4 \leq \arg(z + 1)$ ...(ii)

or $\quad |\arg(z + 1)| \leq \pi/4$

29. Let, $z_1 = \dfrac{w - \bar{w}z}{1 - z}$, be purely real.

$\Rightarrow \quad z_1 = \bar{z}_1$

$\therefore \quad \dfrac{w - \bar{w}z}{1 - z} = \dfrac{\bar{w} - w\bar{z}}{1 - \bar{z}}$

$\Rightarrow \quad w - w\bar{z} - \bar{w}z + \bar{w}z\cdot\bar{z} = \bar{w} - z\bar{w} - w\bar{z} + wz\cdot\bar{z}$

$\Rightarrow \quad (w - \bar{w}) + (\bar{w} - w)|z|^2 = 0$

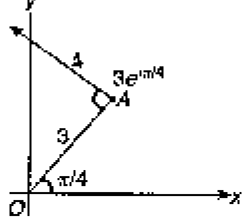
$\Rightarrow \quad (w - \bar{w})(1 - |z|^2) = 0$

$\Rightarrow \quad |z|^2 = 1$ {as, $w - \bar{w} \neq 0$, since $\beta \neq 0$}

$\Rightarrow \quad |z| = 1$ and $z \neq 1$

Hence (b) is the correct answer.

30. Let $OA = 3$, so that the complex number associated with A is $3e^{i\pi/4}$. If z is the complex number associated with P, then



$$\frac{z-3e^{i\pi/4}}{0-3e^{i\pi/4}} = \frac{4}{3}e^{-i\pi/2} = -\frac{4i}{3}$$

$\Rightarrow \quad 3z - 9e^{i\pi/4} = 12ie^{i\pi/4}$

$\Rightarrow \quad z = (3+4i)e^{i\pi/4}$.

31. Let $z = \cos\theta + i\sin\theta$

$$\Rightarrow \quad \frac{z}{1-z^2} = \frac{\cos\theta + i\sin\theta}{1-(\cos 2\theta + i\sin 2\theta)}$$

$$= \frac{\cos\theta + i\sin\theta}{2\sin^2\theta - 2i\sin\theta\cos\theta}$$

$$= \frac{\cos\theta + i\sin\theta}{-2i\sin\theta(\cos\theta + i\sin\theta)}$$

$$= \frac{i}{2\sin\theta}$$

Hence, $\frac{z}{1-z^2}$ lies on the imaginary axis *i.e.*, $x = 0$.

Alter :

Let $E = \frac{z}{1-z^2} = \frac{z}{z\bar{z} - z^2} = \frac{1}{\bar{z} - z}$ which is imaginary.

D OBJECTIVE (MORE THAN ONE OPTION)

1. $z_1 = a + ib$ and $z_2 = c + id$

As, $|z_1|^2 = a^2 + b^2 = 1$ and $|z_2|^2 = c^2 + d^2 = 1$ ...(i)

also $\text{Re}(z_1\bar{z}_2) = 0 \Rightarrow ac + bd = 0$

$\Rightarrow \quad \frac{a}{b} = \frac{-d}{c} = \lambda$ ...(ii)

from (i) and (ii), $b^2\lambda^2 + b^2 = c^2 + \lambda^2c^2$

$\Rightarrow b^2 = c^2$ and $a^2 = d^2$

now, $|w_1| = \sqrt{a^2 + c^2} = \sqrt{a^2 + b^2} = 1$

$|w_2| = \sqrt{b^2 + d^2} = \sqrt{a^2 + b^2} = 1$

$\text{Re}(w_1\bar{w}_2) = ab + cd = (b\lambda)b + c(-\lambda c)$

$= \lambda(b^2 - c^2) = 0$

Hence, (a), (b), (c) are correct answers.

2. $|z_1| = |z_2|$. Thus, $\frac{z_1 + z_2}{z_1 - z_2} \times \frac{\bar{z}_1 - \bar{z}_2}{\bar{z}_1 - \bar{z}_2}$

$$\Rightarrow \quad \frac{z_1\bar{z}_1 - z_1\bar{z}_2 + z_2\bar{z}_1 - z_2\bar{z}_2}{|z_1 - z_2|^2}$$

$$\Rightarrow \quad \frac{|z_1|^2 + (z_2\bar{z}_1 - z_1\bar{z}_2) - |z_2|^2}{|z_1 - z_2|^2}$$

$$\Rightarrow \quad \frac{z_2\bar{z}_1 - \bar{z}_2 z_1}{\text{Real number}} \quad (\text{as, } |z_1|^2 = |z_2|^2)$$

As, we know $z - \bar{z} = 2i\,\text{Im}(z)$

$\therefore \quad z_2\bar{z}_1 - z_1\bar{z}_2 = 2i\,\text{Im}(z_2\bar{z}_1)$

$$\therefore \quad \frac{z_1 + z_2}{z_1 - z_2} = \frac{2i\,\text{Im}(z_2\bar{z}_1)}{|z_1 - z_2|^2},$$

which is purely imaginary or zero.

Hence, (a) and (d) are correct answers.

E SUBJECTIVE QUESTIONS

1. As n is not a multiple of 3, but odd integers and $x^3 + x^2 + x = 0 \Rightarrow x = 0, \omega, \omega^2$

Now when $x = 0 \Rightarrow (x+1)^n - x^n - 1$

$= 1 - 0 - 1 = 0$

$\therefore \quad x = 0$ is root of $(x+1)^n - x^n - 1$

Again when $= \omega$

$\Rightarrow \quad (x+1)^n - x^n - 1$

$\Rightarrow (1+\omega)^n - \omega^n - 1 \Rightarrow -\omega^{2n} - \omega^n - 1 = 0$

(as n is not a multiple of 3 and odd)

Similarly $x = \omega^2$ is root of $\{(x+1)^n - x^n - 1\}$

Hence $x = 0, \omega, \omega^2$ are roots of $(x+1)^n - x^n - 1$

Thus $x^3 + x^2 + x$ divides $(x+1)^n - x^n - 1$.

2. $$\frac{(1+i)x - 2i}{3+i} + \frac{(2-3i)y + i}{3-i} = i$$

$\Rightarrow (1+i)(3-i)x - 2i(3-i) + (3+i)(2-3i)y + i(3+i) = 10i$

$\Rightarrow 4x + 2ix - 6i - 2 + y - 7iy + 3i - 1 = 10i$

$\Rightarrow \quad 4x + y - 3 = 0$ and $2x - 7y - 3 = 10$

$\Rightarrow \quad x = 3$ and $y = -1$.

3. As z_1, z_2, z_3 are vertices of equilateral triangle

$\therefore$ Circumcentre (z_0) = centroid $\left(\frac{z_1 + z_2 + z_3}{3}\right)$ ...(i)

also for equilateral triangle

$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$ ...(ii)

Squaring (i), we get

$9z_0^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_1z_2 + z_2z_3 + z_3z_1)$

$9z_0^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_1^2 + z_2^2 + z_3^2)$ {using (ii)}

$\Rightarrow \quad 3z_0^2 = z_1^2 + z_2^2 + z_3^2$

4. Here, $z_1 R z_2 \Leftrightarrow \frac{z_1 - z_2}{z_1 + z_2}$ is real

(i) Reflexive : $z_1 R z_1 \Leftrightarrow \frac{z_1 - z_1}{z_1 + z_2} = 0$ (purely real)

$\therefore$ $z_1 R z_1$ is reflexive.

(ii) Symmetric : $z_1 R z_2 \Leftrightarrow \frac{z_1 - z_2}{z_1 + z_2}$ is real

$\Rightarrow$ $\frac{-(z_2 - z_1)}{z_1 + z_2}$ is real

$\Rightarrow$ $z_2 R z_1$

$\therefore$ $z_1 R z_2 \Rightarrow z_2 R z_1$

Hence, symmetric.

(iii) Transitive : $z_1 R z_2 \Rightarrow \frac{z_1 - z_2}{z_1 + z_2}$ is real

$z_2 R z_3 \Rightarrow \frac{z_2 - z_3}{z_2 + z_3}$ is real

Here, let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ and $z_3 = x_3 + iy_3$

$\therefore \frac{z_1 - z_2}{z_1 + z_2}$ is real $\Rightarrow \frac{(x_1 - x_2) + i(y_1 - y_2)}{(x_1 + x_2) + i(y_1 + y_2)}$ is real

$$\Rightarrow \frac{\{(x_1 - x_2) + i(y_1 - y_2)\}\{(x_1 + x_2) - i(y_1 + y_2)\}}{(x_1 + x_2)^2 + (y_1 + y_2)^2}$$

$\Rightarrow (y_1 - y_2)(x_1 + x_2) - (x_1 - x_2)(y_1 + y_2) = 0$

$\Rightarrow 2x_2 y_1 - 2y_2 x_1 = 0 \Rightarrow \frac{x_1}{y_1} = \frac{x_2}{y_2}$ ....(i)

Similarly, $z_2 R z_3 \Rightarrow \frac{x_2}{y_2} = \frac{x_3}{y_3}$ ....(ii)

From (i) and (ii), we have $\frac{x_1}{y_1} = \frac{x_3}{y_3}$

$\Rightarrow$ $z_1 R z_3$

Thus $z_1 R z_2$ and $z_2 R z_3 \Rightarrow z_1 R z_3$. (transitive). Hence R is an equivalence relation.

5. As z_1, z_2 and origin forms an equilateral triangle.

{we know if z_1, z_2, z_3 forms equilateral Δ

$\Rightarrow z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$ }

$\therefore z_1^2 + z_2^2 + 0^2 = z_1 z_2 + z_2 . 0 + 0 . z_1$

or $z_1^2 + z_2^2 = z_1 z_2$

$\Rightarrow z_1^2 + z_2^2 - z_1 z_2 = 0$

6. As $1, a_1, a_2, \ldots\ldots, a_{n-1}$ are n^{th} roots of unity

$\Rightarrow (x^n - 1) = (x-1)(x-a_1)(x-a_2)\ldots.(x-a_{n-1})$

or $\frac{x^n - 1}{x-1} = (x-a_1)(x-a_2)\ldots..(x-a_{n-1})$

$$\left\{\text{as } \frac{x^n - 1}{x-1} = x^{n-1} + x^{n-2} + \ldots.. + x + 1\right\}$$

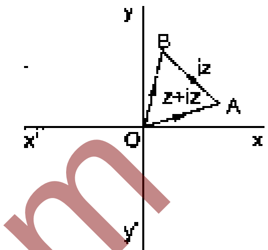
$\therefore x^{n-1} + x^{n-2} + \ldots.. + x^2 + x + 1 = (x-a_1)(x-a_2)\ldots..(x-a_{n-1})$

Putting $x = 1$, we get

$1 + 1 + \ldots\ldots n$ times $= (1-a_1)(1-a_2)\ldots..(1-a_{n-1})$

$\Rightarrow (1-a_1)(1-a_2)\ldots..(1-a_{n-1}) = n$

7. We have : $iz = ze^{i\pi/2}$. This implies that iz is the vector obtained by rotating vector z in anticlockwise direction through 90°. Therefore, $OA \perp AB$. So,

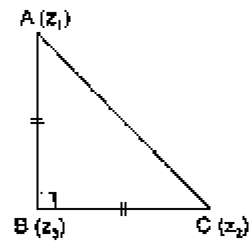


Area of $\Delta OAB = \frac{1}{2} OA \times OB$

$= \frac{1}{2}|z||iz| = \frac{1}{2}|z|^2$

8. Since, Δ is right angled isosceles Δ.

$\therefore$ Rotating z_2 about z_3 in anticlock wise direction through an angle of $\pi/2$, we get



$\frac{z_2 - z_3}{z_1 - z_3} = \frac{|z_2 - z_3|}{|z_1 - z_3|} e^{i\pi/2}$

where, $|z_2 - z_3| = |z_1 - z_3|$

$\Rightarrow (z_2 - z_3) = i(z_1 - z_3)$

squaring both sides we get,

$(z_2 - z_3)^2 = -(z_1 - z_3)^2$

$\Rightarrow z_2^2 + z_3^2 - 2z_2 z_3 = -z_1^2 - z_3^2 + 2z_1 z_3$

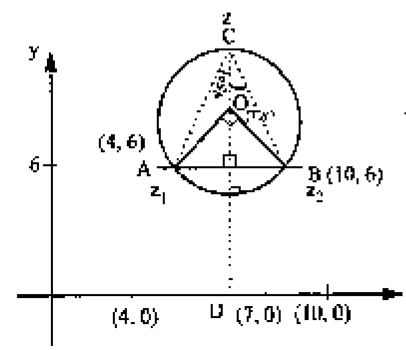
$\Rightarrow z_1^2 + z_2^2 - 2z_1 z_2 = 2z_1 z_3 + 2z_2 z_3 - 2z_3^2 - 2z_1 z_2$

$\Rightarrow (z_1 - z_2)^2 = 2\{(z_1 z_3 - z_3^2) + (z_2 z_3 - z_1 z_2)\}$

$\Rightarrow (z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$

9. As $z_1 = 10 + 6i$, $z_2 = 4 + 6i$

and $\arg\left(\frac{z - z_1}{z - z_2}\right) = \frac{\pi}{4}$ represents locus of z is a circle shown as



As from the figure centre is $(7, y)$ and $\angle AOB = 90°$ clearly $OC = 9$.

$\Rightarrow$ $OD = 6 + 3 = 9$

$\therefore$ Centre $= (7, 9)$ and radius $= \frac{6}{\sqrt{2}} = 3\sqrt{2}$

$\Rightarrow$ Equation of circle : $|z - (7 + 9i)| = 3\sqrt{2}$

10. $iz^3 + z^2 - z + i = 0$ (given)

⇒ $iz^3 - i^2z^2 - z + i = 0$ $(\because i^2 = -1)$

⇒ $iz^2(z - i) - 1(z - i) = 0$

⇒ $(iz^2 - 1)(z - i) = 0$

⇒ $z - i = 0$ or $iz^2 - 1 = 0$

⇒ $z = i$ or $z^2 = 1/i = -i$

if $z = i$, then $|z| = |i| = 1$.

if $z^2 = -i$, then $|z^2| = |-i| = 1$

⇒ $|z|^2 = 1$

⇒ $|z| = 1$ therefore we have $|z| = 1$

11. Let $z = r_1(\cos\theta_1 + i\sin\theta_1)$ and $w = r_2(\cos\theta_2 + i\sin\theta_2)$

We have $|z| = r_1, |w| = r_2$, $\arg z = \theta_1$ and $\arg w = \theta_2$

Since $|z| \le 1, |w| \le 1$ (given) $\Rightarrow r_1 \le 1$ and $r_2 \le 1$

We have

$z - w = (r_1\cos\theta_1 - r_2\cos\theta_2) + i(r_1\sin\theta_1 - r_2\sin\theta_2)$

$\Rightarrow |z - w|^2 = (r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2$

$= r_1^2\cos^2\theta_1 + r_2^2\cos^2\theta_2 - 2r_1r_2\cos\theta_1\cos\theta_2 + r_1^2\sin^2\theta_1 + r_2^2\sin^2\theta_2 - 2r_1r_2\sin\theta_1\sin\theta_2$

$= r_1^2(\cos^2\theta_1 + \sin^2\theta_1) + r_2^2(\cos^2\theta_2 + \sin^2\theta_2) - 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$

$= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)$

$= (r_1 - r_2)^2 + 2r_1r_2\cos(\theta_1 - \theta_2)$

$= (r_1 - r_2)^2 + 2r_1r_2[1 - \cos(\theta_1 - \theta_2)]$

$= (r_1 - r_2)^2 + 4r_1r_2\sin^2\left(\frac{\theta_1 - \theta_2}{2}\right)$

$\le |r_1 - r_2|^2 + 4\left|\sin\frac{\theta_1 - \theta_2}{2}\right|^2$ $[\because r_1, r_2 \le 1]$

and $|\sin\theta| \le |\theta| \;\forall\, \theta \in R$

Therefore $|z - w|^2 \le |r_1 - r_2|^2 + 4\left|\frac{\theta_1 - \theta_2}{2}\right|^2$

$\le |r_1 - r_2|^2 + |\theta_1 - \theta_2|^2$

$\Rightarrow |z - w|^2 \le (|z| - |w|)^2 + (\arg z - \arg w)^2$

Alter

$|z - w|^2 = |z|^2 + |w|^2 - 2|z||w|\cos(\arg z - \arg w)$

$= |z|^2 + |w|^2 - 2|z||w| + 2|z||w| - 2|z||w|\cos(\arg z - \arg w)$

$= (|z| - |w|)^2 + 2|z||w|.2\sin^2\left(\frac{\arg z - \arg w}{2}\right)$ ...(1)

(as, $\sin\theta \le \theta$)

$\therefore\ |z - w|^2 \le (|z| - |w|)^2 + 4.1.1\left(\frac{\arg z - \arg w}{2}\right)^2$

$\Rightarrow\ |z - w|^2 \le (|z| - |w|)^2 + (\arg z - \arg w)^2$

12. Let $z = x + iy$

since $\bar z = iz^2$ (given), we get

$\overline{(x + iy)} = i(x + iy)^2$

⇒ $x - iy = i[x^2 - y^2 + 2ixy]$

⇒ $x - iy = -2xy + i(x^2 - y^2)$

Imp. note : It is a compound equation, therefore, we can generate from it more than one primary equations.

Now here equating the real and imaginary parts, we get

$x = -2xy$ and $-y = x^2 - y^2$

⇒ $x + 2xy = 0$ and $x^2 - y^2 + y = 0$

⇒ $x(1 + 2y) = 0$

⇒ $x = 0$ or $y = -1/2$

putting $x^2 - y^2 + y = 0$

⇒ $0 - y^2 + y = 0$

⇒ $y(1 - y) = 0$

or $y = 0$ or $y = 1$

Now putting $y = -1/2$ in $x^2 - y^2 + y = 0$, we get

$x^2 - \frac{1}{4} - \frac{1}{2} = 0 \Rightarrow x^2 = 3/4$

or $x = \pm\sqrt{3}/2$

Therefore, $z = 0 + i0,\ 0 + i;\ \pm\frac{\sqrt{3}}{2} - \frac{i}{2}$

As $z \ne 0$, we get

$z = i, \pm\sqrt{3}/2 - i/2$

13. $z_1 + z_2 = -p$ and $z_1z_2 = q$ (given)

Now, $\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}(\cos\alpha + i\sin\alpha)$

Applying componendo and dividendo

$\Rightarrow \frac{z_1 + z_2}{z_1 - z_2} = \frac{\cos\alpha + i\sin\alpha + 1}{\cos\alpha + i\sin\alpha - 1}$

$= \frac{2\cos^2(\alpha/2) + 2i\sin(\alpha/2)\cos(\alpha/2)}{-2\sin^2(\alpha/2) + 2i\sin(\alpha/2)\cos(\alpha/2)}$

$= \frac{2\cos(\alpha/2)[\cos(\alpha/2) + i\sin(\alpha/2)]}{2i\sin(\alpha/2)[\cos(\alpha/2) + i\sin(\alpha/2)]}$

$= \frac{\cot(\alpha/2)}{i} = \frac{i\cot(\alpha/2)}{i^2} = -i\cot\alpha/2$

$\Rightarrow \frac{-p}{z_1 - z_2} = -i\cot(\alpha/2)$

Squaring both sides

$\Rightarrow \frac{p^2}{(z_1 - z_2)^2} = -\cot^2(\alpha/2)$

$\Rightarrow \frac{p^2}{(z_1 + z_2)^2 - 4z_1z_2} = -\cot^2(\alpha/2)$

$\Rightarrow \frac{p^2}{p^2 - 4q} = -\cot^2(\alpha/2)$

⇒ $p^2 = -p^2\cot^2(\alpha/2) + 4q\cot^2(\alpha/2)$

⇒ $p^2(1+\cot^2\alpha/2) = 4q\cot^2(\alpha/2)$

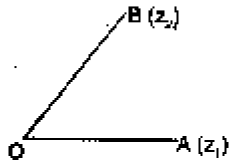
⇒ $p^2\operatorname{cosec}^2(\alpha/2) = 4q\cot^2(\alpha/2)$

⇒ $p^2 = 4q\cos^2\alpha/2$

Alter

Here, $z_1+z_2 = -p$ and $z_1z_2 = q$

∴ $\frac{z_2}{z_1} = \frac{|z_2|}{|z_1|}e^{i\alpha}$, by rotation of A about O.



or $\frac{z_2}{z_1} = \frac{(\cos\alpha + i\sin\alpha)}{1}$

⇒ $\frac{z_1+z_2}{z_1-z_2} = \frac{1+\cos\alpha+i\sin\alpha}{\cos\alpha+i\sin\alpha-1}$

$= \frac{2\cos\alpha/2(\cos\alpha/2+i\sin\alpha/2)}{2i\sin\alpha/2(\cos\alpha/2+i\sin\alpha/2)}$

∴ $\left(\frac{z_1+z_2}{z_1-z_2}\right)^2 = \left(\frac{\cos\alpha/2}{i\sin\alpha/2}\right)^2$

⇒ $\frac{p^2}{p^2-4q} = -\frac{\cos^2\alpha/2}{\sin^2\alpha/2}$

⇒ $p^2\sin^2\alpha/2 = -p^2\cos^2\alpha/2 + 4q\cos^2\alpha/2$

or $p^2(\sin^2\alpha/2+\cos^2\alpha/2) = 4q\cos^2\alpha/2$

⇒ $p^2 = 4q\cos^2\alpha/2$

14. Let Q be z_2 and its reflection be the point $P(z_1)$ in the given line. If $O(z)$ be any point on the given line then by definition OR is right bisector of QP.

∴ $OP = OQ$ or $|z-z_1| = |z-z_2|$

or $|z-z_1|^2 = |z-z_2|^2$

or $(z-z_1)(\bar z-\bar z_1) = (z-z_2)(\bar z-\bar z_2)$

or $z(\bar z_1-\bar z_2)+\bar z(z_1-z_2) = z_1\bar z_1 - z_2\bar z_2$

Comparing with given line $z\bar b+\bar z b = c$

$\frac{\bar z_1-\bar z_2}{\bar b} = \frac{z_1-z_2}{b} = \frac{z_1\bar z_1 - z_2\bar z_2}{c} = \lambda$, say

$\frac{\bar z_1-\bar z_2}{\lambda} = \bar b,\ \frac{z_1-z_2}{\lambda} = b,\ \frac{z_1\bar z_1-z_2\bar z_2}{\lambda} = c$ ...(1)

Also $\bar z_1 b + z_2\bar b = \bar z_1\frac{z_1-z_2}{\lambda}+z_2\frac{\bar z_1-\bar z_2}{\lambda}$

$= \frac{z_1\bar z_1 - z_2\bar z_2}{\lambda} = c$ by (1)

15. $|z|^2w - |w|^2 z = z - w$ (given)

$z\bar z w - w\bar w z = z - w$ ...(1)

∵ $|z|^2 = z\bar z$

Taking modulus of both the sides, we get

⇒ $|zw||\bar z - \bar w| = |z-w|$

⇒ $|zw||\bar z - \bar w| = |z-w|$

∵ $|z| = |\bar z|$

⇒ $|zw||\bar z-\bar w| = |\bar z-\bar w|$

⇒ $|\bar z-\bar w|(|zw|-1) = 0$

⇒ $|\bar z - w| = 0$ or $|zw|-1=0$ ⇔

⇒ $|z-w|=0$ or $|zw|=1$

⇒ $z-w=0$

⇒ $z=w$ or $|zw|=1$

Now suppose $z\ne w$

then $|zw|=1$ or $|z||w|=1$

⇒ $|z| = \frac{1}{|w|} = r$ (say)

Let $z = re^{i\theta}$ and $w = \frac1r e^{i\phi}$

putting these values in (1), we get

$r^2\left(\frac1r e^{i\phi}\right) - \frac{1}{r^2}(re^{i\theta}) = re^{i\theta} - \frac1r e^{i\phi}$

⇒ $re^{i\phi} - \frac1r e^{i\theta} = re^{i\theta} - \frac1r e^{i\phi}$

⇒ $\left(r+\frac1r\right)e^{i\phi} = \left(r+\frac1r\right)e^{i\theta}$

⇒ $e^{i\phi} = e^{i\theta}$

⇒ $\phi=\theta$

Therefore, $z = re^{i\theta}$ and $w = \frac1r e^{i\theta}$

⇒ $z\bar w = re^{i\theta}\cdot\frac1r e^{-i\theta} = 1$

Imp. note : 'if and only if' means we have to prove the relation in both directions.

Conversely

Assuming that $z=w$ or $z\bar w = 1$

If $z=w$, then

L.H.S. $= z\bar z w - w\bar w z = |z|^2 . z - |w|^2 . z$

$= |z|^2 . z - |z|^2 . z = 0$

and R.H.S. $= z - w = 0$

If $z\bar w = 1$ then $\bar z w = 1$ and

L.H.S. $= z\bar z w - w\bar w z = \bar z . 1 - \bar w . 1 = \bar z - \bar w = z - w$

$= 0 =$ R.H.S. Hence proved

Alter

We have, $|z|^2 w - |w|^2 z = z - w$

⇔ $|z|^2 w - |w|^2 z - z + w = 0$

⇔ $(|z|^2+1)w - (|w|^2+1)z = 0$

⇔ $(|z|^2+1)w = (|w|^2+1)z$

⇔ $\frac zw = \frac{|z|^2+1}{|w|^2+1}$

∴ $\frac zw$ is purely real,

⇔ $\frac{\bar z}{\bar w} = \frac zw$ ⇒ $z\bar w = \bar z w$ ...(1)

again, $|z|^2 w - |w|^2 z = z - w$

⇔ $z.\bar z w - w.\bar w z = z - w$

⇔ $z(\bar z w - 1) - w(z\bar w - 1) = 0$

⇔ $(z-w)(z\bar w - 1) = 0$ {using (1)}

$\Rightarrow \quad z = w$ or $z\bar{w} = 1$

Hence, $|z|^2 w - |w|^2 z = z - w$ if and only if $z = w$ or $z\bar{w} = 1$

16. $$z^{p+q} - z^p - z^q - 1 = 0 \qquad \ldots(1)$$

$$\Rightarrow \quad (z^p - 1)(z^q - 1) = 0$$

as α is root of (1), either $\alpha^p - 1 = 0$ or $\alpha^q - 1 = 0$

$\Rightarrow$ either $\dfrac{\alpha^p - 1}{\alpha - 1} = 0$ or $\dfrac{\alpha^q - 1}{\alpha - 1} = 0$ (as $\alpha \neq 1$)

$\Rightarrow$ either $1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1}$

or $\quad 1 + \alpha + \ldots + \alpha^{q-1} = 0$

But $\alpha^p - 1 = 0$ and $\alpha^q - 1 = 0$ cannot occur simultaneously as p and q are distinct primes, so neither p divides q nor q divides p, which is the requirement for $1 = \alpha^p = \alpha^q$.

17. Given $|z_1| < 1$ and $|z_2| > 1$

Then to prove

$$\left|\frac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1 \left\{\text{using } \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right\}$$

$$\Rightarrow \quad |1 - z_1\bar{z}_2| < |z_1 - z_2|$$

Squaring both sides, we get,

$$(1 - z_1\bar{z}_2)(1 - \bar{z}_1 z_2) < (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$$

$$\{\text{using } |z|^2 = z\bar{z}\}$$

$$\Leftrightarrow \quad 1 - z_1\bar{z}_2 - \bar{z}_1 z_2 + z_1\bar{z}_1 z_2\bar{z}_2 < z_1\bar{z}_1 - z_1\bar{z}_2 - \bar{z}_2 z_1 + z_2\bar{z}_2$$

$$\Leftrightarrow \quad 1 + |z_1|^2 |z_2|^2 < |z_1|^2 + |z_2|^2$$

$$\Leftrightarrow \quad 1 - |z_1|^2 - |z_2|^2 + |z_1|^2 |z_2|^2 < 0$$

$$\Leftrightarrow \quad (1 - |z_1|^2)(1 - |z_2|^2) < 0 \qquad \ldots(2)$$

which is true by (1) as $|z_1| < 1$ and $|z_2| > 1$

$\therefore \quad (1 - |z_1|^2) > 0$ and $(1 - |z_2|^2) < 0$

$\therefore$ (2) is true whenever (1) is true.

$$\Rightarrow \quad \left|\frac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$$

18. Given : $a_1 z + a_2 z^2 + \ldots + a_n z^n = 1$

and $$|z| < 1/3 \qquad \ldots(1)$$

$$|a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n| = 1$$

$$\{\text{using } |z_1 + z_2| \le |z_1| + |z_2|\}$$

$$\Rightarrow \quad |a_1 z| + |a_2 z^2| + |a_3 z^3| + \ldots + |a_n z^n| \ge 1$$

$$\Rightarrow \quad 2(|z| + |z|^2 + |z|^3 + \ldots + |z|^n) > 1 \qquad [\text{using } |a_r| < 2]$$

$$\Rightarrow \quad \frac{2|z|(1 - |z|^n)}{1 - |z|} > 1 \qquad \{\text{using sum of } n \text{ terms of G.P.}\}$$

$$\Rightarrow \quad 2|z| - 2|z|^{n+1} > 1 - |z| \quad \Rightarrow \quad 3|z| > 1 + 2|z|^{n+1}$$

$$\Rightarrow \quad |z| > \frac{1}{3} + \frac{2}{3}|z|^{n+1}$$

$$\Rightarrow \quad |z| > \frac{1}{3}, \text{ which contradicts} \qquad \ldots(1)$$

$\therefore$ There exists no complex number z such that

$$|z| < \frac{1}{3} \quad \text{and} \quad \sum_{r=1}^{n} a_r z^r = 1$$

19. As we know; $|z|^2 = z.\bar{z}$

$$\Rightarrow \quad \frac{|z - \alpha|^2}{|z - \beta|^2} = k^2$$

$$\Rightarrow \quad (z - \alpha)(\bar{z} - \bar{\alpha}) = k^2 (z - \beta)(\bar{z} - \bar{\beta})$$

$$|z|^2 - \alpha\bar{z} - \bar{\alpha}z + |\alpha|^2 = k^2(|z|^2 - \beta\bar{z} - \bar{\beta}z + |\beta|^2)$$

or $$|z|^2 (1 - k^2) - (\alpha - k^2\beta)\bar{z} - (\bar{\alpha} - \bar{\beta}k^2) z + (|\alpha|^2 - k^2|\beta|^2) = 0$$

$$\Rightarrow |z|^2 - \frac{(\alpha - k^2\beta)}{(1 - k^2)}\bar{z} - \frac{(\bar{\alpha} - \bar{\beta}k^2)}{(1 - k^2)} z + \frac{|\alpha|^2 - k^2|\beta|^2}{(1 - k^2)} = 0 \qquad \ldots(1)$$

On comparing with equation of circle,

$$|z|^2 + a\bar{z} + \bar{a}z + b = 0$$

whose centre is $(-a)$ and radius $= \sqrt{|a|^2 - b}$

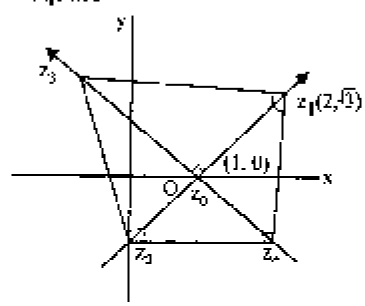
$\therefore$ centre for (i)

$$= \frac{\alpha - k^2\beta}{1 - k^2} \text{ and radius}$$

$$= \sqrt{\left(\frac{\alpha - k^2\beta}{1 - k^2}\right)\left(\frac{\bar{\alpha} - k^2\bar{\beta}}{1 - k^2}\right) - \frac{\alpha\bar{\alpha} - k^2\beta\bar{\beta}}{1 - k^2}}$$

$$\text{radius} = \left|\frac{k(\alpha - \beta)}{1 - k^2}\right|$$

20. Here, centre of circle is (1, 0) is also the mid-point of diagonals of square



$$\Rightarrow \quad \frac{z_1 + z_2}{2} = z_0$$

$$\Rightarrow \quad z_2 = -\sqrt{3}i, \qquad (\text{where } z_0 = 1 + 0\,i)$$

and $$\frac{z_3 - 1}{z_1 - 1} = e^{\pm i\pi/2}$$

$$\Rightarrow \quad z_3 = 1 + (1 + \sqrt{3}i)\cdot\left(\cos\frac{\pi}{2} \pm i\sin\frac{\pi}{2}\right), \text{ as } z_1 = 2 + \sqrt{3}i$$

$$= 1 \pm i(1 + \sqrt{3}i) = (1 \mp \sqrt{3}) \pm i$$

$$z_3 = (1 - \sqrt{3}) + i \quad \text{and} \quad z_4 = (1 + \sqrt{3}) - i$$