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DEPARTMENT OF MATHEMATICS

MATHEMATICS II (MA6251)

FOR

SECOND SEMESTER ENGINEERING STUDENTS
ANNA UNIVERSITY SYLLABUS

This text contains some of the most important short-answer (Part A) and long-answer questions (Part B) and their answers. Each unit contains 30 university questions. Thus, a total of 150 questions and their solutions are given. A student who studies these model problems will be able to get pass mark (hopefully!!).

Prepared by the faculty of Department of Mathematics

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UNIT I ORDINARY DIFFERENTIAL EQUATIONS

Part – A

Problem 1 Solve the equation $\left(D^2 - D + 1\right)y = 0$

Solution:

The A.E is $m^2 - m + 1 = 0 \Rightarrow m = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}i}{2}$.

$m = \frac{1 \pm \sqrt{3}i}{2}$ and $\alpha = \frac{1}{2};\ \beta = \frac{\sqrt{3}}{2}$

$G.S : y = e^{\alpha x}\left(A\cos\beta x + B\sin\beta x\right)$

$G.S : y = e^{\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}x}{2} + B\sin\frac{\sqrt{3}x}{2}\right)$ where A, B are arbitrary constants.

Problem 2 Find the particular integral of $\left(D^2 + a^2\right)y = b\cos ax + c\sin ax$.

Solution:

Given $\left(D^2 + a^2\right)y = b\cos ax + c\sin ax$.

$$\text{P.I} = b\frac{1}{D^2 + a^2}\cos ax + c.\frac{1}{D^2 + a^2}\sin ax.$$

$$= \frac{bx\sin ax}{2a} - \frac{cx\cos ax}{2a}$$

$$= \frac{x}{2a}\left[b\sin ax - c\cos ax\right].$$

Problem 3 Find the particular integral of $(D+1)^2 y = e^{-x}\cos x$.

Solution:

$$\text{P.I} = \frac{1}{(D+1)^2}e^{-x}\cos x$$

$$= \frac{e^{-x}}{(D-1+1)^2}\cos x$$

$$= e^{-x}\frac{1}{D^2}\cos x$$

$$= e^{-x}\frac{1}{D}\sin x$$

$$= -e^{-x}\cos x.$$

Problem 4 Find the particular integral of $(D^2+4)y = x^4$.

Solution:

$$\text{P.I} = \frac{1}{D^2+4}x^4$$

$$= \frac{1}{4\left(1+\frac{D^2}{4}\right)}x^4$$

$$= \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}x^4$$

$$= \frac{1}{4}\left(1-\frac{D^2}{4}+\frac{D^4}{16}\right)x^4$$

$$= \frac{1}{4}\left(x^4-\frac{4.3x^2}{4}+\frac{4.3.2.1}{16}\right)$$

$$= \frac{1}{4}\left(x^4-3x^2+\frac{3}{2}\right).$$

Problem 5 Solve $(D^2+6D+9)y = e^{-2x}x^3$.

Solution:

The A.E is $m^2+6m+9=0$

$\Rightarrow (m+3)^2 = 0$

$m = -3, -3$

C.F: $(A+Bx)e^{-3x}$

$$= \frac{1}{(D+3)^2}e^{-2x}x^3$$

$$= \frac{e^{-2x}}{(D-2+3)^2}x^3$$

$$= \frac{e^{-2x}}{(1+D)^2}x^3 = e^{-2x}(1+D)^{-2}x^3$$

$$\text{P.I} = e^{-2x}\left(1-2D+3D^2-4D^3\right)x^3$$

$$= e^{-2x}\left(x^3-2(3x^2)+3(3.2x)-4(3.2.1)\right)$$

$$= e^{-2x}\left(x^3-6x^2+18x-24\right)$$

G.S. $y = (A+Bx)e^{-3x} + (x^3-6x^2+18x-24)e^{-2x}$.

Problem 6 Solve $\left(D^2+2D-1\right)y = x$

Solution:

The A.E is $m^2+2m-1=0 \Rightarrow m = \dfrac{-2\pm\sqrt{4+4}}{2}$ y

$\Rightarrow m = -1\pm\sqrt{2}$

C.F: $Ae^{(-1+\sqrt{2})x} + Be^{(-1-\sqrt{2})x} = Ae^{-x}e^{\sqrt{2}x} + Be^{-x}e^{-\sqrt{2}x}$

P.I $= \dfrac{1}{\left(D^2+2D-1\right)}x$

$= \dfrac{1}{-\left(1-2D-D^2\right)}x$

$= -\left[1-\left(2D+D^2\right)\right]^{-1}x$

P.I $= -\left[1+2D+D^2\right]x = -x-2$

G.S: $y = e^{-x}\left(Ae^{\sqrt{2}x} + Be^{-\sqrt{2}x}\right) - (x+2)$.

Problem 7 Find the particular integral $\left(D^2+4D+5\right)y = e^{-2x}\cos x$

Solution:

P.I $= \dfrac{1}{D^2+4D+5}e^{-2x}\cos x$

$= \dfrac{1}{(D+2)^2+1}\left(e^{-2x}\cos x\right)$

$= e^{-2x}\dfrac{1}{(D-2+2)^2+1}\cos x$

$= e^{-2x}\dfrac{1}{D^2+1}\cos x$

P.I $= \dfrac{xe^{-2x}}{2}\sin x$.

Problem 8 Solve for x from the equations $x'-y=t$ and $x+y'=1$.

Solution:

$x'-y=t \rightarrow (1) \Rightarrow x''-y'=1 \Rightarrow x''-1=y'$

$x+y'=1 \rightarrow (2) \Rightarrow x+x''-1=1$

Thus $x''+x=2$ (or) $\left(D^2+1\right)x = 2$

$m^2+1=0 \Rightarrow m=\pm i$

C.F. $A\cos t + B\sin t$

$$\text{P.I} = \frac{1}{(D^2+1)}(2) = (D^2+1)^{-1}(2) = 2$$

G.S: $x = A\cos t + B\sin t + 2.$

Problem 9 Solve $[D^3 - 3D^2 - 6D + 8]y = x$.

Solution:

The A.E is $m^3 - 3m^2 - 6m + 8 = 0$

$(m-1)(m+2)(m-4) = 0$

$m = 1, -2, 4$

$\therefore C.F$ is $C_1e^x + C_2e^{-2x} + C_3e^{4x}$

$$\text{P.I} = \frac{1}{D^3 - 3D^2 - 6D + 8}x$$

$$= \frac{1}{8\left[1 + \frac{D^3 - 3D^2 - 6D}{8}\right]}x$$

$$= \frac{1}{8}\left[1 + \frac{D^3 - 3D^2 - 6D}{8}\right]^{-1}x$$

$$= \frac{1}{8}\left[1 - \left(\frac{D^3 - 3D^2 - 6D}{8}\right) + ..\right]x$$

$$= \frac{1}{8}\left[x + \frac{6}{8}\right] = \frac{1}{8}\left[x + \frac{3}{4}\right].$$

Complete solution is $y = C.F + P.I$

$$y = C_1 e^x + C_2 e^{-2x} + C_3e^{4x} + \frac{1}{8}\left[x + \frac{3}{4}\right].$$

Problem 10 Solve the equation $[D^2 - 4D + 13]y = e^{2x}$

Solution:

Given $[D^2 - 4D + 13]y = e^{2x}$

The A.E is $m^2 - 4m + 13 = 0$

$$m = \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm 6i}{2} = 2 \pm 3i$$

C.F $y = e^{2x}(A\cos 3x + B\sin 3x)$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 13} e^{2x}$$

$$= \frac{1}{4 - 8 + 13} e^{2x} = \frac{1}{9} e^{2x}$$

G.S: $y = C.F + P.I$

$$y = e^{2x}\left(A\cos 3x + B\sin 3x\right) + \frac{e^{2x}}{9}.$$

Problem 11 Solve the equation $\left(D^5 - D\right) y = 12e^x$

Solution:

Given $\left(D^5 - D\right) y = 12e^x$

The A.E is $m^5 - m = 0$

$m\left(m^4 - 1\right) = 0$

$m^4 - 1 = 0$

$m = 0 (or) m^4 - 1 = 0$

$\left(m^2 - 1\right)\left(m^2 + 1\right) = 0$

$m = 0,\ m = \pm 1,\ m = \pm i$

C.F $= C_1 e^{0x} + C_2 e^x + C_3 e^{-x} + \left[C_4 \cos x + C_5 \sin x\right]$

$$\text{P.I} = \frac{1}{D^5 - D} 12e^x$$

$$= \frac{1}{1-1} 12e^x \quad \text{(Replacing D by 1)}$$

$$= \frac{x}{5D^4 - 1} 12e^x \quad \text{(Replacing D by 1)}$$

$$= \frac{x}{5-1} 12e^x = \frac{x}{4} 12e^x = 3xe^x$$

G.S. $y = C.F + P.I$

$$= C_1 + C_2 e^x + C_3 e^{-x} + \left[C_4 \cos x + C_5 \sin x\right] + 3xe^x.$$

Problem 12 Solve the equation $\left(D^2 + 5D + 6\right) y = e^{-7x} \sinh 3x$

Solution:

The A.E is $m^5 + 5m + 6 = 0$

$(m+2)(m+3) = 0$

$m = -2, -3$

C.F. is $C_1 e^{-2x} + C_2 e^{-3x}$

$$\text{P.I} = \frac{1}{D^2 + 5D + 6} e^{-7x} \sinh 3x$$

$$= \frac{1}{D^2+5D+6} e^{-7x} \left(\frac{e^{3x}-e^{-3x}}{2} \right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2+5D+6} e^{-4x} - \frac{1}{D^2+5D+6} e^{-10x} \right]$$

$$= \frac{1}{2}\left[\frac{e^{-4x}}{16-20+6} - \frac{e^{-10x}}{10-50+6} \right]$$

$$= \frac{1}{2}\left[\frac{e^{-4x}}{2} + \frac{e^{-10x}}{34} \right]$$

$\therefore$ G.S. $y = C_1 e^{-2x} + C_2 e^{-3x} + \dfrac{e^{-4x}}{4} + \dfrac{e^{-10x}}{68}$.

Problem 13 Solve the equation $\left(D^3 - 3D^2 + 4D - 2\right) y = e^x$

Solution:

Given $m^3 - 3m^2 - 4m - 2 = 0$

$(m-1)\left(m^2 - 2m + 2\right) = 0$

$m = 1\ (or)\ m = 1 \pm i$

Complementary function $= Ae^x + e^x\left(B\cos x + C\sin x\right)$

P.I $= \dfrac{1}{D^3 - 3D^2 + 4D - 2} e^x$

$$= \frac{1}{(1)^3 - 3(1)^2 + 4(1) - 2} e^x \quad \text{(Replacing D by 1)}$$

$$= \frac{1}{1-3+4-2} e^x = \frac{1}{0} e^x$$

$$= \frac{x}{3D^2 - 6D + 4} e^x$$

$$= \frac{1}{3-6+4} e^x \quad \text{(Replacing D by 1)}$$

$$= xe^x$$

G.S: $y = C.F. + P.I$

$$= Ae^x + e^x\left(B\cos x + C\sin x\right) + xe^x.$$

Problem 14 Solve the equation $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = e^{-2x}$

Solution:

Given $\left(D^2 + 4D + 4\right) y = e^{-2x}$

The A.E is $m^2 + 4m + 4 = 0$

$\left(m^2 + 2\right)\left(m + 2\right) = 0$

$m = -2, -2$

C.F.: $(Ax+B)e^{-2x}$

$$\text{P.I} = \frac{1}{D^2+4D+4}e^{-2x}$$

$$= \frac{1}{(-2)^2+4(-2)+4}e^{-2x} \quad (\text{Replacing D by - 2})$$

$$= \frac{1}{4-8+4}e^{-2x} \quad (\therefore \text{Dr is } 0)$$

$$= \frac{x}{2D+4}e^{-2x} \quad (\text{Replacing D by -2})$$

$$= \frac{x}{2(-2)+4}e^{-2x}$$

$$= \frac{x^2}{2}e^{-2x} \quad (\therefore \text{Dr is } 0)$$

G.S is $y = (Ax+B)e^{-2x} + \frac{x^2}{2}e^{-2x}$.

Problem 15 Solve the equation $(D^2+2D+1)y = e^{-x}+3$

Solution:

Given $(D^2+2D+1)y = e^{-x}+3$

The A.E is $m^2+2m+1=0$

$(m+1)(m+1)=0$

$m=-1,-1$

C.F.: $(Ax+B)e^{-x}$

P.I $= P.I_1 + P.I_2$

$$P.I_1 = \frac{1}{D^2+2D+1}e^{-x}$$

$$= \frac{1}{(-1)^2+2(-1)+1}e^{-x} \quad (\text{Replacing D by -1})$$

$$= \frac{1}{1-2+1}e^{-x}$$

$$= \frac{x}{2D+2}e^{-x} \quad (\because \text{Dr is } 0)$$

$$= \frac{x}{2(-1)+2}e^{-x} \quad (\text{Replacing D by -1})$$

$$PI_2 = \frac{1}{D^2+2D+1}3e^{0x}$$

$$= \frac{1}{(0)^2 + 2(0) + 1} 3e^{0x} \quad \text{(Replacing D by 0)}$$

G.S is $y = (Ax + B)e^{-x} + \frac{x^2}{2}e^{-x} + 3.$

Part-B

Problem 1 Solve $(D^2 - 2D - 8)y = -4\cosh x \sinh 3x + (e^{2x} + e^x)^2 + 1.$

Solution:

The A.E. is $(m^2 - 2m - 8) = 0$

$\Rightarrow (m-4)(m+2) = 0$

$\Rightarrow m = -2, 4$

C.F.: $Ae^{-2x} + Be^{4x}$

R.H.S $= -4\cosh x \sin 3x + (e^{2x} + e^x)^2 + 1$

$$= -4\left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^{3x} - e^{-3x}}{2}\right) + (e^{2x} + e^x)^2 + 1$$

$$= -(e^{4x} - e^{-2x} + e^{2x} - e^{-4x}) + e^{4x} + 2e^{3x} + e^{2x} + 1$$

$$= e^{-2x} + e^{-4x} + 2e^{3x} + 1e^{0x}$$

$$\text{P.I.} = \frac{1}{(D-4)(D+2)}(e^{-2x}) + \frac{1}{(D-4)(D+2)}(-e^{-4x} + 2e^{3x} + e^{0x})$$

$$= \frac{-1}{(-2-4)(D+2)}e^{-2x} - \frac{e^{-4x}}{(-8)(-2)} - \frac{2e^{3x}}{(-1)(5)} + \frac{1}{(-4)(2)}$$

$$= \frac{-xe^{-2x}}{6} - \frac{e^{-4x}}{16} - \frac{2e^{3x}}{5} - \frac{1}{8}$$

G.S is $y = Ae^{-2x} + Be^{4x} - \frac{xe^{-2x}}{6} - \frac{e^{-4x}}{16} - \frac{2e^{3x}}{5} - \frac{1}{8}.$

Problem 2 Solve $y'' + y = \sin^2 x + \cos x \cos 2x \cos 3x$

Solution:

The A.E is $m^2 + 1 = 0 \Rightarrow m^2 = -1 \Rightarrow m = \pm i$

C.F.: $A\cos x + B\sin x$

$$\text{R.H.S} = \frac{\cos x(2\cos 2x \cos 3x)}{2} = \frac{\cos x}{2}[\cos 5x + \cos x]$$

$$= \frac{1}{4}[2\cos x \cos 5x + 2\cos^2 x]$$

$$=\frac{1}{4}\left[\cos 6x+\cos 4x+1+\cos 2x\right]$$

$\left[\because 2\cos A\cos B=\cos\left(A+B\right)+\cos\left(A-B\right)\right]$ $A=3x,\ B=2x$

$$\text{P.I.}=\frac{1}{\left(D^2+1\right)}\left[\sin^2 x+\cos x\cos 2x\cos 3x\right]$$

$$=\frac{1}{D^2+1}\left[\frac{e^{0x}}{2}-\frac{\cos 2x}{2}+\frac{\cos 6x}{4}+\frac{\cos 4x}{4}+\frac{\cos 2x}{4}+\frac{e^{0x}}{4}\right]$$

$$=\frac{1}{2}-\frac{\cos 2x}{\left(-4+1\right)^2}+\frac{\cos 6x}{4\left(-36+1\right)}+\frac{\cos 4x}{4\left(-16+1\right)}+\frac{\cos 2x}{4\left(-4+1\right)}+\frac{1}{4}$$

$$=\frac{3}{4}+\frac{\cos 2x}{6}-\frac{\cos 6x}{140}-\frac{\cos 4x}{60}-\frac{\cos 6x}{140}+\frac{3}{4}.$$

G.S. is $y=A\cos x+B\sin x+\frac{\cos 2x}{12}-\frac{\cos 4x}{60}-\frac{\cos 6x}{140}+\frac{3}{4}.$

Problem 3 Solve $\frac{d^2x}{dy^2}+10x=\cos 8y.$

Solution:

Here y is independent and x is dependent variable

Let $D=\frac{d}{dy}$.

The A.E is $m^2+10=0$

$\Rightarrow m^2=-10$

$\Rightarrow m=\pm\sqrt{10}i$

C.F.: $A\cos\sqrt{10}y+B\sin\sqrt{10}y$

$$\text{P.I}=\frac{1}{\left(D^2+10\right)}\cos 8y=\frac{\cos 8y}{-64+10}=\frac{-\cos 8y}{54}$$

G.S. is $x=A\cos\sqrt{10}y+B\sin\sqrt{10}y-\frac{\cos 8y}{54}$

Problem 4 Solve $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=\sin x\cos 2x$.

Solution:

The A.E is $m^2+6m+9=0$

$\left(m+3\right)^2=0$

$m=-3,\ -3.$

C.F.: $\left(A+Bx\right)e^{-3x}$

$$\text{R.H.S}=\frac{2\sin x\cos 2x}{2}=\frac{1}{2}\left[\sin 3x+\sin\left(-x\right)\right]$$

$$=\frac{1}{2}[\sin 3x-\sin x]$$

$$[2\sin A\cos B=\sin(A+B)+\sin(A-B)] \quad A=x,\ B=2x$$

$$\text{P.I.} =\frac{1}{2}\frac{1}{(D+3)^2}\sin 3x-\frac{1}{2}\frac{1}{(D+3)^2}\sin x$$

$$=\frac{1}{2}\frac{1}{D^2+6D+9}\sin 3x-\frac{1}{2}\frac{1}{D^2+6D+9}\sin x$$

$$\text{P.I.}=\frac{1}{2}\times\frac{1}{-9+6D+9}\sin 3x-\frac{1}{2}\times\frac{1}{-1+6D+9}\sin x$$

$$=\frac{1}{12}\times\frac{1}{D}\sin 3x-\frac{1}{2}\times\frac{1}{8+6D}\sin x$$

$$=\frac{-\cos 3x}{12(3)}-\frac{(4-3D)\sin x}{4(4+3D)(4-3D)}$$

$$=\frac{-1}{36}\cos 3x-\frac{1}{4}\ \frac{1}{16-9D^2}(4\sin x-3\cos x)$$

$$=\frac{-1}{36}\cos 3x-\frac{1}{4}\times\frac{4\sin x-3\cos x}{16+9}$$

$$=\frac{-\cos 3x}{36}-\frac{\sin x}{25}+\frac{3\cos x}{100}$$

Problem 5 Solve $(D^2+4)y=x^4+\cos^2 x$

Solution:

The A.E. is $m^2+4=0$

$m=\pm 2i$

C.F.: $A\cos 2x+B\sin 2x$

$$\text{P.I} =\frac{1}{D^2+4}x^4+\frac{1}{D^2+4}\left(\frac{1+\cos 2x}{2}\right)$$

$$=\frac{1}{4}\frac{1}{\left(1+\frac{D^2}{4}\right)}x^4+\frac{1}{2}\frac{1}{(D^2+4)}e^{0x}+\frac{1}{2}\frac{1}{D^2+4}\cos 2x$$

$$=\frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}x^4+\frac{1}{2(4)}+\frac{(x\sin 2x)}{2(2)(2)}$$

$$=\frac{1}{4}\left(1-\frac{D^2}{4}+\frac{D^4}{16}\right)x^4+\frac{1}{8}+\frac{x\sin 2x}{8}$$

$$=\frac{x^4}{4}-\frac{12x^2}{16}+\frac{4.3.2.1}{64}+\frac{1}{8}+\frac{x\sin 2x}{8}$$

G.S. is $y = A\cos 2x + B\sin 2x + \frac{4}{8} - \frac{3x^2}{4} + \frac{x^4}{4} + \frac{x\sin 2x}{8}$

Problem 6 Solve $(D^2+2D-1)y = (x+e^x)^2 + \cos 2x \cosh x$.

Solution:

The A.E is $m^2+2m-1=0$

$$m = \frac{-2\pm\sqrt{4+4}}{2} = -1\pm\sqrt{2}$$

C.F.: $Ae^{(-1+\sqrt{2})x} + Be^{(-1-\sqrt{2})x}$

$$\text{P.I.} = \frac{1}{(D^2+2D-1)}(x^2+2xe^x+e^x) + \frac{1}{(D^2+2D-1)}\cos 2x\frac{(e^x+e^{-x})}{2}$$

$$\frac{1}{(D^2+2D-1)}x^2 = -\frac{1}{[1-(2D+D^2)]}x^2$$

$$= -\left[1+(2D+D^2)+(2D+D^2)^2+....\right]x^2$$

$$= -\left[1+2D+D^2+4D^2\right]x^2 = -x^2$$

$$= -x^2+4x+(5)(2)$$

$$\frac{1}{D^2+2D-1}x^2 = -x^2+4x+10$$

$$\frac{2}{D^2+2D-1}xe^x = \left(\frac{2e^x}{(D+1)^2+2(D+1)-1}\right)x$$

$$= 2e^x\frac{1}{D^2+2D+1+2D+2-1}x$$

$$= \frac{2e^x}{D^2+4D+2}(x)$$

$$= \frac{2e^x}{2}\frac{1}{\left[1+\left(2D+\frac{D^2}{2}\right)\right]}x$$

$$= e^x\left[1+\left(2D+\frac{D^2}{2}\right)\right]^{-1}x$$

$$= e^x\left[1+\left(2D+\frac{D^2}{2}\right)+...\right]x$$

$$= e^x[1+2D]x$$

$$\frac{2}{(D^2+2D-1)}xe^x = e^x[x+2] = (x+2)e^x$$

$$\frac{1}{D^2+2D-1}e^{2x}=\frac{1}{(4+4-1)}e^{2x}=\frac{e^{2x}}{7}$$

$$\frac{1}{D^2+2D-1}\frac{e^x\cos 2x}{2}+\frac{1}{D^2+2D-1}\frac{e^{-x}\cos 2x}{2}$$

$$=\frac{e^x}{2}\frac{1}{(D+1)^2+2(D+1)-1}\cos 2x+\frac{e^{-x}}{2}\frac{1}{(D-1)^2+2(D-1)-1}\cos 2x$$

$$=\frac{e^x}{2}\frac{1}{D^2+2D+1+2D+2-1}\cos 2x+\frac{e^{-x}}{2}\frac{1}{-4-2}\cos 2x$$

$$=\frac{e^x}{2}\frac{(2D+1)\cos 2x}{2(2D-1)(2D+1)}-\frac{e^{-x}}{12}\cos 2x$$

$$=\frac{e^x}{2}\frac{1}{2(4D^2-1)}(-2.2\sin 2x+\cos 2x)-\frac{e^{-x}\cos 2x}{12}$$

$$=\frac{e^x}{4}\frac{(-4\sin 2x+\cos 2x)}{(-16-1)}-\frac{e^{-x}\cos 2x}{12}$$

$$=-\frac{e^x(\cos 2x-4\sin 2x)}{17}-\frac{e^{-x}\cos 2x}{12}$$

The General Solution is

$$y=Ae^{(-1+\sqrt{2})x}+Be^{-(1+\sqrt{2})x}+10+4x-x^2+\frac{e^{2x}}{7}+(x+2)e^x$$

$$-\frac{e^x}{17}(\cos 2x-4\sin 2x)-\frac{e^{-x}}{12}\cos 2x$$

Problem 7 Solve $(D^2+4)y=x^2\cos 2x$

Solution:

The A.E is $m^2+4=0$

$\Rightarrow m^2=-4$

$\Rightarrow m=\pm 2i$

C.F.: $A\cos 2x+B\sin 2x$

$$\text{P.I}=\frac{1}{D^2+4}(x^2\cos 2x)$$

$$=R.P\ of\ \frac{1}{D^2+4}x^2e^{i2x}=R.P.\ of\ \frac{e^{2ix}}{(D+2i)^2+4}x^2$$

$$\text{P.I.}=R.P\ of\ e^{2ix}\frac{1}{D^2+4iD-4+4}x^2$$

$$=R.P\ of\ e^{2ix}\frac{1}{D^2+4iD}x^2=R.P\ \ of\ e^{2ix}\frac{1}{D(D+4i)}x^2$$

$$= R.P\ of\ e^{2ix}\frac{1}{D}\frac{1}{4i\left(1+\frac{D}{4i}\right)}x^2$$

$$= R.P\ \ of\ \ \frac{e^{2ix}}{4i}\frac{1}{D}\left(1+\frac{D}{4i}\right)^{-1}x^2$$

$$= R.P\ \ of\ \ \frac{e^{2ix}}{4i}\frac{1}{D}\left(1-\frac{D}{4i}-\frac{D^2}{16}\right)x^2$$

$$= R.P\ \ of\ \ \left(\frac{-ie^{2ix}}{4}\right)\left(\frac{x^3}{3}-\frac{x^2}{4i}-\frac{x}{8}\right)$$

$$= R.P\ \ of\ \ \left(\frac{-ie^{2ix}}{4}\right)\left(\frac{x^3}{3}+\frac{ix^2}{4}-\frac{x}{8}\right)$$

$$= R.P\ of\ \left(\frac{e^{2ix}}{4}\right)\left(\frac{-x^3 i}{3}+\frac{x^2}{4}+\frac{ix}{8}\right)$$

$$= R.P\ of\ \ \frac{(\cos 2x+\sin 2x)}{4}\left(\frac{-x^3 i}{3}+\frac{x^2}{4}+\frac{ix}{8}\right)$$

$$= \frac{1}{4}\left[\frac{x^2\cos 2x}{4}+\frac{x^3\sin 2x}{3}-\frac{x\sin 2x}{8}\right]$$

P.I. $= \frac{1}{4}\left[\frac{x^2\cos 2x}{4}+\frac{x^3\sin 2x}{3}-\frac{x\sin 2x}{8}\right]$

G.S.: $y = A\cos 2x + B\sin 2x + \frac{x^2\cos 2x}{16}+\frac{x^3\sin 2x}{12}-\frac{x\sin 2x}{32}$.

Problem 8 Solve $\left(D^2+a^2\right)y = \sec a\,x$.

Solution:

The A.E. is $m^2+a^2=0$

$\Rightarrow m^2 = -a^2$

$\Rightarrow m = \pm\, a\,i$

C.F.: $A\cos ax + B\sin a\,x$

P.I $= \frac{1}{(D+ai)(D-ai)}\sec a\,x \rightarrow (1)$

Using partial fractions

$$\frac{1}{D^2+a^2} = \left[\frac{C_1}{D+ai}+\frac{C_2}{D-ai}\right]$$

$$1 = C_1(D-ai)+C_2(D+ai)$$

$$C_1 = -\frac{1}{2ia},\ C_2 = \frac{1}{2ia}$$

$$\text{P.I.} = -\frac{1}{2ia}\frac{1}{(D+ai)}\sec ax + \frac{1}{2ia}\frac{1}{(D-ai)}\sec a\,x$$

$$= -\frac{1}{2ia}\frac{1}{D-(-ai)}\sec ax + \frac{e^{aix}}{2ia}\int e^{-aix}\sec a\,x\,dx$$

$$= -\frac{e^{-aix}}{2ia}\int e^{aix}\sec a\,x\,dx + \frac{e^{aix}}{2ia}\int e^{-aix}\sec a\,x\,dx$$

$$\text{P.I.} = -\frac{e^{-aix}}{2ia}\int\frac{(\cos ax + i\sin ax)}{\cos ax}dx + \frac{e^{aix}}{2ia}\int\frac{(\cos ax - i\sin ax)}{\cos ax}dx$$

$$\text{P.I.} = -\frac{e^{-aix}}{2ia}\int(1+i\tan ax)\,dx + \frac{e^{aix}}{2ia}\int(1-i\tan ax)\,dx$$

$$= -\frac{e^{-aix}}{2ia}\left[x+\frac{i}{a}\log\sec ax\right] + \frac{e^{aix}}{2ia}\left[x-\frac{i}{a}\log\sec ax\right]$$

$$= \frac{2x}{2a}\left[\frac{e^{aix}-e^{-aix}}{2i}\right] - \frac{2i}{2ia^2}[\log\sec ax]\left[\frac{e^{aix}+e^{-aix}}{2}\right]$$

$$= \frac{x}{a}\sin ax - \frac{1}{a^2}(\log\sec ax)(\cos ax)$$

$$= \frac{1}{a^2}[ax\sin ax + \cos ax\log\cos ax]$$

G.S. is $y = C.F + P.I.$

Problem 9 Solve $(D^2+4D+3)y = e^{-x}\sin x + xe^{3x}$.

Solution:

The A.E is $m^2+4m+3=0$

$(m+1)(m+3)=0$

$m=-1,-3$

C.F.: $Ae^{-x}+Be^{-3x}$

$$\text{P.I} = \frac{1}{(D+3)(D+1)}e^{-x}\sin x + \frac{1}{(D+1)(D+3)}xe^{3x}$$

$$= \frac{e^{-x}}{(D-1+3)(D-1+1)}(\sin x) + \frac{e^{3x}}{(D+3+1)(D+3+3)}(x)$$

$$= e^{-x}\frac{1}{(D+2)D}\sin x + e^{3x}\frac{1}{(D+4)(D+6)}x$$

$$= -e^{-x}\frac{D-2}{(D+2)(D-2)}\cos x + e^{3x}\frac{1}{D^2+10D+24}x$$

$$= -e^{-x}\frac{1}{(D^2-4)}(-\sin x - 2\cos x) + \frac{e^{3x}}{24}\frac{1}{1+\frac{10D}{24}+\frac{D^2}{24}}x$$

$$\text{P.I.} = e^{-x}\frac{1}{(D^2-4)}(\sin x + 2\cos x) + \frac{e^{3x}}{24}\left[1+\frac{5D}{12}+\frac{D^2}{24}\right]^{-1} x$$

$$= \frac{e^{-x}(\sin x + 2\cos x)}{(-1-4)} + \frac{e^{3x}}{24}\left[1-\frac{5D}{12}\right]x$$

$$= -\frac{e^{-x}}{5}(\sin x + 2\cos x) + \frac{e^{3x}}{24}\left(x - \frac{5}{12}\right)$$

G.S. is $y = Ae^{-x} + Be^{-3x} - \frac{e^{-x}}{5}(\sin x + 2\cos x) + \frac{e^{3x}}{24}\left(x - \frac{5}{12}\right)$.

Problem 10 Solve the Legendre's linear equation

$$\left[(3x+2)^2 D^2 + 3(3x+2)D - 36\right]y = 3x^2 + 4x + 1$$

Solution:

Let $\left[(3x+2)^2 + D^2 + 3(3x+2)D - 36\right]y = 3x^2 + 4x + 1$

Let $3x + 2 = e^t$ or $t = \log(3x+2)$

$$\Rightarrow \frac{dt}{dx} = -\frac{3}{3x+2}$$

$3x = e^z - 2$

$$x = \frac{1}{3}e^z - \frac{2}{3}$$

$Let\,(3x+2)D = 3D'$

$(3x+2)^2 D^2 = 9D'(D'-1)$

$$\left[9D'(D'-1) + 3(3D') - 36\right]y = 3\left[\frac{1}{3}e^z - \frac{2}{3}\right] + 4\left[\frac{1}{3}e^z - \frac{2}{3}\right] + 1$$

$$\left[9D'^2 - 9D' + 9D' - 36\right]y = 3\left[\frac{1}{9}e^{2z} + \frac{4}{9} - \frac{4}{9}e^z\right] + \frac{4}{3}e^z - \frac{8}{3} + 1$$

$$\left[9D'^2 - 36\right]y = \frac{1}{3}e^{2z} + \frac{4}{3} - \frac{4}{3}e^z + \frac{4}{3}e^z - \frac{8}{3} + 1$$

$$= \frac{1}{3}e^{2z} - \frac{1}{3}$$

A.E is $9m^2 - 36 = 0$

$9m^2 = 36$

$m^2 = 4$

$m = \pm 2$

$C.F = Ae^{2z} + Be^{-2z}$

$$= A(3x+2)^2 + B(3x+2)^{-2}$$

$$P.I_1 = \frac{1}{9D'^2 - 36}\frac{e^{2z}}{3}$$
$$= \frac{1}{3}.\frac{1}{36-36}e^{2z}$$
$$= \frac{1}{3}z\frac{1}{18D'}e^{2z}$$
$$= \frac{1}{54}z\frac{e^{2z}}{2}$$
$$= \frac{1}{108}ze^{2z}$$
$$= \frac{1}{108}[\log 3x+2](3x+2)^2$$
$$P.I_2 = \frac{1}{9D^{12} - 36}\frac{e^{0z}}{3}$$
$$= \frac{1}{3}.\frac{1}{-36}e^{0z} = -\frac{1}{108}$$
$$y = C.F + P.I_1 - PI_2$$
$$= A(3x+2)^2 + B(3x+2)^{-2} + \frac{1}{108}(3x+2)^2\log(3x+2) + \frac{1}{108}$$
$$= A(3x+2)^2 + B(3x+2)^{-2} + \frac{1}{108}\left[(3x+2)^2\log(3x+2)+1\right].$$

Problem 11 Solve $(D^2+5D+4) = e^{-x}\sin 2x + x^2 + 1$ where $D = \frac{d}{dx}$.

Solution:

The $A.E\ m^2+5m+4=0$

$m=-4$ or $m=-1$

$C.F = Ae^{-4x}+Be^{-x}$

$$P.I = \frac{1}{D^2+5D+4}\left(e^{-x}\sin 2x + 2x^2+1\right)$$
$$= e^{3x}\frac{1}{(D-1)^2+5(D-1)+4}\sin 2x + \frac{1}{4\left(1+\frac{D^2+5D}{4}\right)}\left(x^2+1\right)$$
$$= e^{3x}\frac{1}{D^2+3D}\sin 2x + \frac{1}{4}\left(1-\frac{5D}{4}+\frac{5D^2}{16}\right)\left(x^2+1\right)$$
$$= \frac{-e^{-x}}{26}[2\sin 2x + 3cos2x] + \frac{1}{4}\left(x^2-\frac{5}{2}x+\frac{13}{8}\right)$$

$G.S : y = C.F. + P.I$

$$y = Ae^{-4x} + Be^{-x} - \frac{e^{-x}}{26}(2\sin 2x + 3\cos 2x) + \frac{1}{4}\left(x^2 - \frac{5}{2}x + \frac{13}{8}\right).$$

Problem 12 Solve $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = \sin(\log x)$.

Solution:

Given equation is $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = \sin(\log x)$.

$\left(x^2D^2 - 4xD + 6\right)y = \sin(\log x) \to (1)$

Put $x = e^z$ (or) $z = \log x$

$xD = D' \to (2)$

$x^2D^2 = D'(D'-1) \to (3)$ Where D' denotes $\frac{d}{dz}$

Sub (2) & (3) in (1) we get

$\left(D'(D'-1) + 4D' + 2\right)y = \sin z$

$(i.e)\left(D'^2 - D' + 4D' + 2\right)y = \sin z$

$\left(D'^2 + 3D' + 2\right)y = \sin z \to (4)$

The $A.E$ is $m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0$

$m = -1, -2$

C.F.: $Ae^{-z} + Be^{-2z}$

$$\text{P.I.} = \frac{1}{D'^2 + 3D' + 2}\sin z$$

$$= \frac{1}{-1 + 3D' + 2}\sin z$$

$$= \frac{1}{3D' + 1}\sin z$$

$$= \frac{3D' - 1}{9D'^2 - 1}\sin z$$

$$= \frac{(3D'-1)\sin z}{9(-1) - 1} \quad \left[\text{Replace } D'^2 \text{ by } -1\right]$$

$$= \frac{3D'(\sin z) - \sin z}{-10}$$

$$= \frac{3\cos z - \sin z}{-10}$$

$\therefore$ The solution of (4) is

$$y = Ae^{-z} + Be^{-2z} + \frac{3\cos z - \sin z}{-10}$$

Sub $z=\log x$ or $x=e^{z}$, we get

$$y=Ae^{-\log x}+Be^{-2\log x}-\frac{3\cos(\log x)-\sin(\log x)}{10}$$

$$y=Ax^{-1}+Bx^{-2}-\frac{3\cos(\log x)-\sin(\log x)}{10}$$

$$y=\frac{A}{x}+\frac{B}{x^{2}}-\frac{3\cos(\log x)-\sin(\log x)}{10}$$

This gives the solution of the given differential equation.

Problem 13 Solve the simultaneous ordinary differential equation

$(D+4)x+3y=t,\ 2x+(D+5)y=e^{2t}$

Solution:

Given $(D+4)x+3y=t\rightarrow(1)$

$2x+(D+5)y=e^{2t}\rightarrow(2)$

$2\times(1)-(D+4)\times(2)$

$6y-(D+4)(D+5)y=2t-(D+4)e^{2t}$

$\left[6-D^{2}-9D-20\right]y=2t-2e^{2t}-4e^{zt}$

$\left(D^{2}+9D+14\right)y=6e^{2t}-2t$

The A.E. is $m^{2}+9m+14=0$

$(m+7)(m+2)=0$

$m=-2,-7$

C.F.: $Ae^{-2t}+Be^{-7t}$

$$\text{P.I.}=\frac{6}{\left(D^{2}+9D+14\right)}e^{2t}-\frac{2}{\left(D^{2}+9D+14\right)}t$$

$$=\frac{6e^{2t}}{4+18+14}-\frac{2}{14}\frac{1}{1+\frac{9D}{14}+\frac{D^{2}}{14}}(t)$$

$$=\frac{6e^{2t}}{36}-\frac{1}{7}\left(1+\frac{9D}{14}+\frac{D^{2}}{14}\right)^{-1}(t)$$

$$=\frac{e^{2t}}{6}-\frac{1}{7}\left(1-\frac{9D}{14}\right)(t)=\frac{e^{2t}}{6}-\frac{1}{7}\left(t-\frac{9}{14}\right)$$

G.S. is $y=Ae^{-2t}+Be^{-7t}+\frac{e^{2t}}{6}-\frac{t}{7}+\frac{9}{98}$

To Calculate x

$$Dy=-2Ae^{-2t}-7Be^{-7t}+\frac{2e^{2t}}{6}-\frac{1}{7}$$

$$5y = 5Ae^{-2t} + 5Be^{-7t} + \frac{5e^{2t}}{6} - \frac{5t}{7} + \frac{45}{98}$$

$$(D+5)y = 3Ae^{-2t} - 2Be^{-7t} + \frac{7e^{2t}}{6} - \frac{5t}{7} - \frac{1}{7} + \frac{45}{98}$$

$$(2) \Rightarrow 2x = -(D+5)y + e^{2t}$$

$$= -3Ae^{-2t} + 2Be^{-7t} - \frac{7e^{2t}}{6} + \frac{5t}{7} - \frac{31}{98} + e^{2t}$$

$$x = \frac{-3A}{2}e^{-2t} + Be^{-7t} - \frac{7}{72}e^{2t} + \frac{5t}{14} - \frac{31}{196}$$

The General solution is

$$x = \frac{-3A}{2}e^{-2t} + Be^{-7t} - \frac{e^{2t}}{12} + \frac{5t}{14} - \frac{31}{196}$$

$$y = Ae^{-2t} + Be^{-7t} + \frac{e^{2t}}{6} - \frac{t}{7} + \frac{9}{98}.$$

Problem 14 Solve: $\frac{d^2y}{dx^2} + y = \tan x$ by method of variation of parameters

Solution:

A.E is $m^2 + 1 = 0$

$m = \pm i$

C.F = $c_1 \cos x + c_2 \sin x$

$P.I = PI_1 + PI_2$

$f_1 = \cos x;\ f_2 = \sin x$

$f_1' = -\sin x;\ f_2' = \cos x$

$f_2' f_1' - f_1' f_2 = 1$

Now, $P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx$

$$= -\int \sin x \tan x dx$$

$$= -\int \frac{\sin^2 x}{\cos x} dx = \int \frac{(-1 + \cos^2 x)}{\cos x} dx$$

$$= -\int \sec x dx + \int \cos x dx$$

$$= -\log(\sec x + \tan x) + \sin x$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \cos x \tan x dx$$

$$= -\cos x$$

$\therefore$ y = C.F + Pf1 + Qf2

$= c_1\cos x + c_2\sin x + [-\log(\sec x + \tan x) + \sin x]\cos x - \cos x\sin x$

$= c_1\cos x + c_2\sin x - \log(\sec x + \tan x)\cos\cos x$.

Problem 15 Solve by the method of variation of parameters $\frac{d^2y}{dx^2} + 4y = \sec 2x$

Solution:

The A.E is $m^2 + 4 = 0$

$m = \pm 2i$

C.F = $c_1\cos 2x + c_2\sin 2x$

P.I = $Pf_1 + Qf_2$

$f_1 = \cos 2x$; $f_2 = \sin 2x$

$f_1' = -2\sin 2x$; $f_2' = 2\cos 2x$

$f_2' f_1 - f_1' f_2 = 2$

Now, $P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx$

$= -\int \frac{\sin 2x}{2}\sec 2x dx$

$= -\frac{1}{2}\int \tan 2x dx = \frac{1}{4}\log(\cos 2x)$

$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$

$= \frac{1}{2}\int \cos 2x \sec 2x dx = \frac{1}{2}x$

∴ $y = C.F + Pf_1 + Qf_2$

$= c_1\cos 2x + c_2\sin 2x + \frac{1}{4}\log(\cos 2x)\cos 2x + \frac{1}{2}x\sin 2x$.

UNIT II VECTOR CALCULUS

Part-A

Problem 1 Prove that $div(grad\,\phi) = \nabla^2\phi$

Solution:

$$div(grad\,\phi) = \nabla.\nabla\phi$$

$$= \nabla\left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

$$= \nabla^2\phi.$$

Problem 2 Find a, b, c, if $\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ is irrotational.

Solution:

$\vec{F}$ is irrotational if $\nabla\times\vec{F} = \vec{0}$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right] - \vec{j}\left[\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(bx-3y+2z) - \frac{\partial}{\partial y}(x+2y+az)\right]$$

$$= \vec{i}[c+1] + \vec{j}[a-4] + \vec{k}[b-2]$$

$$\because \nabla\times\vec{F} = 0 \Rightarrow 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{i}[c+1] + \vec{j}[a-4] + \vec{k}[b-2]$$

$$\therefore c+1 = 0,\ a-4 = 0,\ b-2 = 0$$

$$\Rightarrow c = -1,\ a = 4,\ b = 2.$$

Problem 3 If S is any closed surface enclosing a volume V and $\vec{r}$ is the position vector of a point, prove $\iint_S (\vec{r}.\hat{n})\,ds = 3V$

Solution:

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

By Gauss divergence theorem

$$\iint_S \overline{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \overline{F} dV \qquad \text{Here } \overline{F} = \nabla . \vec{r}$$

$$\iint_S \vec{r} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{r} dV$$

$$= \iiint_V \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \right) . \left(x\vec{i} + y\vec{j} + z\vec{k} \right) dV$$

$$= \iiint (1+1+1) dV$$

$$\iint_S \vec{r} \cdot \hat{n}\, ds = 3V.$$

Problem 4 If $\vec{r} = \vec{a}\cos nt + \vec{b}\sin nt$, where $\vec{a}, \vec{b}, n$ are constants show that $\vec{r} \times \frac{d\vec{r}}{dt} = n\left(\vec{a} \times \vec{b}\right)$

Solution:

Given $\vec{r} = \vec{a}\cos nt + \vec{b}\sin nt$

$$\frac{d\vec{r}}{dt} = -n\vec{a}\sin nt + n\vec{b}\cos nt$$

$$\vec{r} \times \frac{d\vec{r}}{dt} = \left(\vec{a}\cos nt + \vec{b}\sin nt\right) \times \left(-n\vec{a}\sin nt + n\vec{b}\cos nt\right)$$

$$= n\left(\vec{a} \times \vec{b}\right)\cos^2 nt - \left(\vec{b} \times \vec{a}\right)\sin^2 nt$$

$$= n\left(\vec{a} \times \vec{b}\right)\cos^2 nt + \left(\vec{a} \times \vec{b}\right)\sin^2 nt \qquad \left(\because \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}\right)$$

$$= n\left(\vec{a} \times \vec{b}\right)(1) = n\left(\vec{a} \times \vec{b}\right)$$

Problem 5 Prove that $div\left(curl \vec{A}\right) = 0$

Solution:

$$\nabla . (\nabla \times \mathrm{A}) = \nabla \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \nabla \cdot \left[\vec{i}\left(\frac{\partial \mathrm{A}_3}{\partial y} - \frac{\partial \mathrm{A}_2}{\partial z} \right) - \vec{j}\left(\frac{\partial \mathrm{A}_3}{\partial x} - \frac{\partial \mathrm{A}_1}{\partial z} \right) \right] + \vec{k}\left(\frac{\partial \mathrm{A}_2}{\partial x} - \frac{\partial \mathrm{A}_1}{\partial y} \right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \mathrm{A}_3}{\partial y} - \frac{\partial \mathrm{A}_2}{\partial z} \right) - \frac{\partial}{\partial y}\left(\frac{\partial \mathrm{A}_3}{\partial x} - \frac{\partial \mathrm{A}_1}{\partial z} \right) + \frac{\partial}{\partial z}\left(\frac{\partial \mathrm{A}_2}{\partial x} - \frac{\partial \mathrm{A}_1}{\partial y} \right)$$

$$=\left(\frac{\partial^2 A_3}{\partial x\partial y}-\frac{\partial^2 A_2}{\partial x\partial z}\right)+\left(\frac{\partial^2 A_1}{\partial y\partial z}-\frac{\partial^2 A_3}{\partial y\partial x}\right)+\left(\frac{\partial^2 A_2}{\partial z\partial x}-\frac{\partial^2 A_1}{\partial z\partial y}\right)$$

$$\therefore div\left(curl\vec{A}\right)=0$$

Problem 6 Find the unit normal to surface $xy^3z^2=4$ at $(-1,-1,2)$

Solution:

Let $\phi = xy^3z^2-4$

$$\nabla\phi = y^3z^2\vec{i}+3xy^2z^2\vec{j}+2xy^3z\vec{k}$$

$$\nabla\phi_{(-1,-1,2)}=(-1)^3(2)^2\vec{i}+3(-1)(-1)^2(2)^2\vec{j}+2(-1)(-1)^3(2)\vec{k}$$

$$=-4\vec{i}-12\vec{j}+4\vec{k}$$

Unit normal to the surface is $\hat{n}=\frac{\nabla\phi}{|\nabla\phi|}$

$$=\frac{-4\vec{i}-12\vec{j}+4\vec{k}}{\sqrt{16+144+16}}$$

$$=-\frac{-4\left(\vec{i}+3\vec{j}-\vec{k}\right)}{\sqrt{176}}$$

$$=\frac{-4\left(\vec{i}+3\vec{j}-\vec{k}\right)}{\sqrt{16\times 11}}=\frac{-\left(\vec{i}+3\vec{j}-\vec{k}\right)}{\sqrt{11}}.$$

Problem 7 Applying Green's theorem in plane show that area enclosed by a simple closed curve C is $\frac{1}{2}\int(xdy-ydx)$

Solution:

$$\int_C Pdx+Qdy=\int\int_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

$P=-y,\ Q=x$

$$\frac{\partial P}{\partial y}=-1,\ \frac{\partial Q}{\partial x}=1$$

$$\therefore\int(x\,dy-ydx)=\int\int_R(1+1)\,dxdy=2\int\int_R dx\,dy$$

= 2 Area enclosed by C

$\therefore$ Area enclosed by C $=\frac{1}{2}\int(xdy-ydx)$.

Problem 8 If $\vec{A}$ and $\vec{B}$ are irrotational show that $\vec{A}\times\vec{B}$ is solenoidal

Solution:

Given $\vec{A}$ is irrotational i.e., $\nabla\times\vec{A}=\vec{0}$

$\vec{B}$ is irrotational i.e., $\nabla \times \vec{B} = \vec{0}$

$$\nabla\cdot\left(\vec{A}\times\vec{B}\right) = \vec{B}.\left(\nabla\times\vec{A}\right) - \vec{A}.\left(\nabla\times\vec{B}\right)$$

$$= \vec{B}\cdot\vec{0} - \vec{A}\cdot\vec{0} = \vec{0}$$

$\therefore \vec{A}\times\vec{B}$ is solenoidal.

Problem 9 If $\vec{F} = grad\left(x^3+y^3+z^3-3xyz\right)$ find curl $\vec{F}$

Solution:

$$\vec{F} = \nabla\left(x^3+y^3+z^3-3xyz\right)$$

$$= \left(3x^2-3yz\right)\vec{i} + \left(3y^2-3xz\right)\vec{j} + \left(3z^2-3xy\right)\vec{k}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}\left(3z^2-3xy\right) - \frac{\partial}{\partial z}\left(3y^2-3xz\right)\right] - \vec{j}\left[\frac{\partial}{\partial x}\left(3z^2-3xy\right) - \frac{\partial}{\partial z}\left(3x^2-3yz\right)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}\left(3y^2-3xz\right) - \frac{\partial}{\partial y}\left(3x^2-3yz\right)\right]$$

$$= \vec{i}\left[-3x+3x\right] - \vec{j}\left[-3y+3y\right] + \vec{k}\left[-3z+3z\right]$$

$$= \vec{i}\,0 + \vec{j}\,0 + \vec{k}\,0 = 0.$$

Problem 10 If $\vec{F} = x^2\vec{i} + y^2\vec{j}$, evaluate $\int \vec{F}\cdot d\vec{r}$ along the straight line $y = x$ from $(0,0)$ to $(1,1)$.

Solution:

$$\vec{F}.d\vec{r} = \left(x^2\vec{i} + y^2\vec{j}\right)\cdot\left(dx\vec{i} + dy\right)$$

$$= x^2dx + y^2dy$$

Given $y = x$

$dy = dx$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_0^1\left(x^2dx + y^2dy\right)$$

$$= \int_0^1 x^2dx + x^2dx = 2\int_0^1 x^2dx = 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}$$

Problem 11 What is the unit normal to the surface $\phi(x,y,z) = C$ at the point (x,y,z)?

Solution:

$\hat{n} = \dfrac{\nabla \phi}{|\nabla \phi|}$.

Problem 12 State the condition for a vector $\vec{F}$ to be solenoidal

Solution:

$\nabla . \vec{F} = div\vec{F} = 0$

Problem 13 If $\vec{a}$ is a constant vector what is $\nabla \times \vec{a}$?

Solution:

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$$\nabla \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} = \vec{0}$$

Problem 14 Find grad ϕ at $(2,2,2)$ when $\phi = x^2 + y^2 + z^2 + 2$

Solution:

$grad\phi = \nabla\phi$

$$= \vec{i}\frac{\partial}{\partial x}\left(x^2 + y^2x^2 + 2\right) + \vec{j}\frac{\partial}{\partial y}\left(x^2 + y^2 + z^2 + 2\right) + \vec{k}\frac{\partial}{\partial z}\left(x^2 + y^2 + z^2 + 2\right)$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\nabla\phi_{(2,2,2)} = 4\vec{i} + 4\vec{j} + 4\vec{k}$$

Problem 15 State Gauss Divergence Theorem

Solution:

The surface integral of the normal component of a vector function F over a closed surface S enclosing volume V is equal to the volume integral of the divergence of $\vec{F}$ taken over V. i.e., $\int\int_S \vec{F}.\vec{n}ds = \int\int\int_V \nabla.\vec{F}dV$

Part –B

Problem 1 Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $(1,-2,-1)$ in the direction of the vector $2\vec{i} - \vec{j} - 2\vec{k}$.

Solution:

$\phi = x^2yz + 4xz^2$

$$\nabla\phi = \left(2xyz+4z^2\right)\vec{i}+x^2z\vec{j}+\left(x^2y+8xz\right)\vec{k}$$

$$\nabla\phi_{(1,-2,-1)} = \left[2(1)(-2)(-1)+4(-1)^2\right]\vec{i}+(1)^2(-1)\vec{j}+\left[(1)^2(-2)+8(1)(-1)\right]\vec{k}$$

$$=(4+4)\vec{i}-\vec{j}+(-2-8)\vec{k}$$

$$=8\vec{i}-\vec{j}-10\vec{k}$$

Directional derivative $\vec{a}$ is $=\dfrac{\nabla\phi.\vec{a}}{|\nabla\phi|}$

$$=\frac{\left(8\vec{i}-\vec{j}-10\vec{k}\right).\left(2\vec{i}-\vec{j}-2\vec{k}\right)}{\sqrt{4+1+4}}$$

$$=\frac{16+1+20}{3}=\frac{37}{3}.$$

Problem 2 Find the maximum directional derivative of $\phi = xyz^2$ at $(1,0,3)$.

Solution:

Given $\phi = xyz^2$

$$\nabla\phi = yz^2\vec{i}+xz^2\vec{j}+2xyz\vec{k}$$

$$\nabla\phi_{(1,0,3)} = 0(3)^2\vec{i}+(1)(3)^2\vec{j}+2(1)(0)(3)\vec{k}=9\vec{j}$$

Maximum directional directive of ϕ is $\nabla\phi = 9\vec{j}$

Magnitude of maximum directional directive is $|\nabla\phi| = \sqrt{9^2}=9.$

Problem 3 Find the angle between the surfaces $x^2+y^2+z^2=9$ and $x^2+y^2-z=3$ at the point $(2,-1,2)$.

Solution:

Let $\phi_1 = x^2+y^2+z^2-9$

$$\nabla\phi_1 = 2x\vec{i}+2y\vec{j}+2z\vec{k}$$

$$\nabla\phi_{1(2,-1,2)} = 2(2)\vec{i}+2(-1)\vec{j}+2(2)\vec{k}=4\vec{i}-2\vec{j}+4\vec{k}$$

$$\phi_2 = x^2+y^2-z-3$$

$$\nabla\phi_2 = 2x\vec{i}+2y\vec{j}-\vec{k}$$

$$\nabla\phi_{2(2,-1,2)} = 4\vec{i}-2\vec{j}-2\vec{k}$$

If θ is the angle between the surfaces then

$$\cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$=\frac{\left(4\vec{i}-2\vec{j}+4\vec{k}\right).\left(4\vec{i}-2\vec{j}-2\vec{k}\right)}{\sqrt{16+4+16}\sqrt{16+4+4}}$$

$$= \frac{16+4-8}{\sqrt{36}\sqrt{24}}$$

$$= \frac{12}{6\times 2\sqrt{6}} = \frac{1}{\sqrt{6}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right).$$

Problem 4 Find the work done, when a force $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$ moves a particle from the origin to the point $(1,1)$ along $y^2 = x$.

Solution:

Given $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\vec{F}.d\vec{r} = (x^2 - y^2 + x)dx - (2xy + y)dy$

Given $y^2 = x$

$2ydy = dx$

$\therefore \vec{F}.d\vec{r} = (x^2 - x + x)dx - (2y^3 + y)dy$

$= x^2 dx - (2y^3 + y)dy$

$$\int_C \vec{F}\vec{dr} = \int_0^1 x^2 dx - \int_0^1 (2y^3 + y)dy$$

$$= \left[\frac{x^3}{3}\right]_0^1 - \left[\frac{2y^4}{4} + \frac{y^2}{2}\right]_0^1$$

$$= \left(\frac{1}{3} - 0\right) - \left[\left(\frac{2}{4} + \frac{1}{2}\right) - (0+0)\right]$$

$$= \frac{1}{3} - \left[\frac{1}{2} + \frac{1}{2}\right]$$

$$= \frac{1}{3} - 1 = \frac{-2}{3}$$

$$\therefore \text{Work done} = \int_C \vec{F}.\vec{dr} = \frac{2}{3}$$

Problem 5 Prove that $\vec{F} = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$ is irrotational and find its scalar potential.

Solution:

Given $\vec{F} = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y^2\cos x+z^3 & 2y\sin x-4 & 3xz^2\end{vmatrix}$$

$$=\vec{i}[0-0]-\vec{j}\left[3z^2-3z^2\right]+\vec{k}[2y\cos x-2y\cos]$$

$$=0\vec{i}-0\vec{j}+0\vec{k}=0$$

$\nabla\times\vec{F}=0$

Hence $\vec{F}$ is irrotational

$\vec{F}=\nabla\phi$

$$\left(y^2\cos x+z^3\right)\vec{i}\left(2y\sin x-4\right)\vec{j}+3xz^2\vec{k}=\vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\frac{\partial\phi}{yz}$$

Equating the coefficient $\vec{i},\vec{j},\vec{k}$

$$\frac{\partial\phi}{\partial x}=y^2\cos x+z^3\Rightarrow\int\partial\phi=\int y^2\cos x+z^3dx$$

$$\phi_1=y^2\sin x+z^3x+C_1$$

$$\frac{\partial\phi}{\partial x}=2y\sin x-4\Rightarrow\int\partial\phi=\int\left(2y\sin x-4\right)dy$$

$$\phi_2=2\left(\sin x\right)\frac{y^2}{2}-4y+C_2$$

$$\frac{\partial\phi}{\partial x}=3xz^2\Rightarrow\int\partial\phi=\int 3xz^2dy$$

$$\phi_3=3x\frac{z^3}{3}+C_3$$

$$\therefore\phi=y^2\sin x+xz^3-4y+C$$

Problem 6 If $\vec{F}=3xy\vec{i}-y^2\vec{j}$ evaluate $\int\vec{F}.\vec{dr}$ when C is curve in the xy plane $y=2x^2$, from $(0,0)$ to $(1,2)$

Solution:

$\vec{F}=3xy\vec{i}-y^2\vec{j}$

$\vec{dr}=dx\vec{i}+dy\vec{j}+dz\vec{k}$

$\vec{F}.\vec{dr}=3xydx-y^2dy$

Given $y=2x^2$

$dy=4xdx$

$$\therefore\vec{F}.\vec{dr}=3x(2x^2)dx-\left(2x^2\right)^2 4x\,dx$$

$$=6x^3dx-4x^4(4x)dx$$

$$=6x^3dx-16x^5dx$$

$$\int_C \vec{F}d\vec{r} = \int_0^1 \left(6x^3 - 16x^5\right)dx$$

$$= \left[6\frac{x^4}{4} - \frac{16x^6}{6}\right]_0^1$$

$$= \frac{6}{4} - \frac{16}{6} = -\frac{-7}{6}.$$

Problem 7 Find $\int_C \vec{F}.d\vec{r}$ when $\vec{F} = \left(x^2 + y^2\right)\vec{i} - 2xy\vec{j}$ where the cure C is the rectangle in the xy plane bounded by $x = 0,\ x = a,\ y = b,\ y = 0$.

Solution:

Given $\vec{F}\left(x^2 + y^2\right)\vec{i} - 2xy\vec{j}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\vec{F}d\vec{r} = (x^2 + y^2)dx - 2xy\,dy$

C is the rectangle $OABC$ and C consists of four different paths.

OA (y = 0)

AB (x = a)

BC (y = b)

CO (x = 0)

$\therefore \int_C \vec{F}.d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$

Along

$OA,\quad y = 0,\quad dy = 0$

$AB,\quad x = a,\quad dx = 0$

$BC,\quad y = b,\quad dy = 0$

$CO,\quad x = 0,\quad dx = 0$

$$\therefore C\int_C \vec{F}.d\vec{r} = \int_{OA} x^2dx \int_{AB} -2aydy + \int_{BC}\left(x^2 + b^2\right)dx + \int_{CO} 0$$

$$= \int_0^a x^2dx - 2a\int_0^b ydy + \int_a^0 \left(x^2 + b^2\right)dx$$

$$= \left[\frac{x^3}{3}\right]_0^a - 2a\left[\frac{y^2}{2}\right]_o^b + \left[\frac{x^3}{3} + b^2x\right]_a^o$$

$$= \left(\frac{a^3}{3} - 0\right) - 2a\left(\frac{b^2}{2} - 0\right) + \left(\left(0 + 0\right) - \left(\frac{a^3}{3} + ab^2\right)\right) = -2ab^2.$$

Problem 8 If $\vec{F} = \left(4xy - 3x^2z^2\right)\vec{i} + 2x^2\vec{j} - 2x^3zk$ check whether the integral $\int_C \vec{F}.d\vec{r}$ is independent of the path C.

Solution:

Given

$\vec{F} = \left(4xy - 3x^2z^2\right)\vec{i} + 2x^2\vec{j} - 2x^3zk$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\int_C \vec{F}.d\vec{r} = \int_C \left(4xy - 3x^2z^2\right)dx + \int_C 2x^2dy - \int_C 2x^3zdz$

This integral is independent of path of integration if

$\vec{F} = \nabla\phi \Rightarrow \nabla\times\vec{F} = 0$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy - 3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix}$$

$$= \vec{i}(0,0) - j\left(-6x^2z + 6x^2z\right) + \vec{k}\left(4x - 4x\right)$$

$$= 0\vec{i} - 0\vec{i} - 0\vec{j} + 0\vec{k} = 0.$$

Hence the line integral is independent of path.

Problem 9 Verify Green's Theorem in a plane for $\int_C \left(x^2(1+y)dx + (y^3 + x^3)dy\right)$ where C is the square bounded $x = \pm a, y = \pm a$

Solution:

Let $P = x^2(1+y)$

$\frac{\partial P}{\partial y} = x^2$

$Q = y^3 + x^3$

$\frac{\partial Q}{\partial x} = 3x^2$

By green's theorem in a plane

$$\int_C \left(Pdx + Qdy\right) = \iint_C \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$

Now $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$

$$= \int_{-a}^{a}\int_{-a}^{a}\left(3x^2 - x^2\right)dx\,dy$$

$$= \int_{-a}^{a} dy \int_{-a}^{a} 2x^2dx$$

$$= (y)_{-a}^{a}\left(\frac{2x^3}{3}\right)_{-a}^{a}$$

$$= (a+a)\frac{2}{3}\left(a^3 + a^3\right)$$

$$=\frac{8a^4}{3}-(1)$$

Now $\int_C (Pdx+Qdy) = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$

Along $AB,\ y=-a,\ dy=0$

X varies from $-a$ to a

$$\int_{AB} (Pdx+Qdy) = \int_{-a}^{a} \left(x^2(1+y)dx + \left(x^3+y^3\right)dy\right)$$

$$= \int_{-a}^{a} x^2(1-a)dx + 0$$

$$=(1-a)\left[\frac{x^3}{3}\right]_{-a}^{a}$$

$$=\left(\frac{1-a}{3}\right)\left(a^3+a^3\right)=\frac{2a^3}{3}-\frac{2a^4}{3}$$

Along BC

$x=a,\ dx=0$

Y varies from $=-a$ to a

$$\int_{BC} (Pdx+Qdy) = \int_{-a}^{a} \left(x^2(1+y)dx + \left(x^3+y^3\right)dy\right)$$

$$= \int_{-a}^{a} (a^3+y^3)dy$$

$$=\left[a^3y+\frac{y^4}{4}\right]_{-a}^{a}$$

$$=\left(a^4+\frac{a^4}{4}\right)-\left(-a^4+\frac{a^4}{4}\right)=2a^4$$

Along CD

$y=a,\ dy=0$

X varies from a to $-a$

$$\int_{CD} (Pdx+Qdy) = \int_{a}^{-a} \left(x^2(1+y)dx + \left(x^3+y^3\right)dy\right)$$

$$= \int_{a}^{-a} x^2(1+a)dx$$

$$=(1+a)\left(\frac{x^3}{3}\right)_{a}^{-a} dx$$

$$=(1+a)\left[\frac{-a^3-a^3}{3}\right]$$

$$= -\frac{2a^3}{3} - \frac{2a^4}{3}$$

Along DA,

$x = -a,\ dx = 0$

Y Varies from a to $-a$

$$\int_{DA} (Pdx + Qdy) = \int_a^{-a} \left(x^2(1+y)dx + (x^3 + y^3)dy \right)$$

$$= \int_{+a}^{-a} \left(a^2(1+y)dx + (y^3 - a^3)dy \right)$$

$$= \left[\frac{y^4}{4} - a^3 y \right]_a^a$$

$$= \left(\frac{a^4}{4} + a^4 \right) - \left(\frac{a^4}{4} - a^4 \right) = 2a^4$$

$$\int_C (Pdx + Qdy) = \frac{2a^3}{3} - \frac{2a^4}{3} + 2a^4 - \frac{2a^3}{3} - \frac{2a^4}{3} + 2a^4$$

$$= 4a^4 - \frac{4}{3}a^4$$

$$= \frac{8a^4}{3} ......(2)$$

From (1) and (2)

$$\int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \frac{8a^4}{3}.$$

Hence Green's theorem verified.

Problem 10 Verify Green's theorem in a plane for $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the region defined by $x = y^2,\ y = x^2$.

Solution:

Green's theorem states that

$$\int_C udx + vdy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy$$

$$\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial p}{\partial y} \right) dxdy$$

Given $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$

$P = 3x^2 - 8y^2$

$$\frac{\partial P}{\partial y} = -16y$$

$$Q = 4y - 6xy$$

$$\frac{\partial Q}{\partial x} = -6y$$

Evaluation of $\int_C Pdx + Qdy$

(i) Along OA

$$y = x^2 \Rightarrow dy = 2xdx$$

$$\int_{OA} Pdx + Qdy = \int_{OA} \left(3x^2 - 8x^4\right)dx + \left(4x^2 - 6x^3\right)2xdx$$

$$= \int_0^1 \left(3x^2 - 8x^4 + 8x^3 - 12x^4\right)dx$$

$$= \int_0^1 \left(-20x^4 + 8x^3 + 3x^2\right)dx$$

$$= \left[-20\frac{x^5}{5} + 8\frac{x^4}{4} + \frac{3x^3}{3}\right]_0^1$$

$$= \frac{-20}{5} + \frac{8}{5} + \frac{3}{3}$$

$$= -4 + 2 + 1 = -1$$

Along AO

$$y^2 = x \Rightarrow 2ydy = dx$$

$$\int_{Ao} Pdx + Qdy = \int_{Ao} \left(3y^4 - 8y^2\right)2y\,dy + \left(4y - 6y^3\right)dy$$

$$= \int_{AO} (6y^5 - 16y^3 + 4y - 6y^3)dy$$

$$= \int_1^0 \left(6y^5 - 22y^3 + 4y\right)dy$$

$$= \left[6\frac{y^6}{6} - 22\frac{y^4}{4} + \frac{4y^2}{2}\right]_1^0$$

$$= \left[y^6 - \frac{11}{2}y^4 + 2y^2\right]_1^0 = \frac{5}{2}$$

$$\therefore \int_C Pdx + Qdy = \int_{OA} + \int_{AO} = -1 + \frac{5}{2} = \frac{3}{2} \rightarrow (1)$$

Evaluation of $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$

$$\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy = \iint_R \left(-6y + 16y\right)dxdy$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} 10y\,dx\,dy = \int_0^1 [10\,xy]_{x=y}^{x=\sqrt{y}}\,dy$$

$$= \int_0^1 10y\left(\sqrt{y} - y^2\right)dy$$

$$= 10\int_0^1 \left(y^{\frac{3}{2}} - y^3\right)dy$$

$$= 10\left[\frac{y^{\frac{5}{2}}}{\frac{5}{2}} - \frac{y^4}{4}\right]_0^1$$

$$= 10\left[\frac{2}{5} - \frac{1}{4}\right]$$

$$= 10\left[\frac{8-5}{20}\right]$$

$$= \frac{30}{20} = \frac{3}{2} \rightarrow (2)$$

For (1) and (2)

Hence Green's theorem is verified.

Problem 11 Verify Gauss divergence theorem for $\vec{F} = y\vec{i} + x\vec{j} + z^2\vec{k}$ over the cylindrical region bounded by $x^2 + y^2 = 9,\ Z = 0$ and $Z = 2$.

Solution:

Gauss divergence theorem is

$$\iint_S \vec{F}.\vec{n}\ \ ds = \iiint_V div\,\vec{F}\,dV$$

$$div\,\vec{F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z^2) = 2z$$

$$\iiint_V div\,\vec{F}\,dV = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^2 2z\,dzdy\,dx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 2\left(\frac{z^2}{2}\right)_0^2 dydx$$

$$= 4\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dydx$$

= 4 (Area of the circular region)

$$= 4\left(\pi\,(3)^2\right)$$

$$= 36\pi\ldots\ldots\ldots\ldots\ldots(1)$$

$$\iint_S \vec{F}.\vec{n}\ ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3}$$

S_1 is the bottom of the circular region, S_2 is the top of the circular region and S_3 is the cylindrical region

On $S_1, \vec{n} = -\vec{k}, \quad ds = dxdy, \quad z = 0$

$$\iint_{S_1} \vec{F}.\vec{n}\, ds = \iint -z^2 dxdy = 0$$

On S_2, $\vec{n} = \vec{k}$, $ds = dxdy$, $z = 2$

$$\iint_{s_2} \vec{F}.\vec{n}\ ds = \iint z^2 dx\, dy$$

$$= 4\iint dxdy$$

$$= 4\,(\text{Area of circular region})$$

$$= 4\left(\pi\,(3)^2\right) = 36\pi$$

On S_3, $\phi = x^2 + y^2 - 9$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i} + 2y\vec{j}}{\sqrt{4\left(x^2+y^2\right)}}$$

$$= \frac{x\vec{i} + y\vec{j}}{3}$$

$$\iint_{S_3} \vec{F}.\vec{n}\, ds = \iint \left(y\vec{i} + x\vec{j} + z^2\vec{k}\right)\left(\frac{x\vec{i} + y\vec{j}}{3}\right) ds$$

$$= \iint \frac{yx + yx}{3}\, ds = \frac{2}{3}\iint_S xy\, ds$$

Let $x = 3\cos\theta, \quad y = 3\sin\theta$

$ds = 3\ d\theta dy$

θ varies from 0 to 2π

z varies from 0 to 2π

$$= \frac{2}{3}\int_0^2\int_0^{2\pi} (9\sin\theta\cos\theta) 3\, d\theta\, dz$$

$$= \frac{18}{2}\int_0^2\int_0^{2\pi} \sin 2\theta\, d\theta\, dz$$

$$= 9\int_0^2 \left(-\frac{\cos 2\theta}{2}\right)_0^{2\pi} dz$$

$$= -\frac{9}{2}\int_0^2 [1-1]\ dz = 0$$

$$\therefore \iint_S \vec{F}.\vec{n}\, ds = 0 + 36\pi + 0 = 36\pi \ldots\ldots\ldots\ldots..(2)$$

from (1) and (2)

$$\int_C \vec{F}.\vec{n}\ ds = \iiint_V div\ \vec{F} dV$$

Problem 12 Verify Stoke's theorem for the vector field defined by $\vec{F} = (x^2 - y^2)\vec{i} - 2xy\vec{j}$ in the rectangular region in the xy plane bounded by the lines $x = 0,\ x = a,\ y = 0,\ y = b.$

Solution:

$$\vec{F} = (x^2 - y^2)\vec{i} - 2xy\vec{j}$$

By Stoke's theorem $\int_C \vec{F}.d\vec{r} = \iint_S curl\,\vec{F}.\vec{n}\,ds$

$$curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix}$$

$$= \vec{i}[0-0] - \vec{j}[0-0] + \vec{k}[-2y-2y] = -4y\vec{k}$$

As the region is in the xy plane we can take $\vec{n} = \vec{k}$ and $ds = dxdy$

$$\iint_S curl\,\vec{F}.\vec{n}\,ds = \iint -4y\vec{k}.\vec{k}\,dx\,dy$$

$$= -4\int_0^b\int_0^a y\ dx\,dy$$

$$= -4\left(\frac{y^2}{2}\right)_0^b (x)_0^a$$

$$= -2ab^2 \text{..............(1)}$$

$$\int_C \vec{F}.dr = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

<u>Along OA</u>

$y = 0 \Rightarrow dy = 0,$

x varies from 0 to a

$$\therefore \int_{OA} = \int_0^a (x^2 + y^2)dx - 2xy\ dy$$

$$= \int_0^a x^2 dx = \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3}$$

<u>Along AB</u>

$x = a \Rightarrow dx = o,\ \ y$ varies from 0 to b

$$\int_{AB} = \int_0^b \left(a^2 + y^2\right).0 - 2ay\ dy$$

$$= -2a\left(\frac{y^2}{2}\right)_0^b = -ab^2$$

Along BC

$y = b,\ dy = 0$

x varies from a to 0

$$\int_{BC} = \int_a^0 \left(x^2 + b^2\right)dx - 0$$

$$= \left(\frac{x^3}{3} + b^2 x\right)_a^0$$

$$= -\frac{a^3}{3} - ab^2$$

Along CO

$x = 0,\ dx = 0,$

y varies from b to 0

$$\int_{CO} = \int_b^0 \left(0 + y^2\right)0 + 0 = 0$$

$$\therefore \int_c \vec{F}.d\vec{r} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 + 0$$

$$= -2ab^2 ....(2)$$

For (1) and (2)

$$\int_C \vec{F}.d\vec{r} = \iint_S curl\vec{F}.\vec{n}ds$$

Here Stoke's theorem is verified.

Problem 13 Find $\iint_S \vec{F}.\vec{n}ds$ if $\vec{F} = \left(x + y^2\right)\vec{i} - 2x\vec{j} + 2yz\vec{k}$ where S is the surface in the plane $2x + y + 2z = 6$ in the first octant.

Solution:

Let $\phi = 2x + y + 2z - 6$ be the given surface

Then $\nabla\phi = 2\vec{i} + \vec{j} + 2\vec{k}$

$$|\nabla\phi| = \frac{2\vec{i} + \vec{j} + 2\vec{k}}{\sqrt{4+1+4}} = \frac{2\vec{i} + \vec{j} + 2\vec{k}}{3}$$

$\therefore$ The unit outward normal $\vec{n}$ to the surface S is $\hat{n} = \frac{1}{3}\left[2\vec{i} + \vec{j} + 2\vec{k}\right]$

Let R be the projection of S on the xy plane

$$\therefore \iint_S \vec{F}.\vec{n}\, ds = \iint_{R_1} \vec{F}.\vec{n} \frac{dx\, dy}{|\vec{n}.\vec{k}|}$$

$$\vec{n}.\vec{k} = \frac{1}{3}\left(2\vec{i} + \vec{j} + 2\vec{k}\right).\vec{k} = \frac{2}{3}$$

$$\vec{F}.\vec{n} = \left[\left(x + y^2\right)\vec{i} - 2x\vec{j} + 2yz\vec{k}\right].\frac{1}{3}\left(2\vec{i} + \vec{j} + 2\vec{k}\right)$$

$$= \frac{2}{3}\left(x + y^2\right) - \frac{2}{3}x + \frac{4}{3}yz$$

$$= \frac{2}{3}\left(y^2 + 2yz\right)$$

$$= \frac{2}{3}y\left(y + 2z\right)$$

$$= \frac{2}{3}y\left[y + 6 - y - 2x\right]$$

$$= \frac{2}{3}y\left[6 - 2x\right]$$

$$= \frac{4}{3}y\left(3 - x\right)$$

$$\therefore \iint_S \vec{F}.\vec{n}\, ds = \iint_S \vec{F}.\vec{n} \frac{dx\, dy}{|\vec{n}.\vec{k}|}$$

$$= \iint \frac{4}{3}y\left(3 - x\right)\frac{dxdy}{2/3}$$

$$= 2\iint_{R_1}\left(3 - x\right) dx\, dy$$

$$= 2\int_0^3 \int_0^{6-2-x} \left(3 - x\right)\, dx\, dy$$

$$= 2\int_0^3 \left(3 - x\right)\left(\frac{y^2}{2}\right)_0^{6-2-x} dx$$

$$= 4\int_0^3 \left(3 - x\right)^3 dx$$

$$= 4\left[\frac{(3 - x)^4}{-4}\right]_0^3$$

$$= 81 \text{ units.}$$

Problem 14 Evaluate $\int_C \left[\left(x + y\right)dx + \left(2x - 3xy\right)\right]$ where C is the boundary of the triangle with vertices $\left(2,0,0\right),\left(0,3,0\right) \& \left(0,0,6\right)$ using Stoke's theorem.

Solution:

Stoke's theorem is $\int_C \vec{F}.\vec{dr} = \iint_S curl\vec{F}.\hat{n}ds$ where S is the surface of the triangle and $\hat{n}$ is the unit vector normal to surface S.

Given $\vec{F}.\vec{dr} = (x+y)dx + (2x-z)dy + (y+z)dz$

$$\therefore \vec{F} = (x+y)\vec{i} + (2x-z)\vec{j} + (y+z)\vec{k}$$

$$\vec{dr} = \vec{i}\,dx + \vec{j}dy + \vec{k}\,dz$$

$$curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= \vec{i}(1+1) - \vec{j}(0-0) + \vec{k}(2-1)$$

$$curl\,\vec{F} = 2\vec{i} + \vec{k}$$

Equation of the plane ABC is

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$$

$$3x + 2y + z = 6$$

Let $\phi = 3x + 2y + z - 6$

$$\nabla\phi = 3\vec{i} + 2\vec{j} + \vec{k}$$

Unit normal vector to the surface ABC $(or\,\phi)$ is

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{3\vec{i} + 2\vec{j} + \vec{k}}{\sqrt{14}}$$

$$curl\,\vec{F}.\hat{n} = (2\vec{i} + \vec{k}).\left(\frac{3\vec{i} + 2\vec{j} + \vec{k}}{\sqrt{14}}\right) = \frac{6+1}{\sqrt{14}} = \frac{7}{\sqrt{14}}$$

Hence $\iint_S curl\vec{F}.\hat{n}ds = \iint_S \frac{7}{\sqrt{14}}ds$

$$= \frac{7}{\sqrt{14}} \iint_R \frac{dxdy}{|\hat{n}\vec{k}|}$$ where R is the projection of surface ABC on XOY plane

$$= \frac{7}{\sqrt{14}} \iint_R \frac{dxdy}{\frac{1}{\sqrt{14}}} \qquad \left(\because \vec{n}\,\vec{k} = \left(\frac{3i+2j+k}{\sqrt{14}}\right).k = \frac{1}{\sqrt{14}}\right)$$

$$= 7\iint_R dxdy$$

$$= 7 \times \left(Area\ of\ \Delta^{le}\ OAB\right)$$

$$= 7 \times 3 = 21.$$

Problem 15 Verify Stoke's theorem for $\vec{F}=(y-z)\vec{i}+yz\vec{j}-xz\vec{k}$ where S is the surface bounded by the planes $x=0,\ x=1,\ y=0,\ y=1,\ z=0$ and $z=1$ above the XOY plane.

Solution:

Stoke's theorem is $\int_C \vec{F}.d\vec{r}=\iint_S \nabla\times\vec{F}.\hat{n}ds$

$$\vec{F}=(y-z)\vec{i}+yz\vec{j}-xz\vec{k}$$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y-z & yz & -xz\end{vmatrix}$$

$$=-y\vec{i}+(z-1)\vec{j}-\vec{k}$$

$$\iint_S \nabla\times\vec{F}.\hat{n}\ ds=\iint_{S_1}+\iint_{S_2}+\iint_{S_3}+\iint_{S_4}+\iint_{S_5}$$

$\iint_{S_6}$ is not applicable, since the given condition is above the XOY plane.

$$\iint_{S_1}=\iint_{AEGD}\left[-y\vec{i}+(z-1)\vec{j}-\vec{k}\right].\vec{i}dydz$$

$$=\iint_{AEGD}-ydy\,dz$$

$$=\int_0^1\int_0^1 -y\,dy\,dz=\int_0^1\left[-\frac{y^2}{2}\right]_0^1 dz$$

$$=-\frac{1}{2}(z)_0^1=-\frac{1}{2}$$

$$\iint_{S_2}=\iint_{OBFC}\left[-y\vec{i}+(z-1)\vec{j}-\vec{k}\right].(-\vec{i})\,dydz$$

$$=\int_0^1\int_0^1 y\,dy\,dz=\int_0^1\left[\frac{y^2}{2}\right]_0^1 dz=\frac{1}{2}$$

$$\iint_{S_3}=\iint_{EBFG}\left[-y\vec{i}+(z-1)\vec{j}-\vec{k}\right]\vec{j}dxdz$$

$$=\int_0^1\int_0^1(z-1)\,dx\,dz=\int_0^1(xz-x)_0^1\,dz$$

$$=\left(\frac{z^2}{2}-z\right)_0^1=\frac{1}{2}-1=-\frac{1}{2}$$

$$\iint_{S_4}=\iint_{OADC}\left[-y\vec{i}+(z-1)\vec{j}-\vec{k}\right](-\vec{j}\ )dxdz$$

$$= \int_0^1 \int_0^1 (-z+1)\, dx\, dz$$

$$= \int_0^1 (-xz+x)_0^1 = \int_0^1 (-z+1)\, dz$$

$$= \left(\frac{-z^2}{2} + z \right)_0^1 = \frac{-1}{2} + 1 = \frac{1}{2}$$

$$\iint_{S_5} = \iint_{DGFC} \left(-y\vec{i} + (z-1)\vec{j} - \vec{k}\right).\vec{k}dxdy$$

$$= \int_0^1 \int_0^1 (-1)dxdy = \int_0^1 (-x)_0^1\, dy$$

$$= \int_0^1 (-1)\, dy = (-y)_0^1 = -1$$

$$\iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5}$$

$$= -\frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - 1 = -1$$

$$L.H.S = \int_C \vec{F}.\vec{dr} = \int_{OA} + \int_{AE} + \int_{EB} + \int_{BO}$$

$$\int_{OA} = \int_{OA} (y-z)\, dx + yzdy - xzdz$$

$$= \int_{OA} 0 = 0 \qquad [\because y=0,\ z=0,\ dy=0,\ dz=0]$$

$$\int_{AE} = \int_{AE} (y-z)\, dx + yzdy - xzdz$$

$$= \int_{AE} 0 = 0 \qquad [\because x=1,\ z=0,\ dx=0,\ dz=0]$$

$$\int_{EB} = \int_{EB} (y-z)\, dx + yzdy - xzdz$$

$$= \int_1^0 1\, dx \qquad (y=1,\ z=0,)$$

$$= [x]_1^0 = 0 - 1 = -1$$

$$\int_{BO} = \int_{BO} (y-z)\, dx + yzdy - xzdz$$

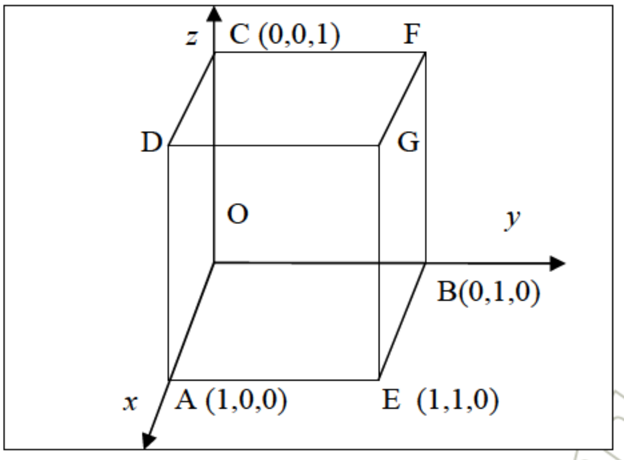
$$= \int_{BO} 0 = 0 \qquad [x=0,\ z=0]$$

$$= 0$$

$$\therefore \int_C = \int_{OA} + \int_{AE} + \int_{EB} + \int_{BO}$$

$$= 0 + 0 - 1 + 0 = -1$$

$\therefore$ L.HS = R.HS.
Hence Stoke's theorem is verified.



UNIT III ANALYTIC FUNCTIONS

Part-A

Problem 1 State Cauchy – Riemann equation in Cartesian and Polar coordinates.
Solution:
Cartesian form:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

Polar form:

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\ \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Problem 2 State the sufficient condition for the function f(z) to be analytic.
Solution:
The sufficient conditions for a function $f(z)=u+iv$ to be analytic at all the points in a region R are

(1) $u_x = v_y,\quad u_y = -v_x$

(2) u_x, u_y, v_x, v_y are continuous functions of x and y in region R.

Problem 3 Show that $f(z)=e^z$ is an analytic Function.
Solution:

$$f(z)=u+iv=e^z$$
$$=e^{x+iy}$$
$$=e^x e^{iy}$$
$$=e^x[\cos y+i\sin y]$$

$u=e^x\cos y,\ v=e^x\sin y$

$u_x=e^x cosy,\ v_x=e^x siny$

$u_y=-e^x siny,\ v_y=e^x cosy$

i.e., $u_x=v_y,\ u_y=-v_x$

Hence C-R equations are satisfied.

$\therefore f(z)=e^z$ is analytic.

Problem 4 Find whether $f(z)=\overline{z}$ is analytic or not.
Solution:
Given $f(z)=\overline{z}=x-iy$

i.e., $u = x,\ v = -y$

$\frac{\partial u}{\partial x}=1,\ \frac{\partial v}{\partial x}=0$

$\frac{\partial u}{\partial y}=0,\ \frac{\partial v}{\partial y}=-1$

$\therefore u_x \neq v_y$

C-R equations are not satisfied anywhere.

Hence $f(z)=\bar{z}$ is not analytic.

Problem 5 State any two properties of analytic functions

Solution:

(i) Both real and imaginary parts of any analytic function satisfy Laplace equation.

i.e., $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ or $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$.

(ii) If $w=u+iv$ is an analytic function, then the curves of the family $u(x,y)=c$, cut orthogonally the curves of the family $v(x,y)=c$.

Problem 6 Show that $f(z)=|z|^2$ is differentiable at $z=0$ but not analytic at $z=0$.

Solution:

$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=\lim_{z\to 0}\frac{|z|^2}{z}=\lim_{z\to 0}\frac{z\bar{z}}{z}=\lim_{z\to 0}\bar{z}=0$

$\therefore f(z)$ is differentiable at $z=0$.

Let $z=x+iy$

$\bar{z}=x-iy$

$|z|^2=z\bar{z}=(x+iy)(x-iy)=x^2+y^2$

$f(z)=x^2+y^2+i0$

$u=x^2+y^2,\ v=0$

$u_x=2x,\ v_x=0$

$u_y=2y,\ v_y=0$

The C-R equation $u_x=v_y$ and $u_y=-v_x$ are not satisfied at points other than z = 0. Therefore $f(z)$ is not analytic at points other than $z=0$. But a function can not be analytic at a single point only. Therefore $f(z)$ is not analytic at z = 0 also.

Problem 7 Determine whether the function $2xy+i(x^2-y^2)$ is analytic.

Solution:

Given $f(z)=2xy+i(x^2-y^2)$

i.e., $u=2xy$, $v=x^2-y^2$

$$\frac{\partial u}{\partial x} = 2y, \quad \frac{\partial v}{\partial x} = 2x$$

$$\frac{\partial u}{\partial y} = 2x, \frac{\partial v}{\partial y} = -2y$$

$\therefore u_x \neq v_y$ and $u_y \neq -v_x$

C-R equations are not satisfied.

Hence $f(z)$ is not analytic function.

Problem 8 Show that $v = sinhx\,cosy$ is harmonic

Solution:

$v = sinhxcosy$

$$\frac{\partial v}{\partial x} = coshxcosy, \frac{\partial v}{\partial y} = -sinhxsiny$$

$$\frac{\partial^2 v}{\partial x^2} = sinhxcosy, \frac{\partial^2 v}{\partial y^2} = -sinhycosy$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = sinhxcosy - sinhycosy = 0$$

Hence v is a harmonic function.

Problem 9 Construct the analytic function $f(z)$ for which the real part is $e^x cosy$.

Solution:

$u = e^x \cos y$

$$\frac{\partial u}{\partial x} = e^x \cos y$$

Assume $\frac{\partial u}{\partial x}(x, y) = \phi_1(z, 0)$

$\therefore \phi_1(z, 0) = e^z$

$$\frac{\partial u}{\partial y} = -e^x siny$$

Assume $\frac{\partial u(x, y)}{\partial y} = \phi_2(z, 0)$

$\therefore \phi_2(z, 0) = 0$

$$f(z) = \int f'(z) dz = \int \phi_1(z, 0) dz - i\int \phi_2(z, 0) dz$$

$$= \int e^z\, dz - i\int 0$$

$f(z) = e^z + C$.

Problem 10 Prove that an analytic function whose real part is constant must itself be a constant.

Solution:

Let $f(z)=u+iv$ be an analytic function

$\Rightarrow u_x=v_y,\ u_y=-v_x$............(1)

Given

$u=c$ (a constant)

$u_x=0,\ u_y=0$

$\Rightarrow v_y=0$ & $v_x=0$ by(1)

We know that $f(z)=u+iv$

$f'(z)=u_x+iv_x$

$f'(z)=0+i0$

$f'(z)=0$

Integrating with respect to z, $f(z)=C$

Hence an analytic function with constant real part is constant.

Problem 11 Define conformal mapping

Solution:

A transformation that preserves angle between every pair of curves through a point both in magnitude and sense is said to be conformal at that point.

Problem 12 If $w=f(z)$ is analytic prove that $\frac{dw}{dz}=\frac{\partial w}{\partial x}=-i\frac{\partial w}{\partial y}$ where $w=u+iv$ and prove that $\frac{\partial^2 w}{\partial z\partial \bar{z}}=0$

Solution:

$w=u(x,y)+iv(x,y)$ is an analytic function of z.

As $f(z)$ is analytic we have $u_x=v_y$, $u_y=-v_x$

Now $\frac{dw}{dz}=f'(z)=u_x+iv_x=v_y-iu_y=-i(u_y+iv_y)$

$$=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=-i\left[\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right]$$

$$=\frac{\partial}{\partial x}(u+iv)=-i\frac{\partial}{\partial y}(u+iv)$$

$$=\frac{\partial w}{\partial x}=-i\frac{\partial w}{\partial y}$$

W.K.T. $\frac{\partial w}{\partial \bar{z}} = 0$

$\therefore \frac{\partial^2 w}{\partial z \, \partial \bar{z}} = 0$

Problem 13

Define bilinear transformation. What is the condition for this to be conformal?

Solution:

The transformation $w = \frac{az+b}{cz+d}$, $ad - bc \neq 0$ where a, b, c, d are complex numbers is called a bilinear transformation.

The condition for the function to be conformal is $\frac{dw}{dz} \neq 0$.

Problem 14

Find the invariant points or fixed points of the transformation $w = 2 - \frac{2}{z}$.

Solution:

The invariant points are given by $z = 2 - \frac{2}{z}$

i.e., $z = 2 - \frac{2}{z}$

$z^2 = 2z - 2$

$z^2 - 2z + 2 = 0$

$$z = \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2}$$

$$= 1 \pm i$$

The invariant points are $z = 1+i,\ 1-i$

Problem 15

Find the critical points of (i) $w = z + \frac{1}{z}$ (ii) $w = z^3$.

Solution:

(i). Given $w = z + \frac{1}{z}$

For critical point $\frac{dw}{dz} = 0$

$\frac{dw}{dz} = 1 - \frac{1}{z^2} = 0$

$z = \pm i$ are the critical points

(ii). Given $w = z^3$

$$\frac{dw}{dz} = 3z^2 = 0$$

$z = 0$

$\therefore z = 0$ is the critical point.

Part-B

Problem 1 Determine the analytic function whose real part is $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

Solution:

Given $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$$

$$\phi_1(z,0) = 3z^2 + 6z$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = 6xy - 6y$$

$$\phi_2(z,0) = 0$$

By Milne Thomason method

$$f(z) = \int \varphi_1(z,0)\,dz - i\int \varphi_2(z,0)\,dz$$

$$= \int (3z^2 + 6z)\,dz - 0$$

$$= 3\frac{z^3}{3} + 6\frac{z^2}{2} + C = z^3 + 3z^2 + C$$

Problem 2 Find the regular function $f(z)$ whose imaginary part is $v = e^{-x}[x\cos y + y\sin y]$

Solution:

$$v = e^{-x}(x\cos y + y\sin y)$$

$$\phi_2(x,y) = \frac{\partial v}{\partial x} = e^{-x}[\cos y] + (x\cos y + y\sin y)(-e^{-x})$$

$$\phi_2(z,0) = e^{-z} + (z)(-e^{-z}) = e^{-z} - ze^{-z} = e^{-z}(1-z)$$

$$\phi_1(x,y) = \frac{\partial u}{\partial y} = e^{-x}[-x\sin y + y\cos y + \sin y(1)]$$

$$\phi_1(z,0) = e^{-z}[0+0+0] = 0$$

By Milne's Thomson Method

$$f(z) = \int \phi_1(z,0)\,dz + i\int \phi_2(z,0)\,dz$$

$$= \int 0\,dz + i\int (1-z)e^{-z}dz$$

$$= i\left[(1-z)\left[\frac{e^{-z}}{-1}\right] - (-1)\left[\frac{e^{-z}}{(-1)^2}\right]\right] + C$$

$$= i\left[-(1-z)e^{-z} + e^{-z}\right] + C$$

$$= i\left[-e^{-z} + ze^{-z} + e^{-z}\right] + C = i\left[ze^{-z}\right] + C$$

Problem 3 Determine the analytic function whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$.

Solution:

Given $u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_1(z,0) = \frac{(1-\cos 2z)(2\cos 2z) - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{(1-\cos 2z)(2\cos 2z) - 2(1-\cos^2 2z)}{(1-\cos 2z)^2}$$

$$= \frac{(1-\cos 2z)(2\cos 2z) - 2(1-\cos 2z)(1+\cos 2z)}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2(1+\cos 2z)}{1-\cos 2z} = \frac{2\cos 2z - 2 - 2\cos 2z}{1-\cos 2z}$$

$$= \frac{-2}{1-\cos 2z} = -\frac{1}{\left(\frac{1-\cos 2z}{2}\right)}$$

$$= -\frac{1}{\sin^2 z} = -\cos ec^2 z$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x[2\sinh 2y]}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$$\varphi_2(z,0) = 0$$

By Milne's Thomson method

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$$

$$= \int -cosec^2 z\,dz - 0 = cot\,z + C$$

Problem 4 Prove that the real and imaginary parts of an analytic function $w = u + iv$ satisfy Laplace equation in two dimensions viz $\nabla^2 u = 0$ and $\nabla^2 v = 0$.

Solution:

Let $f(z) = w = u + iv$ be analytic

To Prove: u and v satisfy the Laplace equation.

i.e., To prove: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Given: $f(z)$ is analytic

$\therefore$ u and v satisfy C-R equations

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ….. (1)

and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ….. (2)

Diff. (1) p.w.r to x we get $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$ ….. (3)

Diff. (2) p.w.r. to y we get $\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$ ….. (4)

The second order mixed partial derivatives are equal

i.e., $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$

(3) + (4) $\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$

$\therefore$ u satisfies Laplace equation

Diff. (1) p.w.r to y we get $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2}$ … (5)

Diff. (2) p.w.r. to x we get $\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$ … (6)

(5) + (6) $\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 v}{\partial x \partial y} = 0$

i.e., $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$\therefore$ v Satisfies Laplace equation

Problem 5 If f(z) is analytic, prove that $\left(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}\right)|f(z)|^2 = 4.|f'(z)|^2$

Solution:

Let f(z) = u + iv be analytic.

Then $u_x = v_y$ and $u_y = -v_x$ (1)
Also $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$ (2)
Now $|f(z)|^2 = u^2 + v^2$ and $f'(z) = u_x + iv_x$

$$\therefore \frac{\partial}{\partial x}|f(z)|^2 = 2u.u_x + 2v.v_x$$

and $\frac{\partial^2}{\partial x^2}|f(z)|^2 = 2\left[u_x^2 + u.u_{xx} + v_x^2 + v.v_{xx}\right]$ (3)

Similarly $\frac{\partial^2}{\partial y^2}|f(z)|^2 = 2\left[u_y^2 + u.u_{yy} + v_y^2 + v.v_{yy}\right]$ (4)

Adding (3) and (4)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 2\left[u_x^2 + u_y^2 + u(u_{xx} + u_{yy}) + v_x^2 + v_y^2 + v(v_{xx} + v_{yy})\right]$$

$$= 2\left[u_x^2 + v_x^2 + u(0) + v_x^2 + u_x^2 + v(0)\right]$$

$$= 4\left[u_x^2 + v_x^2\right]$$

$$= 4.|f'(z)|^2$$

Problem 6 Prove that $\nabla^2 |\text{Re} f(z)|^2 = 2|f'(z)|^2$

Solution:

Let $f(z) = u + iv$

$$|\text{Re} f(z)|^2 = u^2$$

$$\frac{\partial}{\partial x}(u^2) = 2uu_x$$

$$\frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial}{\partial x}(2uu_x)$$

$$= 2[uu_{xx} + u_x u_x]$$

$$= 2\left[uu_{xx} + u_x^{\ 2}\right]$$

$$\frac{\partial^2}{\partial y^2}(u^2) = 2\left[uu_{yy} + u_y^2\right]$$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2) = 2\left[u(u_{xx} + u_{yy}) + u_x^2 + u_y^2\right]$$

$$= 2\left[u(0) + u_x^2 + u_y^2\right]$$

$$= 2|f'(z)|^2$$

Problem 7 Find the analytic function $f(z)=u+iv$ given that $2u+v=e^x[\cos y-\sin y]$

Solution:

Given $2u+v=e^x[\cos y-\sin y]$

$f(z)=u+iv$..............(1)

$if(z)=iu-v$..............(2)

$(1)\times 2\Rightarrow 2f(z)=2u+i2v$...........(3)

$(3)-(2)\Rightarrow (2-i)f(z)=(2u+v)+i(2v-u)$............(4)

$F(z)=U+iV$

$\therefore 2u+v=U=e^x[\cos y-\sin y]$

$\phi_1(x,y)=\dfrac{\partial U}{\partial x}=e^x\cos y-e^x\sin y$

$\phi_1(z,o)=e^z$

$\phi_2(x,y)=\dfrac{\partial u}{\partial x}=-e^x\sin y-e^x\cos y$

$\phi_2(z,o)=-e^z$

By Milne Thomson method

$F'(z)=\phi_1(z,o)-i\phi_2(z,o)$

$\int F'(z)\,dz=\int e^z dz-i\int -e^z dz$

$F(z)=(1+i)e^z+C$------(5)

From (4) & (5)

$(1+i)e^z+C=(2-i)f(z)$

$f(z)=\dfrac{1+i}{2-i}e^z+\dfrac{C}{2-i}$

$f(z)=\dfrac{1+3i}{5}e^z+\dfrac{C}{2-i}$

Problem 8 Find the Bilinear transformation that maps the points $1+i,\ -i,\ 2-i$ of the z-plane into the points $0,\ 1,\ i$ of the w-plane.

Solution:

Given $z_1=1+i,\ w_1=0$

$z_2=-i,\ w_2=1$

$z_3=2-i,\ w_3=i$

Cross-ratio

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-0)(1-i)}{(0-1)(i-w)}=\frac{[z-(1+i)][-i-(2-i)]}{[(1+i)-(-i)][(2-i)-z]}$$

$$\frac{w(1-i)}{(-1)(i-w)}=\frac{(z-1-i)(-i-2+i)}{(1+i+i)(2-i-z)}$$

$$\frac{w(1-i)}{(w-i)}=\frac{(z-1-i)(-2)}{(1+2i)(2-i-z)}$$

$$\frac{w(1-i)}{(w-i)}=\frac{(-2)(z-1-i)}{(1+2i)(2-i-z)}$$

$$\frac{w}{w-i}=\frac{(-2)}{(1+2i)(1-i)}\frac{(z-1-i)}{(2-i-z)}$$

$$\frac{w}{w-i}=\frac{(-2)}{(1-i+2i+2)}\frac{(z-1-i)}{(2-i-z)}$$

$$\frac{w}{w-i}=\frac{(-2)}{(3+i)}\frac{(z-1-i)}{(2-i-z)}$$

$$\frac{w-i}{w}=\frac{(3+i)(2-i-z)}{(-2)(z-1-i)}$$

$$1-\frac{i}{w}=\frac{(3+i)(2-i-z)}{(-2)(z-1-i)}$$

$$\frac{i}{w}=1-\frac{3+i}{(-2)}\frac{(2-i-z)}{(z-1-i)}$$

$$\frac{i}{w}=1+\frac{3+i}{2}\frac{(2-i-z)}{(z-1-i)}$$

$$\frac{i}{w}=\frac{2(z-1-i)+(3+i)(2-i-z)}{2(z-1-i)}$$

$$\frac{w}{i}=\frac{2(z-1-i)}{2(z-1-i)+(3+i)(2-i-z)}$$

$$w=\frac{2i(z-1-i)}{2(z-1-i)+(3+i)(2-i-z)}$$

$$w=\frac{2i(z-1-i)}{2z-2-2i+6-3i-3z+2i+1-zi}$$

$$w=\frac{2i(z-1-i)}{-z+5-3i-zi}.$$

Problem 9 Prove that an analytic function with constant modulus is constant.

Solution:

Let $f(z)=u+iv$ be analytic

By C.R equations satisfied
i.e., $u_x = v_y,\ u_y = -v_x$

$\therefore f(z) = u + iv$

$\Rightarrow |f(z)| = \sqrt{u^2 + v^2} = C$

$\Rightarrow |f(z)|^2 = u^2 + v^2 = C^2$

$u^2 + v^2 = C^2$...............(1)

Diff (1) with respect to x

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

$uu_x + vv_x = 0$............(2)

Diff (1) with respect to y

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

$-uv_x + vu_x = 0$............(3)

$(2)\times u + (3)\times v \Rightarrow (u^2 + v^2)u_x = 0$

$\Rightarrow u_x = 0$

$(2)\times v - (3)\times u \Rightarrow (u^2 + v^2)v_x = 0$

$\Rightarrow v_x = 0$

W.K.T $f'(z) = u_x + iv_x = 0$

$f'(z) = 0$

Integrate w.r.to z

$f(z) = C$

Problem 10 When the function $f(z) = u + iv$ is analytic show that $u(x, y) = C_1$ and $v(x, y) = C_2$ are Orthogonal.

Solution:

If $f(z) = u + iv$ is an analytic function of z, then it satisfies C-R equations

$u_x = v_y,\ u_y = -v_x$

Given $u(x, y) = C_1$.............(1)

$v(x, y) = C_2$.............(2)

By total differentiation

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

Differentiate equation (1) & (2) we get $du = 0$, $dv = 0$

$$\therefore \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

$$\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\partial u/\partial x}{\partial u/\partial y} = m_1(say)$$

$$\frac{dy}{dx} = \frac{-\partial v/\partial x}{\partial v/\partial y} = m_2(say)$$

$$\therefore m_1\, m_2 = -\frac{-\partial u/\partial x}{\partial u/\partial y} \times \frac{-\partial v/\partial x}{\partial v/\partial y} \quad (\therefore u_x = v_y\, u_y = -v_x)$$

$$\therefore m_1\, m_2 = -1$$

The curves $u(x,y) = C_1$ and $v(x,y) = C_2$ cut orthogonally.

Problem 11 Show that the function $u = \frac{1}{2}\log\left(x^2+y^2\right)$ is harmonic and determine its conjugate.

Solution:

Given $u = \frac{1}{2}\log\left(x^2+y^2\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{2}.\frac{1}{x^2+y^2}(2x) = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}.\frac{1}{x^2+y^2}(2y) = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\left(x^2+y^2\right) - x(2x)}{\left(x^2+y^2\right)^2} = \frac{y^2-x^2}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\left(x^2+y^2\right)(1) - 2y^2}{\left(x^2+y^2\right)^2} = \frac{x^2-y^2}{\left(x^2+y^2\right)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2+x^2-y^2}{\left(x^2+y^2\right)^2} = 0$$

Hence u is harmonic function

To find conjugate of u

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}$$

$$\phi_1(z,o) = \frac{1}{z}$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

$$\phi_2(z,o) = 0$$

By Milne Thomson Methods

$$f'(z) = \phi_1(z,o) - i\phi_2(z,o)$$

$$\int f'(z)\,dz = \int \frac{1}{z}dz + 0$$

$$= \log z + c$$

$$f(z) = \log re^{i\theta}$$

$$f(z) = u + iv = \log r + i\theta$$

$u = \log r$, $v = \theta$

$$u = \log\sqrt{x^2+y^2} \qquad \left[\therefore r^2 = x^2+y^2,\ \ \theta = \tan^{-1}\left(\frac{y}{x}\right)\right]$$

$v = \tan^{-1}\left(\frac{y}{x}\right)$ $\therefore$ Conjugate of u is $\tan^{-1}\left(\frac{y}{x}\right)$.

Problem 12 Find the image of the infinite strips $\frac{1}{4} < y < \frac{1}{2}$ under the transformation $w = \frac{1}{z}$.

Solution: $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

$$z = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$

$$\Rightarrow x = \frac{u}{u^2+v^2} \text{........(1)}$$

$$y = -\frac{v}{u^2+v^2} \text{........(2)}$$

Given strip is $\frac{1}{4} < y < \frac{1}{2}$ when $y = \frac{1}{4}$

$$\frac{1}{4} = -\frac{v}{u^2+v^2} \quad (by\ 2)$$

$$u^2 + (v+2)^2 = 4 \text{..........(3)}$$

which is a circle whose centre is at $(0,-2)$ in the w-plane and radius 2.

When $y = \frac{1}{2}$

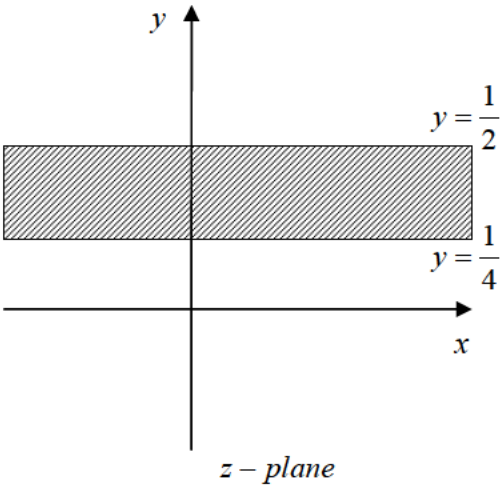
$$\frac{1}{2} = \frac{-v}{u^2+v^2} \quad (by\ 2)$$

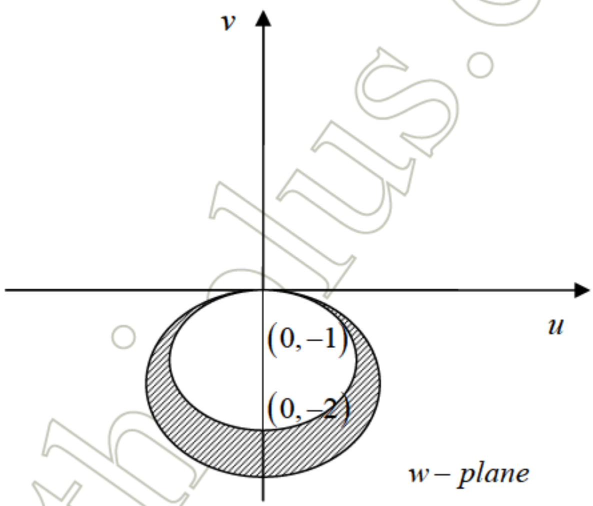
$$u^2 + v^2 + 2v = 0$$

$$u^2 + (v+1)^2 = 1 \text{..........(4)}$$

which is a circle whose centre is at $(0,-1)$ and radius is 1 in the w-plane.

Hence the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ is transformed into the region between circles $u^2 + (v+1)^2 = 1$ and $u^2 + (v+2)^2 = 4$ in the w-plane.





Problem 13 Obtain the bilinear transformation which maps the points $z = 1, i, -1$ into the points $w = 0, 1, \infty$.

Solution: We know that

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-0)(1-\infty)}{(0-1)(\infty-w)} = \frac{(z-1)(i+1)}{(1-i)(-1-z)}$$

$$\frac{w}{-1}(-1) = \frac{z-1}{1-i} \cdot \frac{i+1}{-(1+z)}$$

$$w = -\frac{z-1}{z+1} \cdot \frac{1+i}{1-i}$$

$$w = (-i)\frac{z-1}{z+1}$$

Problem 14 Find the image of $|z - 2i| = 2$ under the transform $w = \frac{1}{z}$

Solution:

Given $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

Now $w = u + iv$

$$z = \frac{1}{w} = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)(u-iv)} = \frac{u-iv}{u^2+v^2}$$

i.e., $x+iy = \dfrac{u-iv}{u^2+v^2}$

$\therefore\ x = \dfrac{u}{u^2+v^2}$..........(1)

$y = \dfrac{-v}{u^2+v^2}$..........(2)

Given $|z-2i| = 2$

$|x+iy-2i| = 2$

$|x+i(y-2)| = 2$

$x^2+(y-2)^2 = 4$

$x^2+y^2-4y = 0$..................(3)

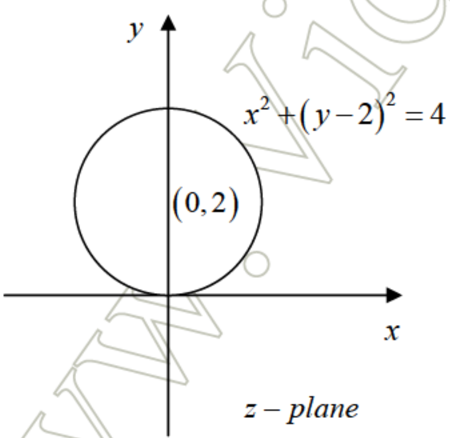
Sub (1) and (2) in (3)

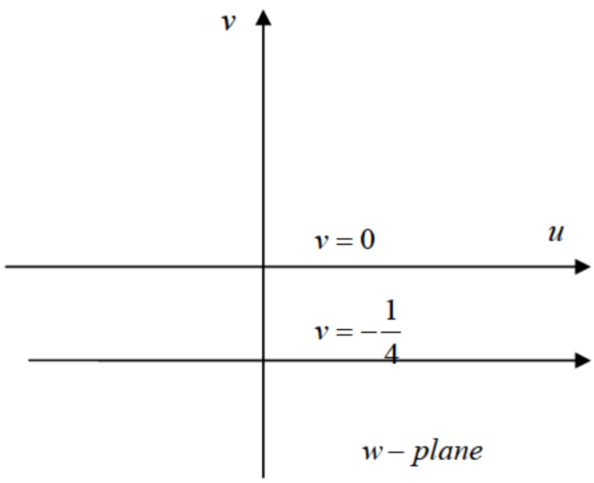
$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 - 4\left[\frac{-v}{u^2+v^2}\right] = 0$$

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + 4\left[\frac{4v}{u^2+v^2}\right] = 0$$

$$\frac{(u^2+v^2)+4v(u^2+v^2)}{(u^2+v^2)^2} = 0 \qquad \frac{(1+4v)(u^2+v^2)}{(u^2+v^2)^2} = 0$$

$1+4v = 0 \Rightarrow v = -\dfrac{1}{4}$ $\quad (\therefore u^2+v^2 \neq 0)$ This is straight line in w-plane.





Problem 15 Prove that $w = \dfrac{z}{1-z}$ maps the upper half of the z-plane onto the upper half of the w-plane.

Solution:

$$w = \frac{z}{1-z} \Rightarrow w(1-z) = z$$

$$w - wz = z$$

$$w = (w+1)z$$

$$w = (w+1)z$$

$$z = \frac{w}{w+1}$$

Put $z = x+iy,\quad w = u+iv$

$$x+iy = \frac{u+iv}{u+iv+1}$$

$$= \frac{(u+iv)(u+1)-iv}{(u+iv+1)(u+1)-iv}$$

$$= \frac{u(u+1)-iuv+iv(u+1)+v^2}{(u+1)^2+v^2}$$

$$= \frac{(u^2+v^2+u)+iv}{(u+1)^2+v^2}$$

Equating real and imaginary parts

$$x = \frac{u^2+v^2+u}{(u+1)^2+v^2},\quad y = \frac{v}{(u+1)^2+v^2}$$

$$y = 0 \Rightarrow \frac{v}{(u+1)^2+v^2} = 0$$

$$y > 0 \Rightarrow \frac{v}{(u+1)^2+v^2} > 0 \Rightarrow v > 0$$

Thus the upper half of the z plane is mapped onto the upper half of the w plane.

UNIT IV COMPLEX INTEGRATION

Part-A

Problem 1 Evaluate $\int_C \frac{z}{(z-1)^3}\,dz$ where C is $|z|=2$ using Cauchy's integral formula

Solution:

Given $\int_C \frac{z}{(z-1)^3}\,dz$

Here $f(z)=z$, $a=1$ lies inside $|z|=2$

$$\therefore \int_C \frac{zdz}{(z-1)^3}=\frac{2\pi i}{2!}f''(1)$$

$$=\pi i[0] \because f''(1)=0$$

$$\therefore \int_C \frac{zdz}{(z-1)^3}=0.$$

Problem 2 State Cauchy's Integral formula

Solution:

If $f(z)$ is analytic inside and on a closed curve C that encloses a simply connected region R and if $'a'$ is any point in R, then $f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$.

Problem 3 Evaluate $\int_C e^{\frac{1}{z}}dz$ where C is $|z-2|=1$.

Solution:

$e^{\frac{1}{z}}$ is analytic inside and on C.

Hence by Cauchy's integral theorem $\int_C e^{\frac{1}{z}}dz=0$

Problem 4 Classify the singularities of $f(z)=\frac{e^{\frac{1}{z}}}{(z-a)^2}$.

Solution:

Poles of $f(z)$ are obtained by equating the denominator to zero.

i.e., $(z-a)^2=0$, $z=a$ is a pole of order 2

The principal part of the Laurent's expansion of $e^{1/z}$ about z = 0 contains infinite number terms. Therefore there is an essential singularity at z = 0.

Problem 5 Calculate the residue of $f(z)=\dfrac{1-e^{2z}}{z^3}$ at the poles.

Solution:

Given $f(z)=\dfrac{1-e^{2z}}{z^3}$

Here $z=0$ is a pole of order 3

$$\therefore\left[\operatorname{Res} f(z)\right]_{z=0}=\underset{z\to 0}{Lt}\frac{1}{2!}\frac{d^2}{dz^2}\left[(z-0)^3\frac{1-e^{2z}}{z^3}\right]$$

$$=\frac{1}{2!}\underset{z\to 0}{Lt}\frac{d^2}{dz^2}\left[1-e^{2z}\right]$$

$$=\frac{1}{2!}\underset{z\to 0}{Lt}\frac{d}{dz}\left[-2e^{2z}\right]$$

$$=\frac{1}{2!}\underset{z\to 0}{Lt}-4e^{2z}$$

$$=\frac{1}{2}(-4)=-2.$$

Problem 6 Evaluate $\int_C\dfrac{\cos\pi z}{z-1}dz$ if C is $|z|=2$.

Solution:

We know that, Cauchy Integral formula is $\int_C\dfrac{f(z)}{z-a}dz=2\pi i f(a)$ if 'a' lies inside C

$\int_C\dfrac{\cos\pi z}{z-1}dz$, Here $f(z)=\cos\pi z$

$\therefore z=1$ lies inside C

$\therefore f(1)=\cos\pi(1)=-1.$

$$\therefore\int_C\frac{\cos\pi z}{z-1}dz=2\pi i(-1)=-2\pi i.$$

Problem 7 Define Removable singularity

Solution:

A singular point $z=z_0$ is called a removable singularity of $f(z)$ is $\underset{z\to z_0}{Lt}f(z)$ exists finitely

Example: For $f(z)=\frac{\sin z}{z}$, z = 0 is a removable singularity since $\underset{z\to 0}{Lt} f(z)=1$

Problem 8 Test for singularity of $\frac{1}{z^2+1}$ and hence find corresponding residues.

Solution:

Let $f(z)=\frac{1}{z^2+1}=\frac{1}{(z+i)(z-i)}$

Here $z=-i$ is a simple pole
$z=i$ is a simple pole

$$Res(z=i)=\underset{z\to i}{Lt}(z-i)\frac{1}{(z+i)(z-i)}$$

$$=\underset{z\to i}{Lt}\frac{1}{(z+i)}=\frac{1}{2i}$$

$$Res(z=-i)=\underset{z\to -i}{Lt}(z+i)\frac{1}{(z+i)(z-i)}=\frac{1}{-2i}.$$

Problem 9 What is the value of $\int_C e^z dz$ where C is $|z|=1$.

Solution:

Put $z=e^{i\theta}$

$dz=ie^{i\theta}d\theta$

$$\int_C e^z dz=\int_0^{2\pi} e^{e^{i\theta}} ie^{i\theta}d\theta..........(1)$$

Put $t=e^{i\theta} \Rightarrow dt=e^{i\theta}d\theta$

When $\theta=0,\ t=1,\ \theta=2\pi,\ sssst=1$

$$\therefore (1)\Rightarrow \int_C e^z dz=\int_1^1 e^t dt=\left[e^t\right]_1^1=0$$

Problem 10 Evaluate $\int_C \frac{3z^2+7z+1}{z+1}dz$, where $|z|=\frac{1}{2}$.

Solution:

Given $\int_C \frac{3z^2+7z+1}{z+1}dz$

Here $f(z)=3z^2+7z+1$

$z=-1$ lies outside $|z|=\frac{1}{2}$

Here $\int_C \frac{3z^2+72+1}{z+1}dz = 0$.(By Cauchy Theorem)

Problem 11 State Cauchy's residue theorem

Solution:

If $f(z)$ be analytic at all points inside and on a simple closed curve C, except for a finite number of isolated singularities $z_1, z_2, ..., z_n$ inside C then $\int_C f(z)dz = 2\pi i \times [\textit{sum of the residue of } f(z) \textit{ at } z_1, z_2, ..., z_n]$.

Problem 12 Calculate the residue of $f(z) = \frac{e^{2z}}{(z+1)^2}$ at its pole.

Solution:

Given $f(z) = \frac{e^{2z}}{(z+1)^2}$

Here $z = -1$ is a pole of order 2

$$[Res f(z)]_{z=-1} = \underset{z \to -1}{Lt} \frac{1}{1!}\frac{d}{dz}(z+1)^2 \frac{e^{2z}}{(z+1)^2}$$

$$= \underset{z \to -1}{Lt} 2e^{2z} = 2e^{-2}.$$

Problem 13 Using Cauchy integral formula evaluate $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}dz$, where $|z| = \frac{3}{2}$

Solution:

$$\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}dz = \int_C \frac{-\cos \pi z^2}{z-1}dz + \int_C \frac{\cos \pi z^2}{(z-2)}dz$$

$$\left[\therefore \frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{z-2}, \quad A = -1 \quad B = 1\right]$$

Here $f(z) = \cos \pi z^2$

$z = 1$ lies inside $|z| = \frac{3}{2}$

$z = 2$ lies outside $|z| = \frac{3}{2}$

Hence by Cauchy integral formula

$$\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz = -2\pi i \, f(z)$$
$$= -2\pi i(-1)$$
$$= 2\pi i \qquad [\because f(z) = \cos \pi z, \; f(1) = \cos \pi = -1]$$

Problem 14 State Laurent's series

Solution:

If C_1 and C_2 are two concentric circles with centres at $z = a$ and radii r_1 and r_2 $(r_1 < r_2)$ and if $f(z)$ is analytic on C_1 and C_2 and throughout the annular region R between them, then at each point z in R,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n},$$

where $a_n = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(z)dz}{(z-a)^{n+1}}, \; n = 0,1,2,\ldots, \; b_n = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(z)dz}{(z-a)^{-n+1}}, \; n = 1,2,3,\ldots$

Problem 15 Find the zeros of $\dfrac{z^3-1}{z^3+1}$.

Solution:

The zeros of $f(z)$ are given by $f(z) = 0, \; \dfrac{z^3-1}{z^3+1} = 0$

i.e., $z^3 - 1 = 0, \; z = (1)^{\frac{1}{3}}$

$z = 1, \; w, w^2$ (Cubic roots of unity)

Part-B

Problem 1 Using Cauchy integral formula evaluate $\displaystyle\int_C \frac{dz}{(z+1)^2(z-2)}$ where C the circle $|z| = \dfrac{3}{2}$.

Solution:

Here $z = -1$ is a pole lies inside the circle

$z = 2$ is a pole lies out side the circle

$$\therefore \int_C \frac{dz}{(z+1)^2(z-2)} = \int \frac{\frac{1}{z-2}}{(z+1)^2} dz$$

Here $\mathrm{f}(\mathrm{z}) = \dfrac{1}{\mathrm{z}-2}$

$$f'(z) = -\frac{1}{(z-2)^2}$$

Hence by Cauchy integral formula

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a)$$

$$\int_C \frac{dz}{(z+1)^2 (z-2)} = \int_C \frac{\frac{1}{z-2}}{[z-(-1)]^2} dz$$

$$= \frac{2\pi i}{1!} f'(-1)$$

$$= 2\pi i \left[\frac{-1}{(-1-2)^2} \right] \quad \left(\therefore f'|z| = \frac{-1}{(z-2)^2} \right)$$

$$= 2\pi i \left[\frac{-1}{9} \right]$$

$$= \frac{-2}{9} \pi i .$$

Problem 2 Evaluate $\int_C \frac{z-2}{z(z-1)} dz$ where C is the circle $|z| = 3$.

Solution:

W.K.T $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$

Given $\int_C \frac{z-2}{z(z-1)} dz$ Here $z = 0$, $z = 1$ lies inside the circle

Also $f(z) = z - 2$

Now $\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$

Put $z = 0 \Rightarrow A = -1$

$z = 1 \Rightarrow B = 1$

$$\frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$

$$\int_C \frac{z-2}{z(z-1)} dz = -\int_C \frac{z-2}{z} dz + \int_C \frac{z-2}{z-1} dz$$

$$= -2\pi i \; f(0) + 2\pi i \; f(1)$$

$$= 2\pi i [f(1) - f(0)]$$

$$= 2\pi i [-1 - (-2)]$$

$$= 2\pi i [2 - 1] = 2\pi i .$$

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Problem 3 Find the Laurent's Series expansion of the function $\frac{z-1}{(z+2)(z+3)}$, valid in the region $2<|z|<3$.

Solution:

Let $f(z)=\frac{z-1}{(z+2)(z+3)}$

$\frac{z-1}{(z+2)(z+3)}=\frac{A}{z+2}+\frac{B}{z+3}$

$z-1=A(z+3)+B(z+2)$

Put $z=-2$

$-2-1=A(-2+3)+0$

$A=3$

Put $z=-3$

$-3-1=A(0)+B(-3+2)$

$-4=-B$

$B=4$

$\therefore f(z)=\frac{-3}{z+2}+\frac{4}{z+3}$

Given region is $2<|z|<3$

$2<|z|$ and $|z|<3$

$\left|\frac{2}{z}\right|<1$ and $\left|\frac{z}{3}\right|<1$

$\therefore f(z)=\frac{-3}{z\left(1+\frac{2}{z}\right)}+\frac{4}{3\left(1+\frac{z}{3}\right)}$

$=\frac{-3}{z}\left(1+\frac{2}{z}\right)^{-1}+\frac{4}{3}\left(1+\frac{z}{3}\right)^{-1}$

$=\frac{-3}{z}\left[\left(1-\frac{2}{z}+\left(\frac{2}{z}\right)^{2}-....\right)\right]+\frac{4}{3}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^{2}-...\right]$

Problem 4 Expand $f(z)=\frac{7z-2}{z(z-2)(z+1)}$ valid in $1<|z+1|<3$

Solution:

Given $f(z)=\frac{7z-2}{z(z-2)(z+1)}$

$f(z)=\frac{7z-2}{z(z-2)(z+1)}=\frac{A}{z}+\frac{B}{z-2}+\frac{C}{z+1}$

$7z-2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$

Put $z = 2$

$B = 2$

Put $z = 0$

$A = 1$

Put $z = -1$

$C = -3$

$\therefore f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$

Given region is $1 < |z+1| < 3$

Let $u = z+1 \Rightarrow 1 < |u| < 3$

$z = u-1 \Rightarrow 1 < |u| \ \& \ |u| < 3$

$\Rightarrow \frac{1}{|u|} < 1 \ \& \ \left|\frac{u}{3}\right| < 1$

$$\therefore f(z) = \frac{1}{u-1} + \frac{2}{u-3} - \frac{3}{u}$$

$$= \frac{1}{u(1-\frac{1}{u})} + \frac{2}{-3(1-\frac{u}{3})} - \frac{3}{u}$$

$$= \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1-\frac{u}{3}\right)^{-1} - \frac{3}{u}$$

$$= \frac{1}{u}\left[1+\frac{1}{u}+\left(\frac{1}{u}\right)^2+\dots\right] - \frac{2}{3}\left(1+\frac{u}{3}+\left(\frac{u}{3}\right)^2+\dots\right) - \frac{3}{u}$$

$$= \frac{1}{z+1}\left[1+\frac{1}{z+1}+\left(\frac{1}{z+1}\right)^2+\dots\right] - \frac{2}{3}\left[1+\left(\frac{z+1}{3}\right)+\left(\frac{z+1}{3}\right)^2+\dots\right] - \frac{3}{z+1}$$

$$\therefore f(z) = -\frac{2}{z+1} + \sum_{n=2}^{\infty}\frac{1}{(z+1)^n} - \frac{2}{3}\sum_{n=0}^{\infty}\frac{(z+1)^n}{3^n}.$$

Problem 5

Expand $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ as a Taylor series valid in the region $|z| < 2$.

Solution:

Given $f(z) = \frac{z^2-1}{(z+2)(z+3)}$

Now $(z+2)(z+3) = z^2+5z+6$

$$\therefore \frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{-5z-7}{(z+2)(z+3)}$$

Now $\frac{-5z-7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$

$-5z-7 = A(z+3) + B(z+2)$

Put $z=-2$

$A=3$

Put $z=-3$

$B=-8$

$\therefore f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$

Given $|z|<2$

$$f(z) = 1 + \frac{3}{2\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left(1-\frac{z}{2}+\left(\frac{z}{2}\right)^2+\ldots\right) - \frac{8}{3}\left(1-\frac{z}{3}+\left(\frac{z}{3}\right)^2+\ldots\right)$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

$$f(z) = 1 + \sum_{n=0}^{\infty}(-1)^n\left[\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}}\right]z^n .$$

Problem 6 Using Cauchy Integral formula Evaluate $\int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz$ where C is circle $|z|=1$.

Solution:

Here $f(z) = \sin^6 z$

$f'(z) = 6\sin^5 \cos z$

$f''(z) = 6\left[-\sin^6 z + \cos^2 z.5\sin^4 z\right]$

Here $a = \frac{\pi}{6}$, clearly $a = \frac{\pi}{6}$ lies inside the circle $|z|=1$

By Cauchy integral formula

$$\int_C \frac{f(z)}{(z-a)^3} = \frac{2\pi i}{2!} f''(a)$$

$$\therefore \int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} = \frac{2\pi i}{2!} f''\left(\frac{\pi}{6}\right)$$

$$= \pi i 6\left[-\sin^6\left(\frac{\pi}{6}\right)+5\cos^2\left(\frac{\pi}{6}\right)\sin^4\frac{\pi}{6}\right]$$

$$= 6\pi i\left[-\frac{1}{64}+\frac{5}{16}\times\frac{3}{4}\right]$$

$$= 6\pi i\left[-\frac{1}{64}+\frac{15}{64}\right]$$

$$= 6\pi i\left[\frac{15-1}{64}\right] = \frac{21\pi i}{16}$$

Problem 7 Expand $f(z)=\sin z$ into a Taylor's series about $z=\frac{\pi}{4}$.

Solution:

Given $f(z)=\sin z$

$f'(z)=\cos z$

$f''(z)=-\sin z$

$f'''(z)=-\cos z$

Here $a=\frac{\pi}{4}$

$$\therefore f\left(\frac{\pi}{4}\right)=\sin\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

$$f'\left(\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

$$f''\left(\frac{\pi}{4}\right)=-\sin\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$$

$$f'''\left(\frac{\pi}{4}\right)=-\cos\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$$

W.K.T Taylor's series of $f(z)$ at $z=a$ is

$$f(z)=f(a)+\frac{z-a}{1!}f'(a)+\frac{(z-a)^2}{2!}f''(a)+\ldots$$

$$f(z)=f\left(\frac{\pi}{4}\right)+\frac{z-\frac{\pi}{4}}{1!}f'\left(\frac{\pi}{4}\right)+\frac{\left(z-\frac{\pi}{4}\right)^2}{2!}f''\left(\frac{\pi}{4}\right)+\ldots$$

$$=\frac{1}{\sqrt{2}}+\left(z-\frac{\pi}{4}\right)\frac{1}{\sqrt{2}}-\left(\frac{z-\frac{\pi}{4}}{2}\right)^2\left(\frac{1}{\sqrt{2}}\right)+\ldots$$

Problem 8 Evaluate $\int_C \frac{z\sec z}{(1-z^2)}dz$ where C is the ellipse $4x^2+9y^2=9$, using Cauchy's residue theorem.

Solution:

Equation of ellipse is

$4x^2+9y^2=9$

$\frac{x^2}{9/4}+\frac{y^2}{1}=1$

i.e., $\frac{x^2}{\left(3/2\right)^2}+\frac{y^2}{1}=1$

$\therefore$ Major axis is $\frac{3}{2}$, Minor axis is 1.

The ellipse meets the x axis at $\pm\frac{3}{2}$ and the y axis at ± 1

Given $f(z)=\frac{z\sec z}{1-z^2}$

$=\frac{z}{(1+z)(1-z)\cos z}$

The poles are the solutions of $(1+z)(1-z)\cos z=0$

i.e., $z=-1,\ z=1$ are simple poles and $z=(2n+1)\frac{\pi}{2}$

Out of these poles $z\pm 1$ lies inside the ellipse

$z=\pm\frac{\pi}{4},\ \pm 3\frac{\pi}{4}$ lies outside the ellipse

$$\left[\operatorname{Res} f(z)\right]_{z=1}=\underset{z\to 1}{Lt}(z-1)\frac{z}{(1+z)(1-z)\cos z}$$

$$=\underset{z\to 1}{Lt}\frac{-z}{(1+z)\cos z}=\frac{-1}{2\cos 1}$$

$$\left[\operatorname{Res} f(z)\right]_{z=-1}=\underset{z\to -1}{Lt}(z+1)\frac{z}{(1+z)(1-z)\cos z}$$

$$=\underset{z\to -1}{Lt}\frac{z}{(1-z)\cos z}$$

$$= \frac{-1}{2\cos 1} = \frac{-1}{2\cos 1}$$

$$\therefore \int_C \frac{z\sec z}{1-z^2}\,dz = 2\pi i\left[\text{sum of the residues }\right]$$

$$= 2\pi i\left[\frac{-1}{2\cos 1} - \frac{1}{2\cos 1}\right]$$

$$= -2\pi i\left[\sec 1\right].$$

Problem 9 Using Cauchy integral formula evaluate (i) $\int_C \frac{z+4}{z^2+2z+5}\,dz$, where C is the circle $|z+1-i| = 2$ (ii) $\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$, C is the circle $|z| = \frac{3}{2}$.

Solution:

(i) Given $|z+1-i| = 2$

$|z-(-1+i)| = 2$ is a circle whose centre is $-1+i$ and radius 2.

i.e., centre $(-1,1)$ and radius 2

$z^2+2z+5 = \left[z-(-1+2i)\right]\left[z-(-1-2i\right]$

$-1+2i$ i.e., $(-1,2)$ lies inside the C

$-1-2i$ i.e., $(-1,-2)$ lies out side the C

$$\left[\therefore z^2+2z+5 = 0 \Rightarrow z = -2 \pm \sqrt{\frac{4-20}{2}},\ z = -1 \pm 2i\right]$$

$$\therefore \int_C \frac{z+4}{\left[z-(-1+2i)\right]\left[z-(-1-2i)\right]}\,dz$$

$$= \int_C \frac{\dfrac{z+4}{\left[z-(-1-2i)\right]}}{z-(-1+2i)}\,dz$$

Hence $f(z) = \frac{z+4}{\left[z-(-1-2i)\right]}$

Here by Cauchy integral formula

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\int_C \frac{z+4}{z^2+2z+5} = 2\pi i\, f(-1+2i)$$

$$= 2\pi i\left[\frac{-1+2i+4}{(-1+2i)-(-1-2i)}\right]$$

$$= 2\pi i\left[\frac{3+2i}{4i}\right] = \frac{\pi}{2}[3+2i].$$

(ii) $\int_C \frac{4-3z}{z(z-1)(z-2)}dz$

$z=0,\ z=1$ lie inside the circle $|z| = \frac{3}{2}$

$z=2$ lies outside the circle

$\therefore \frac{4-3z}{z(z-1)(z-2)} = \frac{A}{z}+\frac{B}{z-1}+\frac{C}{z-2}$

$4-3z = A(z-1)(z-2)+B(z)(z-2)+C(z)(z-1)$

Put $z=0$

$4=4A$

$A=1$

Put $z=1$

$B=-1$

Put $z=2$

$C=-1$

$\therefore \frac{4-3z}{z(z-1)(z-2)} = \frac{2}{z}-\frac{1}{z-1}-\frac{1}{z-2}$

$\int_C \frac{4-3z}{z(z-1)(z-2)}dz = \int_C \frac{2}{z}\,dz - \int_C \frac{1}{z-1}dz - \int_C \frac{1}{z-2}dz$

$\int_C \frac{f(z)}{(z-a)^{n+1}} = \frac{2\pi i}{n!}f^n(0)$

$= 2\,[2\pi i\, f(0)] - 2\pi i\ f(1) - 0$

$= 4\pi i\ f(0) - 2\pi i\ f(1)$

$= 4\pi i\ (1) - 2\pi i\ (1)$

$= 2\pi i \qquad (\therefore f(0)=1\ f(1)=1)$

Problem 10 Using Cauchy's integral formula evaluate $\int_C \frac{dz}{(z^2+4)^2}$ where C is circle $|z-i|=2$

Solution:

$\frac{1}{(z^2+4)^2} = \frac{1}{(z+2i)^2(z-2i)^2}$

Given $|z-i|=2$, centre $(0,1)$, radius 2

$\therefore z=-2i$ lies outside the circle

$z=2i$ lies inside the circle

$$\therefore \int_C \frac{dz}{(z^2+4)^2} = \int_c \frac{\frac{1}{(z+2i)^2}}{(z-2i)^2}dz$$

Here $f(z) = \frac{1}{(z+2i)^2}$

$$f'(z) = \frac{-2}{(z+2i)^3}$$

$$f'(2i) = -\frac{2}{(2i+2i)^3} = -\frac{2}{(4i)^3}$$

$$= -\frac{2i}{64} = -\frac{i}{32}$$

Hence by Cauchy Integral Formula

$$\int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!} f^n(a)$$

$$\int_C \frac{f(z)}{(z^2+4)^2} = \frac{2\pi i}{1!} f'(2i) = \frac{\pi}{16}.$$

Problem 11

Find the Laurent's series which represents the function $\frac{z}{(z+1)(z+2)}$ in (i) $|z|>2$ (ii) $|z+1|<1$

Solution:

(i). Let $f(z) = \frac{z}{(z+1)(z+2)}$

Now $\frac{z}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$

$z = A(z+2) + B(z+1)$

Put $z=-1$

$A=-1$

Put $z=-2$

$B=1$

$$\therefore f(z) = \frac{-1}{z+1} + \frac{2}{z+2}$$

Given $|z|>2,\ 2<|z|$ i.e., $\left|\frac{2}{z}\right| < 1 \Rightarrow \frac{1}{|z|} < 1$

$$\therefore f(z) = \frac{-1}{z+1} + \frac{2}{z+2}$$

$$= \frac{-1}{z\left(1+\frac{1}{z}\right)} + \frac{2}{z\left(1+\frac{2}{z}\right)}$$

$$= \frac{-1}{z}\left(1+\frac{1}{z}\right)^{-1} + \frac{2}{z}\left(1+\frac{2}{z}\right)^{-1}$$

(ii). $|z+1|<1$

Let $u = z+1$

i.e., $|u|<1$

$$f(z) = \frac{-1}{z+1} + \frac{2}{z+2}$$

$$= \frac{-1}{u} + \frac{2}{1+u}$$

$$= \frac{-1}{u} + 2(1+u)^{-1}$$

$$= \frac{-1}{u} + 2(1-u+u^2-...)$$

$$= \frac{-1}{1+z} + 2\left[1-(1+z)+(1+z)^2-...\right]$$

Problem 12 Prove that $\int_0^{2\pi} \frac{d\theta}{a^2 - 2a\cos\theta + 1} = \frac{2\pi}{1-a^2}$, given $a^2 < 1$.

Solution: Let $I = \int_0^{2\pi} \frac{d\theta}{a^2 - 2a\cos\theta + 1}$

Put $z = e^{i\theta}$

Then $d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$$\therefore \; I = \int_C \frac{dz/iz}{a^2 - a\left(z+\frac{1}{z}\right)+1} \quad \text{where C is } |z| = 1.$$

$$= \frac{1}{ai}\int_C \frac{dz}{\left(a+\frac{1}{a}\right)z - z^2 - 1}$$

$$= \frac{i}{a}\int_C \frac{dz}{z^2 - \left(a+\frac{1}{a}\right)z + 1}$$

$$= \int_C f(z)dz \quad \text{where } f(z) = \left(\frac{i}{a}\right)\frac{1}{z^2 - \left(a+\frac{1}{a}\right)z+1}$$

$$= \left(\frac{i}{a}\right)\frac{1}{(z-a)\left(z-\frac{1}{a}\right)}$$

The singularities of f(z) are simple poles at a and $\frac{1}{a}$. $a^2 < 1$ implies $|a| < 1$ and $\frac{1}{|a|} > 1$

$\therefore$ The pole that lies inside C is z = a.

$$\text{Res}[f(z);\,a] = \lim_{z\to a}(z-a).\left(\frac{i}{a}\right)\frac{1}{(z-a)\left(z-\frac{1}{a}\right)}$$

$$= \left(\frac{i}{a}\right)\frac{1}{\left(a-\frac{1}{a}\right)}$$

$$= \frac{i}{a^2-1}$$

$$\text{Hence } I = 2\pi i.\frac{i}{a^2-1} = \frac{2\pi}{1-a^2}$$

Problem 13 Show that $\displaystyle\int_0^{2\pi}\frac{\cos 2\theta.d\theta}{5+4\cos\theta} = \frac{\pi}{6}$

Solution: Let $I = \displaystyle\int_0^{2\pi}\frac{\cos 2\theta.d\theta}{5+4\cos\theta}$

Put $z = e^{i\theta}$

Then $d\theta = \dfrac{dz}{iz}$ and $\cos\theta = \dfrac{1}{2}\left(z+\dfrac{1}{z}\right)$

$$I = \text{Real Part of } \int_0^{2\pi}\frac{e^{i2\theta}.d\theta}{5+4\cos\theta}$$

$$= \text{Real Part of } \int_C \frac{z^2.dz/iz}{5+2\left(z+\frac{1}{z}\right)} \text{ where C is } |z| = 1.$$

$$= \text{Real Part of } \frac{1}{2i}\int_C \frac{z^2.dz}{z^2+\frac{5}{2}z+1}$$

$$= \text{Real Part of } \frac{1}{2i}\int_C \frac{z^2.dz}{\left(z+\frac{1}{2}\right)(z+2)}$$

$$= \text{Real Part of } \int_C f(z)dz \text{ where } f(z) = \frac{1}{2i}.\frac{z^2}{\left(z+\frac{1}{2}\right)(z+2)}$$

z = -½ and z = -2 are simple poles of f(z).

z = -½ lies inside C.

$$\text{Res}[f(z);\,-\tfrac{1}{2}] = \lim_{z\to -\frac{1}{2}}(z+\tfrac{1}{2}).\frac{1}{2i}.\frac{z^2}{\left(z+\frac{1}{2}\right)(z+2)}$$

$$= \frac{1}{2i}.\frac{1/4}{3/2} = \frac{1}{12i}$$

$$\therefore I = \text{Real Part of } 2\pi i.\frac{1}{12i}$$

$= \text{Real Part of } \pi/6$

$= \pi/6$.

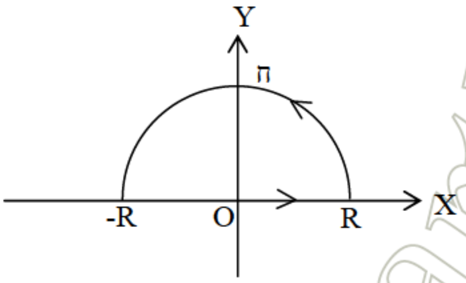
Problem 14 Prove that $\int_0^\infty \frac{dx}{\left(x^2+1\right)^2} = \frac{\pi}{4}$

Solution:

Let $\int_C \phi(z)\,dz = \int_C \frac{dz}{\left(z^2+1\right)^2}$

Where $\phi(z) = \frac{1}{\left(z^2+1\right)^2}$

Here C is the semicircle Γ bounded by the diameter $[-R, R]$



By Cauchy residue theorem,

$\int_C \phi(z)\,dz = \int_{-R}^{R} \phi(x)\,dx + \int_\Gamma \phi(z)\,dz$......(1)

To evaluate of $\int_C \phi(z)\,dz$

The poles of $\phi(z) = \frac{1}{\left(z^2+1\right)^2}$ is the solution of $\left(z^2+1\right)^2 = 0$

i.e., $(z+i)^2(z-i)^2 = 0$

i.e., the poles are $z = i, z = -i$

$z = i$ lies with inside the semi circle

$z = -i$ lies outside the semi circle

Now $\left[Res\,\phi(z)\right]_{z=i} = \underset{z\to i}{Lt} \frac{1}{1!}\frac{d}{dz}(z-i)^2\phi(z)$

$$= \underset{z \to i}{Lt} \frac{1}{1!}\left[(z-i)^2 \frac{1}{(z^2+1)^2}\right]$$

$$= \underset{z \to i}{Lt} \frac{1}{1!} \frac{d}{dz}\left[\frac{1}{(z+i)^2}\right] \qquad \because z^2+1=(z+i)(z-i)$$

$$= \underset{z \to i}{Lt} \frac{-2}{(z+i)^3}$$

$$= \frac{-2}{i+i} = \frac{-2}{(2i)^3} = \frac{1}{4i}$$

$$\therefore \int_C \phi(z)\,dz = 2\pi i\left[Sum\ of\ \ residues\ of\ \phi(z) \text{ at } its\ \ poles\ \ which\ lies\ in\ C\right]$$

$$= 2\pi i\left[\frac{1}{4i}\right] = \frac{\pi}{2} ....(2)$$

Let $R \to \infty$, then $|z| \to \infty$ so that $\phi(z) = 0$

$$\therefore \underset{|z| \to \infty}{Lt} \int_\Gamma \phi(z) dz = 0.........(3)$$

Sub (2) and (3) in (1)

$$\int_C \phi(z) dz = \int_{-\infty}^{\infty} \phi(x)\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2}$$

$$\Rightarrow 2\int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2}$$

$$\Rightarrow \int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}.$$

Problem 15 Evaluate $\int_0^{\infty} \frac{x \sin x}{x^2+a^2} dx$

Solution:

$$2\int_0^{\infty} \frac{x \sin x}{x^2+a^2} dx = \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2} dx$$

$$\int_0^{\infty} \frac{x \sin x}{x^2+a^2} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2} dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{z \sin z}{z^2+a^2} dz$$

$$= \frac{1}{2} I ....(1)$$

Now $z \sin z$ is the imaginary part of ze^{iz}

$$\therefore \text{I} = \int_{-\infty}^{\infty} \frac{z \sin z}{z^2 + a^2} dz$$

$$= \text{I.P.} \int_{-\infty}^{\infty} \frac{ze^{iz}}{z^2 + a^2} dz$$

Let $\phi(z) = \frac{z\,e^{iz}}{z^2 + a^2} = \frac{ze^{iz}}{(z+ia)(z-ia)}$

The poles are $z = -ia$, $z = ia$

Now the poles $z = ia$ lies in the upper half – plane

But $z = -ia$ lies in the lower half – plane.

Hence

$$\left[Res\phi(z)\right]_{z=ia} = \underset{z \to ia}{Lt} (z-ia) \frac{ze^{iz}}{(z+ia)(z-ia)}$$

$$= \underset{z \to ia}{Lt} \frac{ze^{iz}}{(z+ia)}$$

$$= \frac{iae^{-a}}{2ia} = \frac{e^{-a}}{2}$$

$$\therefore \int_{-\infty}^{\infty} \frac{ze^{iz}}{z^2 + a^2} dz = 2\pi i \left[\text{Sum of the residues at each poles in the upper half plane }\right]$$

$$= 2\pi i \left[\frac{e^{-a}}{2}\right]$$

$$= \pi i e^{-a}$$

$$\text{I=I.P. of} \int_{-\infty}^{\infty} \frac{ze^{iz}}{z^2 + a^2} dz$$

$$= \text{I.P. of } \left(\pi i e^{-a}\right)$$

$$\text{I} = \pi e^{-a} ....(2)$$

Sub (2) in (1)

$$\int_{0}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \frac{1}{2} x = \frac{1}{2} \pi e^{-a}$$

UNIT V LAPLACE TRANSFORM

Part – A

Problem 1 State the conditions under which Laplace transform of $f(t)$ exists

Solution:

(i) $f(t)$ must be piecewise continuous in the given closed interval $[a,b]$ where $a>0$ and

(ii) $f(t)$ should be of exponential order.

Problem 2 Find (i) $L\left[t^{3/2}\right]$ (ii) $L\left[e^{-at}\cos bt\right]$

Solution:

(i) We know that

$$L\left[t^n\right]=\frac{\Gamma(n+1)}{s^{n+1}}$$

$$L\left[t^{n/3}\right]=\frac{\Gamma\left(\frac{3}{2}+1\right)}{s^{\frac{3}{2}+1}}=\frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{s^{5/2}}\quad\left[\because \Gamma(n+1)=n\Gamma(n)\right]$$

$$=\frac{\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)}{s^{5/2}}$$

$$=\frac{\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{S^{5/2}}$$

$$=\frac{3\sqrt{\pi}}{4s^{5/2}}\quad\left[\therefore \Gamma(1/2)=\sqrt{\pi}\right]$$

ii)

$$L\left[e^{-at}\cos bt\right]=\left[L(\cos bt)\right]_{s\to s+a}$$

$$=\left[\frac{s}{s^2+b^2}\right]_{s\to s+a}$$

$$=\left[\frac{s+a}{(s+a)^2+b^2}\right]$$

Problem 3 Find $L\left[\sin 8t\cos 4t+\cos^3 4t+5\right]$

Solution:

$$L\left[\sin 8t\cos 4t+\cos^3 4t+5\right]=L\left[\sin 8t\cos 4t\right]+L\left[\cos^3 4t\right]+L[5]$$

$$L\left[\sin 8t+\cos 4t\right]=L\left[\frac{\sin 12t+\sin 4t}{2}\right]\quad\left[\because \sin A\cos B=\frac{\sin(A+B)+\sin(A-B)}{2}\right]$$

$$= \frac{1}{2}\left\{L[\sin 12t] + L(\sin 4t)\right\}$$

$$= \frac{1}{2}\left\{\frac{12}{s^2+144} + \frac{4}{s^2+16}\right\}$$

$$L\left[\cos^3 4t\right] = L\left[\frac{\cos 12t + 3\cos 4t}{4}\right]\left[\because \cos^3\theta = \frac{\cos 3\theta + 3\cos\theta}{4}\right]$$

$$= \frac{1}{4}\left\{L(\cos 12t) + 3L(\cos 4t)\right\}$$

$$= \frac{1}{4}\left[\frac{s}{s^2+144} + \frac{3s}{s^2+16}\right]$$

$$L[5] = 5L[1] = 5\left[\frac{1}{s}\right] = \frac{5}{s}.$$

$$L\left[\sin 8t\cos 4t + \cos^3 4t + 5\right] = \frac{1}{2}\left\{\frac{12}{s^2+144} + \frac{4}{s^2+16}\right\} + \frac{1}{4}\left\{\frac{s}{s^2+144} + \frac{3s}{s^2+16}\right\} + \frac{5}{s}.$$

Problem 4 Find $L\{f(t)\}$ where $f(t) = \begin{cases} 0 & ;when \quad 0 < t < 2 \\ 3 & ;when \quad t > 2 \end{cases}$.

Solution:

W.K.T $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^2 e^{-st} f(t)\,dt + \int_2^\infty e^{-st} f(t)\,dt$$

$$= \int_0^2 e^{-st}\,0.dt + \int_2^\infty e^{-st}\,3dt$$

$$= 3\int_2^\infty e^{-st}\,dt = 3\left[\frac{e^{-st}}{-s}\right]_2^\infty$$

$$= 3\left[\frac{e^{-\infty} - e^{-2s}}{-s}\right] = \frac{3e^{-2s}}{s}.$$

Problem 5 If $L[f(t)] = F(s)$ show that $L\{f(\mathrm{at})\} = \frac{1}{a}F\left(\frac{s}{a}\right)$.

(OR)

State and prove change of scale property.

Solution:

W.K.T $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$L\{f(\text{at})\} = \int_0^\infty e^{-st} f(\text{at})\, dt$$

Put $at = x$ when $t = 0$, $x = 0$

$adt = dx$ when $t = \infty$, $x = \infty$

$$L\{f(\text{at})\} = \int_0^\infty e^{-s\left(\frac{x}{a}\right)} f(x) \frac{dx}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)t} f(t)\, dt \quad [\because x \text{ is a dummy variable}]$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right).$$

Problem 6 Does Laplace transform of $\frac{\cos \text{at}}{t}$ exist? Justify

Solution:

If $L\{f(t)\} = F(s)$ and $\frac{1}{t} f(t)$ has a limit as $t \to 0$ then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\, ds$.

Here $\lim_{t\to 0} \frac{\cos \text{at}}{t} = \frac{1}{0} = \infty$

$\therefore L\left\{\frac{\cos \text{at}}{t}\right\}$ does not exist.

Problem 7 Using Laplace transform evaluate $\int_0^\infty te^{-3t} \sin 2t\, dt$

Solution:

W.K.T $L\{f(t)\} = \int_0^\infty e^{st} f(t)\, dt$

$$= \int_0^\infty e^{-3t} t \sin 2t\, dt = L\left[(t \sin 2t)\right]_{s=3}$$

$$= \left[-\frac{d}{ds} L(\sin 2t)\right]_{s=3} = \left[-\frac{d}{ds}\left(\frac{2}{s^2+4}\right)\right]_{s=3}$$

$$= \left[-\left(\frac{-4s}{(s^2+4)^2}\right)\right]_{s=3}$$

$$= \left[-\left(\frac{4s}{(s^2+4)^2}\right)\right]_{s=3} = \frac{12}{169}.$$

Problem 8 Find $L\left[\int_0^t \frac{\sin u}{u}\,du\right]$

Solution:

By Transform of integrals, $L\left[\int_0^t f(x)\,dx\right] = \frac{1}{s}L\{f(t)\}$

$$L\left[\int_0^t \frac{\sin u}{u}\,du\right] = \frac{1}{s}L\left[\frac{\sin t}{t}\right] = \frac{1}{s}\int_s^\infty L[\sin t]\,ds = \frac{1}{s}\int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \frac{1}{s}\left[\tan^{-1} s\right]_s^\infty = \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1} s\right]$$

$$= \frac{1}{s}\cot^{-1} s$$

Problem 9 Find the Laplace transform of the unit step function.

Solution:

The unit step function (Heaviside's) is defined as

$$U_a(t) = \begin{cases} 0 & ; \quad t < a \\ 1 & ; \quad t > a \end{cases}, \text{ where } a \geq 0$$

W.K.T $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$L\{U_a(t)\} = \int_0^\infty e^{-st} U_a(t)\,dt$$

$$= \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st}(1)\,dt$$

$$= \int_a^\infty e^{-st}\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty = \left[\frac{e^{-\infty} - e^{-as}}{-s}\right] = \frac{e^{-as}}{s}$$

Thus $L\{U_a(t)\} = \frac{e^{-as}}{s}$

Problem 10 Find the inverse Laplace transform of $\frac{1}{(s+a)^n}$

Solution:

$$L(t^n) = \frac{n!}{s^{n+1}}$$

$$L\left(t^{n-1}\right)=\frac{(n-1)!}{s^n}$$

$$L\left(e^{-at}t^{n-1}\right)=\left[\frac{(n-1)!}{s^n}\right]_{s\to s+a}=\frac{(n-1)!}{(s+a)^n}$$

$$e^{-at}t^{n-1}=L^{-1}\left(\frac{(n-1)!}{(s+a)^n}\right)$$

$$e^{-at}t^{n-1}=(n-1)!L^{-1}\left[\frac{1}{(s+a)^n}\right]$$

$$\therefore L^{-1}\left[\frac{1}{(s+a)^n}\right]=\frac{1}{(n-1)!}e^{-at}t^{n-1}$$

Problem 11 Find the inverse Laplace Transform of $\frac{1}{s\left(s^2+a^2\right)}$

Solution:

W.K.T $L^{-1}\left[\frac{1}{s}F(s)\right]=\int_0^t L^{-1}[F(s)]dt$

$$L^{-1}\left[\frac{1}{s\left(s^2+a^2\right)}\right]=\int_0^t L^{-1}\left[\frac{1}{s^2+a^2}\right]dt$$

$$=\int_0^t \frac{1}{a}L^{-1}\left[\frac{a}{s^2+a^2}\right]dt$$

$$=\frac{1}{a}\int_0^t \sin at\ dt$$

$$=\frac{1}{a}\left[\frac{\cos at}{a}\right]_0^t$$

$$=-\frac{1}{a^2}[\cos at-1]$$

$$=\frac{1}{a^2}[1-\cos at].$$

Problem 12 Find $L^{-1}\left[\frac{s}{(s+2)^2}\right]$

Solution:

$$L^{-1}\left[\frac{s}{(s+2)^2}\right]=L^{-1}[sF(s)]=\frac{d}{dt}\quad L^{-1}[F(s)]$$

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Where $F(s)=\dfrac{1}{(s+2)^2}$, $L\left[t^n\right]=\dfrac{n!}{s^{n+1}}$

$$L^{-1}\left[F(s)\right]=L^{-1}\left[\frac{1}{(s+2)^2}\right],\ L(t)=\frac{1}{s^2}$$

$$L^{-1}\left[F(s)\right]=e^{-2t}\,L^{-1}\left[\frac{1}{s^2}\right]=e^{-2t}\,t$$

$$L^{-1}\left[\frac{s}{(s+2)^2}\right]=\frac{d}{dt}\left[e^{-2t}\,t\right]=t\left(-2e^{-2t}\right)+e^{-2t}$$

$$L^{-1}\left[\frac{s}{(s+2)^2}\right]=e^{-2t}\,(1-2t)$$

Problem 13 Find $L^{-1}\left[\dfrac{s+2}{\left(s^2+4s+5\right)^2}\right]$.

Solution:

$$L^{-1}\left[\frac{s+2}{\left(s^2+4s+5\right)^2}\right]=L^{-1}\left[\frac{s+2}{\left((s+2)^2+1\right)^2}\right]$$

$$=e^{-2t}L^{-1}\left[\frac{s}{\left(s^2+1\right)^2}\right]\text{.......(1)}$$

$$L^{-1}\left[\frac{s}{\left(s^2+1\right)^2}\right]=t\ L^{-1}\int_s^{\infty}\frac{s}{\left(s^2+1\right)^2}\,ds$$

$$=t\,L^{-1}\int\frac{du}{2u^2}\qquad let\ u=s^2+1,\ du=2sds$$

$$=\frac{t}{2}L^{-1}\left(\frac{-1}{u}\right)$$

$$=\frac{t}{2}L^{-1}\left(\frac{-1}{s^2+1^2}\right)_s^{\infty}$$

$$=\frac{t}{2}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$=\frac{t}{2}\sin t\text{........(2)}$$

Using (2) in (1)

$$L^{-1}\left[\frac{s+2}{\left(s^2+4s+5\right)^2}\right]=e^{-2t}.\frac{t}{2}\sin t=\frac{1}{2}te^{-2t}\sin t.$$

Problem 14 Find the inverse Laplace transform of $\dfrac{100}{s\left(s^2+100\right)}$

Solution:

Consider $\dfrac{100}{s\left(s^2+100\right)}=\dfrac{A}{s}+\dfrac{Bs}{s^2+100}$

$100=A\left(s^2+100\right)+\left(Bs+C\right)(s)$

Put $s=0,\ 100=A(100)$

$A=1$

$s=1,\quad 100=A(101)+B+C$

$B+C=-1$

Equating s^2 term

$0=A+B$

$\Rightarrow B=-1$

$\therefore B+C=-1$ i.e., $-1+C=-1$

$C=0$

$$\therefore L^{-1}\left[\frac{100}{s(s^2+100)}\right]=L^{-1}\left[\frac{1}{s}-\frac{s}{s^2+100}\right]$$

$$=L^{-1}\left[\frac{1}{s}\right]-L^{-1}\left[\frac{s}{s^2+100}\right]$$

$$=1-\cos 10t$$

Problem 15 Solve $\dfrac{dx}{dy}-2y=\cos 2t$ and $\dfrac{dy}{dt}+2x=\sin 2t$ given $x(0)=1;\ y(0)=0$

Solution:

$x'-2y=cos2t$

$y'+2x=\sin 2t$ given $x(0)=1;\ y(0)=0$

Taking Laplace Transform we get

$$\left[sL\left(x\right)-x(0)\right]-2L\left[y\right]=L\left[\cos 2t\right]=\frac{s}{s^2+4}$$

$$\therefore sL\left[x\right]-2L\left[y\right]=\frac{s}{s^2+4}+1..........(1)$$

$$\left[sL\left(y\right)-y(0)\right]+2L\left[x\right]=L\left[\sin 2t\right]=\frac{2}{s^2+4}$$

$$2L\left[x\right]+sL\left[y\right]=\frac{2}{s^2+4}........(2)$$

$(1)\times 2-s\times(2)$ gives,

$$-\left(s^2+4\right)L\left(y\right)=\frac{-2}{s^2+4}$$

$$\therefore y = -L^{-1}\left[\frac{2}{s^2+4}\right]$$
$$= -\sin 2t$$
$$2x = \sin 2t - \frac{dy}{dt}$$
$$= \sin 2t + 2\cos 2t$$
$$\therefore x = \cos 2t + \frac{1}{2}\sin 2t$$

Part-B

Problem 1 Find the Laplace transform of $e^{-t}\int_0^t \frac{\sin t}{t}dt$

Solution:

$$L\left(\int_0^t \frac{\sin t}{t}dt\right) = \frac{1}{s}L\left(\frac{\sin t}{t}\right)$$
$$L\left(\frac{\sin t}{t}\right) = \int_s^\infty L(\sin t)\,ds$$
$$= \int_s^\infty \frac{1}{s^2+1}ds$$
$$= \left(\tan^{-1}(s)\right)_s^\infty$$
$$= \frac{\pi}{2} - \tan^{-1}(s)$$
$$= \cot^{-1}(s)$$
$$\therefore L\left(\int_0^t \frac{\sin t}{t}dt\right) = \frac{1}{s}\cot^{-1}(s)$$
$$L\left\{e^{-t}\int_0^t \frac{\sin t}{t}dt\right\} = \left[\frac{1}{s}\cot^{-1}(s)\right]_{s\to s+1}$$
$$= \frac{\cot^{-1}(s+1)}{s+1}.$$

Problem 2 Find $\int_0^\infty te^{-2t}\sin 3t\ dt$ using Laplace transforms.

Solution: $L(\sin 3t) = \frac{3}{s^2+9}$

$$L[t\sin 3t] = -\frac{d}{ds}\left(\frac{1}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}$$

$$\int_0^\infty e^{-st}(t\sin 3t)dt = L[t\sin 3t] \quad (\text{by definition})$$

$$= \frac{6s}{(s^2+9)^2}$$

i.e., $\int_0^\infty te^{-st}\sin 3t\,dt = \frac{6s}{(s^2+9)^2}$

Putting $s=2$ we get $\int_0^\infty te^{-2t}\sin 3t\,dt = \frac{12}{169}$

Problem 3 Find the Laplace transform of $t\int_0^t te^{-4t}\cos 3t\,dt + \frac{\sin 5t}{t}dt$

Solution:

$$L\left[t\int_0^t e^{-4t}\cos 3t\,dt\right] = -\frac{d}{ds}L\left[\int_0^t e^{-4t}\cos 3t\,dt\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s}\left[L\left(e^{-4t}\cos 3t\right)\right]\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s}\left[L(\cos 3t)\right]_{s\to s+4}\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s}\left(\frac{s}{s^2+9}\right)_{s\to s+4}\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s}\frac{s+4}{(s+4)^2+9}\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s}\frac{(s+4)}{(s^2+8s+25)}\right]$$

$$= -\frac{d}{ds}\left[\frac{s+4}{s^3+8s^2+25s}\right]$$

$$= -\left[\frac{(s^3+8s^2+25s)(1)-(s+4)(3s^3+16s+25)}{(s^3+8s^2+25s)^2}\right]$$

$$= -\left[\frac{s^3+8s^2+25s-3s^3+16s^2-25s-12s^2-64s-100}{(s^3+8s^2+25s)^2}\right]$$

$$= -\left[\frac{-2s^3-20s^2-64s-100}{(s^3+8s^2+25s)^2}\right]$$

$$= 2\left[\frac{s^3+10s^2+32s+50}{\left(s^3+8s^2+25s\right)^2}\right]$$

$$L\left[\frac{\sin 5t}{t}\right] = \int_s^\infty L(\sin 5t)\,ds$$

$$= \int_s^\infty \frac{5}{s^2+25}\,ds$$

$$= \left[5.\frac{1}{5}\tan^{-1}\left(\frac{s}{5}\right)\right]_s^\infty$$

$$= \left[\tan^{-1}\left(\frac{s}{5}\right)\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{5}\right)$$

$$= \cot^{-1}\left(\frac{s}{5}\right)$$

$$\therefore L\left[t\int_0^t e^{-4t}\cos 3t\,dt + \frac{\sin 5t}{t}\right] = L\left[t\int_0^t e^{-4t}\cos 3t\,dt\right] + L\left[\frac{\sin 5t}{t}\right]$$

$$= \frac{2\left(s^3+10s^2+32s+50\right)}{\left(s^3+8s^2+25s\right)^2} + \cot^{-1}\left(\frac{s}{5}\right)$$

Problem 4 Find $L\left[t^2e^{2t}\cos 2t\right]$.

Solution:

$$L\left[t^2e^{2t}\cos 2t\right] = (-1)^2\frac{d^2}{ds^2}L\left[e^{2t}\cos 2t\right]$$

$$= \frac{d^2}{ds^2}\left[\left(\frac{s}{s^2+4}\right)_{s\to s-2}\right]$$

$$= \frac{d^2}{ds^2}\left[\left(\frac{s-2}{(s-2)^2+4}\right)\right]$$

$$= \frac{d^2}{ds^2}\left[\frac{s-2}{s^2-4s+8}\right]$$

$$= \frac{d}{ds}\left[\frac{\left(s^2-4s+8\right)(1)-(s-2)(2s-4)}{\left(s^2-4s+8\right)^2}\right]$$

$$=\frac{d}{ds}\left[\frac{s^2-4s+8-2s^2+4s+4s-8}{\left(s^2-4s+8\right)^2}\right]$$

$$=\frac{d}{ds}\left[\frac{-s^2+4s}{\left(s^2-4s+8\right)^2}\right]$$

$$=\frac{\left(s^2-4s+8\right)\left(-2s+4\right)-\left(s^2+4s\right)2\left(s^2-4s+8\right)\left(2s-4\right)}{\left(s^2-4s+8\right)^4}$$

$$=\frac{\left(s^2-4s+8\right)\left(-2s+4\right)-\left(s^2+4s\right)\left(2s-4\right)}{\left(s^2-4s+8\right)^3}$$

$$=\frac{-2s^3+8s^2-16s+4s^2-16s+32+4s^3-8s^2-16s^2+32s}{\left(s^2-4s+8\right)^3}$$

$$=\frac{2s^3-12s^2+32}{\left(s^2-4s+8\right)^3}.$$

Problem 5 Verify the initial and final value theorems for the function $f(t)=1+e^{-t}\left(\sin t+\cos t\right)$

Solution: Given $f(t)=1+e^{-t}\left(\sin t+\cos t\right)$

$$L\{f(t)\}=L\{1+e^{-t}\sin t+e^{-t}\cos t\}$$

$$=L(1)+L\left(e^{-t}\sin t\right)+L\left(e^{-t}\cos t\right)$$

$$=\frac{1}{s}+\frac{1}{(s+1)^2+1}+\frac{s+1}{(s+1)^2+1}$$

$$=\frac{1}{s}+\frac{s+2}{(s+1)^2+1}$$

Initial value theorem:

$$\underset{t\to 0}{Lt}\, f(t)=\underset{s\to\infty}{Lt}\, sF(s)$$

$$LHS=\underset{t\to 0}{Lt}\left[1+e^{-t}\left(\sin t+\cos t\right)\right]=1+1=2$$

$$RHS=\underset{s\to\infty}{Lt}\, sF(s)$$

$$=\underset{s\to\infty}{Lt}\, s\left[\frac{1}{s}+\frac{s+2}{(s+1)^2+1}\right]$$

$$=\underset{s\to\infty}{Lt}\left[1+\frac{s^2+2s}{s^2+2s+2}\right]$$

$$= \underset{s\to\infty}{Lt} \left[\frac{2s^2+4s+2}{s^2+2s+2} \right]$$

$$= \underset{s\to\infty}{Lt} \left[\frac{2+4/s+2/s^2}{1+2/s+2/s^2} \right] = 2$$

$LHS = RHS$

Hence initial value theorem is verified.

Final value theorem:

$$\underset{t\to\infty}{Lt} f(t) = \underset{s\to 0}{Lt} sF(s)$$

$$LHS = \underset{t\to\infty}{Lt} f(t)$$

$$= \underset{t\to\infty}{Lt} \left(1+e^{-t}\sin t + e^{-t}\cos t\right) = 1 \quad \left(\because e^{-\infty} = 0\right)$$

$$RHS = \underset{s\to 0}{Lt} sF(s)$$

$$= \underset{s\to 0}{Lt} s\left[\frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1} \right]$$

$$= \underset{s\to 0}{Lt} \left[1 + \frac{s^2+2s}{(s+1)^2+1} \right] = 1$$

$LHS = RHS$

Hence final value theorem is verified.

Problem 6 Find $L^{-1}\left[\log\left(\frac{s^2+1}{s^2}\right)\right]$.

Solution:

$$L^{-1}\left[F(s)\right] = -\frac{1}{t}L^{-1}\left[F'(s)\right] \text{.......(1)}$$

$$F(s) = \log\left(\frac{s^2+1}{s^2}\right)$$

$$F'(s) = \frac{d}{ds}\log\left[\left(s^2+1\right) - \log\left(s^2\right)\right]$$

$$= \frac{2s}{s^2+1} - \frac{2s}{s^2}$$

$$L^{-1}\left[F'(s)\right] = L^{-1}\left[\frac{2s}{s^2+1} - \frac{2s}{s^2}\right]$$

$$= 2L^{-1}\left[\frac{s}{s^2+1} - \frac{1}{s}\right]$$

$$= 2\left[\cos t - 1\right]$$

$$L^{-1}\left[\log\left(\frac{s^2+1}{s^2}\right)\right] = -\frac{1}{t}2\left[\cos t - 1\right]$$

$$= \frac{2(1-\cos t)}{t}.$$

Problem 7 Find the inverse Laplace transform of $\frac{s+3}{(s+1)(s^2+2s+3)}$

Solution: $\frac{s+3}{(s+1)(s^2+2s+3)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+3}$ ....(1)

$$s+3 = A(s^2+2s+3)+(Bs+c)(s+1)$$

Put $s=-1$

2=2A

A = 1

Equating the coefficients of s^2

0 = A+B $\Rightarrow$ B = -1

Put s = 0

3 = 3A+C

C = 0

$$(1) \Rightarrow \frac{s+3}{(s+1)(s^2+2s+3)} = \frac{1}{s+1} - \frac{s}{s^2+2s+3}$$

$$= \frac{1}{s+1} - \frac{s}{(s+1)^2+2}$$

$$= \frac{1}{s+1} - \frac{s+1}{(s+1)^2+2} + \frac{1}{(s+1)^2+2}$$

$$L^{-1}\left[\frac{s+3}{(s+1)(s^2+2s+3)}\right] = L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{s+1}{(s+1)^2+2}\right] + L^{-1}\left[\frac{1}{(s+1)^2+2}\right]$$

$$= e^{-t} - e^{-t}L^{-1}\left[\frac{s}{s^2+2}\right] + e^{-t}L^{-1}\left[\frac{1}{s^2+2^2}\right]$$

$$= e^{-t} - e^{-t}\cos\sqrt{2}t + e^{-t}\sin\sqrt{2}t$$

$$= e^{-t}\left[1-\cos\sqrt{2}t + \sin\sqrt{2}t\right].$$

Problem 8 Find $L^{-1}\left[s\log\left(\frac{s-1}{s+1}\right)+2\right]$

Solution:

$$L^{-1}\left[s\log\left(\frac{s-1}{s+1}\right)+2\right] = f(t)$$

$$\therefore L\left[f(t)\right] = s\log\left(\frac{s-1}{s+1}\right)+2$$

$$= s\log(s-1) - s\log(s+1) + 2$$

$$L\{t\,f(t)\} = -\frac{d}{ds}\left[s\log(s-1) - s\log(s+1) + 2\right]$$

$$= -\left[\frac{s}{s-1} + \log(s-1) - \frac{s}{s+1} - \log(s+1)\right]$$

$$= -\left[\log\left(\frac{s-1}{s+1}\right) + \frac{s(s+1) - s(s-1)}{s^2-1}\right]$$

$$= \log\left(\frac{s-1}{s+1}\right) - \left(\frac{s^2+s-s^2+s}{s^2-1}\right)$$

$$= \log\left(\frac{s+1}{s-1}\right) - \frac{2s}{s^2-1}$$

$$tf(t) = L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right] - 2L^{-1}\left[\frac{s}{s^2-1}\right]$$

$$= L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right] - 2\cosh t \ldots.(1)$$

To find $L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$

Let $f(t) = L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$

$$L\left[f(t)\right] = \log\left(\frac{s+1}{s-1}\right)$$

$$L\{t\,f(t)\} = -\frac{d}{ds}\left[\log(s+1) - \log(s-1)\right]$$

$$= \frac{1}{s-1} - \frac{1}{s+1} = \frac{2}{s^2-1}$$

$$\therefore t\ f(t) = 2L^{-1}\left[\frac{1}{s^2-1}\right] = 2\sinh t$$

$$f(t) = \frac{2\sinh t}{t} \ldots..(2)$$

Using (2) in (1)

$$tf(t) = \frac{2\sinh t}{t} - 2\cosh t$$

$$f(t) = \frac{2\sinh t}{t^2} - \frac{2\cosh t}{t}$$

$$= 2\left[\frac{\sinh t - t\cosh t}{t^2}\right].$$

Problem 9 Using convolution theorem find $L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]$

Solution:

$$L^{-1}\left[F(s)G(s)\right]=L^{-1}\left[F(s)\right]*L^{-1}\left[G(s)\right]$$

$$L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]=L^{-1}\left[\frac{s}{s^2+a^2}\right]*L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$=L^{-1}\left[\frac{s}{s^2+a^2}\right]*\frac{1}{a}L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$=\cos\text{at}*\frac{1}{a}\sin\text{at}$$

$$=\frac{1}{a}\left[\cos\text{at}*\sin\text{at}\right]$$

$$=\frac{1}{a}\int_0^t\cos\text{a}\,u\sin a\left(t-u\right)du$$

$$=\frac{1}{a}\int_0^t\sin\left(\text{at}-au\right)\cos\text{a}\,u\,du$$

$$=\frac{1}{a}\int_0^t\frac{\sin\left(\text{at}-au+au\right)+\sin\left(\text{at}-au-au\right)}{2}du$$

$$=\frac{1}{2a}\int_0^t\left[\sin\text{at}+\sin a\left(t-2u\right)\right]du$$

$$=\frac{1}{2a}\left[\left(\sin\text{at}\right)u-\frac{\cos a\left(t-2u\right)}{-2a}\right]_0^t$$

$$=\frac{1}{2a}\left[t\sin\text{at}+\frac{\cos\text{at}}{2a}-\frac{\cos\text{at}}{2a}\right]$$

$$=\frac{t\sin\text{at}}{2a}.$$

Problem 10 Find the Laplace inverse of $\frac{1}{(s+1)\left(s^2+9\right)}$ using convolution theorem.

Solution:

$$L^{-1}\left[F(s).G(s)\right]=L^{-1}\left[F(s)\right]*L^{-1}\left[G(s)\right]$$

$$L^{-1}\left[\frac{1}{(s+1)\left(s^2+9\right)}\right]=L^{-1}\left[\frac{1}{(s+1)}.\frac{1}{\left(s^2+9\right)}\right]$$

$$= L^{-1}\left[\frac{1}{(s+1)}\right] * L^{-1}\left[\frac{1}{(s^2+9)}\right]$$

$$= e^{t} * \frac{1}{3}\sin 3t$$

$$= \frac{1}{3}\int_0^t e^{-u}\sin\left[3(t-u)\right]du$$

$$= \frac{1}{3}\int_0^t e^{-u}\sin(3t-3u)\,du$$

$$= \frac{1}{3}\int_0^t e^{-u}\left[\sin 3t\cos 3u - \cos 3t\sin 3u\right]du$$

$$= \frac{1}{3}\sin 3t\int_0^t e^{-u}\cos 3u\;du - \frac{1}{3}\cos 3t\int_0^t e^{-u}\sin 3u\;du$$

$$= \frac{\sin 3t}{3}\left[\frac{e^{-u}}{10}(-\cos 3u + 3\sin 3u)\right]_0^t - \frac{\cos 3t}{3}\left[\frac{e^{-u}}{10}(-\sin 3u - 3\cos 3u)\right]_0^t$$

$$= \frac{\sin 3t}{3}\left[\frac{e^{-u}}{10}(-\cos 3t + 3\sin 3t) - \frac{1}{10}(-1)\right]$$

$$= \frac{\sin 3t}{3}\left[\frac{e^{-u}}{10}(-\sin 3t - 3\cos 3t) - \frac{1}{10}(-3)\right]$$

Problem 11 Find $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$ using convolution theorem

Solution:

$$L^{-1}\left[F(s).G(s)\right] = L^{-1}\left[F(s)\right] * L^{-1}\left[G(s)\right]$$

$$L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+b^2}\right]$$

$$= \frac{1}{2}\left[\frac{\sin\left[(a-b)u+bt\right]}{a-b} + \frac{\sin\left[(a+b)u-bt\right]}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin(at-bt+bt)}{a-b} + \frac{\sin(at+bt-bt)}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{2a\sin at}{a^2-b^2} - \frac{2b\sin bt}{a^2-b^2}\right]$$

$$= \frac{1}{2}\left[\frac{2a\sin at - 2b\sin bt}{a^2-b^2}\right]$$
$$= \frac{a\sin at - b\sin bt}{a^2-b^2}.$$

Problem 12 Using convolution theorem find the inverse Laplace transform of $\frac{1}{\left(s^2+a^2\right)^2}$.

Solution:

$$L^{-1}\left[F(s).G(s)\right] = L^{-1}\left[F(s)\right] * L^{-1}\left[G(s)\right]$$

$$L^{-1}\left[\frac{1}{\left(s^2+a^2\right)^2}\right] = L^{-1}\left[\frac{1}{s^2+a^2}\right] * L^{-1}\left[\frac{1}{s^2+a^2}\right]$$
$$= \frac{\sin at}{a} * \frac{\sin at}{a}$$
$$= \frac{1}{a^2}\int_0^t \sin au \sin a(t-u)\,du$$
$$= \frac{1}{2a^2}\int_0^t \left[\cos(2au - at) - \cos at\right]du \quad \left[\because 2\sin A\sin B = \cos(A-B) - \cos(A+B)\right]$$
$$= \frac{1}{2a^2}\left[\frac{\sin(2au-at)}{2a} - (\cos at)u\right]_0^t$$
$$= \frac{1}{2a^2}\left[\frac{\sin at}{2a} - t\cos at - \left(\frac{-\sin at}{2a}\right)\right]$$
$$= \frac{1}{2a^2}\left[\frac{2\sin at}{2a} - t\cos at\right]$$
$$= \frac{1}{2a^3}\left[\sin at - at\cos at\right]$$

Problem 13 Solve the equation $y'' + 9y = \cos 2t;\ y(0) = 1 \text{ and } y(\pi/2) = -1$

Solution:

Given $y'' + 9y = \cos 2t$

$$L\left[y''(t) + 9y(t)\right] = L\left[\cos 2t\right]$$
$$L\left[y''(t)\right] + 9L\left[y(t)\right] = L\left[\cos 2t\right]$$
$$\left[s^2 L\left[y(t)\right] - sy(0) - y'(0)\right] + 9L\left[y(t)\right] = \frac{s}{s^2+4}$$

As $y'(0)$ is not given, it will be assumed as a constant, which will be evaluated at the end. $\therefore y'(0) = A$.

$$L[y(t)][s^2+9]-s-A=\frac{s}{s^2+4}$$

$$L[y(t)][s^2+9]=\frac{s}{s^2+4}+s+A$$

$$L[y(t)]=\frac{s}{(s^2+4)(s^2+9)}+\frac{s}{s^2+9}+\frac{A}{s^2+9}.$$

Consider $\frac{s}{(s^2+4)(s^2+9)}=\frac{As+B}{s^2+4}+\frac{Cs+D}{s^2+9}$

$$s=(As+B)(s^2+9)+(Cs+B)(s^2+4)$$

$$= As^3+9As+Bs^2+9B+Cs^3+4(s+1)s^2+4$$

Equating coefficient of s^3 A + C = 0 …….(1)
Equating coefficient of s^2 B + D = 0 …….(2)
Equating coefficient of s 9A+ 4C = 1 …….(3)
Equating coefficient of constant 9B +4D = 0 …….(4)
Solving (1) & (3)

$$4A+4C=0$$
$$-9A+4C=-1$$
$$-5A=-1$$
$$A=\frac{1}{5}$$
$$\frac{1}{5}+C=0$$
$$C=-\frac{1}{5}$$

Solving (2) & (4)

$$9B+9D=0$$
$$9B+AD=0$$
$$D=0$$
$$\therefore B=0 \,\&\, D=0.$$

$$\therefore \frac{s}{(s^2+4)(s^2+9)}=\frac{1}{5}\frac{s}{s^2+4}-\frac{s}{5(s^2+9)}$$

$$\therefore L[y(t)]=\frac{1}{5}\left\{\frac{s}{s^2+4}-\frac{s}{s^2+9}\right\}+\frac{s}{s^2+9}+\frac{A}{s^2+4}$$

$$\therefore y(t)=\frac{1}{5}\cos 2t-\frac{1}{5}\cos 3t+\cos 3t+\frac{A}{3}\sin 3t$$

$$=\frac{1}{5}\cos 2t+\frac{4}{5}\cos 3t+\frac{A}{3}\sin 3t$$

Given $y\left(\frac{\pi}{2}\right)=-1$

$-1=-\frac{1}{5}-\frac{A}{5}$

$\therefore A=\frac{12}{5}$

$\therefore y(t)=\frac{1}{5}\cos 2t+\frac{4}{5}\cos 3t+\frac{4}{5}\sin 3t$

Problem 14 Using Laplace transform solve $\frac{d^2y}{dx^2}-\frac{3dy}{dx}+2y=4$ given that $y(0)=2,\ y'(0)=3$

Solution:

$L[y''(t)]-3L[y'(t)]+2L[y(t)]=L[4]$

$s^2L[y(t)]-sy(0)-y'(0)-3sL[y(t)]+3y(0)+2L[y(t)]=\frac{4}{5}$

$(s^2-3s+2)L[y(t)]-2s-3+6=\frac{4}{s}$

$(s^2-3s+2)L[y(t)]=\frac{4}{s}+2s-3$

$L[f(t)]=\frac{2s^2-3s+4}{s(s^2-3s+2)}$

$L[f(t)]=\frac{2s^2-3s+4}{s(s-1)(s-2)}$

$\frac{2s^2-3s+4}{s(s-1)(s-2)}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s-2}$

$2s^2-3s+4=A(s-1)(s-2)+Bs(s-2)+Cs(s-1)$

Put s = 0 4 = 2A => A = 2

s = 1 $3=-B \Rightarrow B=-3$

$s=2$ $6=2c \Rightarrow C=3$

$\therefore L[y(t)]=\frac{2}{s}-\frac{3}{s-1}+\frac{3}{s-2}$

$y(t)=2-3e^t+3e^{2t}$

Problem 15 Solve $\frac{dx}{dy}+y=\sin t;\ x+\frac{dy}{dt}\cos t$ with $x=2$ and $y=0$ when $t=0$

Solution:

Given $x'(t)+y(t)=\sin t$

$$x(t)+y'(t)=\cos t$$

$$L[x'(t)]+L[y(t)]=L[\sin t]$$

$$sL[x'(t)]-x(0)+L[y(t)]=\frac{1}{s^2+1}$$

$$sL[x(t)]+L[y(t)]=\frac{1}{s^2+1}+2.........(1)$$

$$L[x(t)]+L[y'(t)]=L[\cos 2t]$$

$$L[x(t)]+sL[y(t)]-=y(0)=\frac{1}{s^2+1}....(2)$$

Solving (1) & (2)

$$(1-s^2)L[y(t)]=2+\frac{1-s^2}{s^2+1}$$

$$(1-s^2)L[y(t)]=\frac{2s^2+2+1-s^2}{s^2+1}$$

$$L[y(t)]=\frac{2s^2+3}{(s^2+1)(1-s^2)}$$

$$\frac{s^2+3}{(s^2+1)(1-s^2)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{1-s^2}$$

$$s^2+3=(As+B)(1-s^2)+(Cs+D)(s^2+1)$$

Equating s^3 on both sides

$0=-A+C$ $\quad$ $put\, s=0$

$A=c$ $\quad$ $3=B+D$

$A=0\ C=0$

Equating s^2 on both sides

$1=-B+D$ $\quad$ $D=2$

$B=1$

Equation son both sides 0 = A + B

$$\Rightarrow y(t)=L^{-1}\left[\frac{1}{s^2+1}\right]-2L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$=\sin t-2\sinh t$$

To find $x(t)$ we have $x(t)+y'(t)=\cos t$, $x(t)=\cos t-y'(t)$, $y(t)=\sin t-2\sinh t$

$$\frac{dy}{dt}=\cos t-2\cosh t$$

$$x(t)=\cos t-\cos t+2\cosh t$$

$$=2\cosh t$$

Hence $x(t)=2\cosh t$

$$y(t)=\sin t-2\sinh t$$