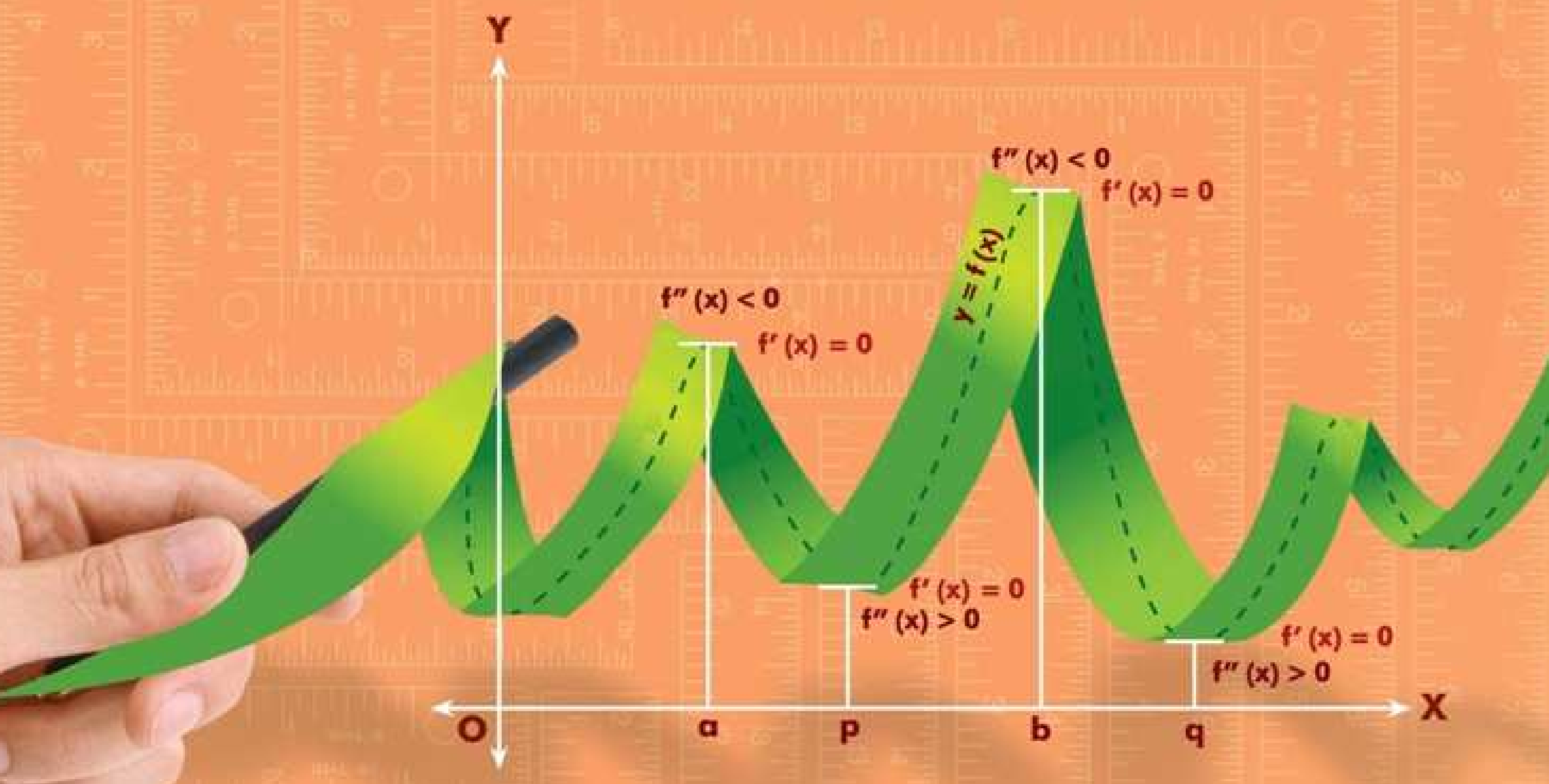
PERFECT

MATHEMATICS - II

Std XII Sci.

Concept of local Maxima and Minima



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STD. XII Sci.
Perfect Maths – II

Eleventh Edition: March 2016

Salient Features :

- Exhaustive coverage of entire syllabus.
- Covers answers to all Textual and Miscellaneous Exercises.
- Precise theory for every topic.
- Neat, labelled and authentic diagrams.
- Written in a systematic manner.
- Self evaluative in nature.
- Multiple choice questions for effective preparation.
- Includes Board Question Papers of March and October 2013, 2014, 2015 and March 2016

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Preface

In the case of good books, the point is not how many of them you can get through, but rather how many can get through to you.

"Std. XII Sci. : PERFECT MATHEMATICS - II" is a complete and thorough guide critically analysed and extensively drafted to boost the students confidence. The book is prepared as per the Maharashtra State board syllabus and provides answers to all **textual questions**. At the beginning of every chapter, topic – wise distribution of all textual questions has been provided for simpler understanding of different types of questions. Neatly labelled diagrams have been provided wherever required.

Multiple Choice Questions help the students to test their range of preparation and the amount of knowledge of each topic. Important theories and formulae are the highlights of this book. The steps are written in systematic manner for easy and effective understanding.

The journey to create a complete book is strewn with triumphs, failures and near misses. If you think we've nearly missed something or want to applaud us for our triumphs, we'd love to hear from you.

Please write to us on : mail@targetpublications.org

Best of luck to all the aspirants!

Yours faithfully,
Publisher

PAPER PATTERN

- There will be one single paper of 80 Marks in Mathematics.
- Duration of the paper will be 3 hours.
- Mathematics paper will consist of two parts viz: Part-I and Part-II.
- Each Part will be of 40 Marks.
- Same Answer Sheet will be used for both the parts.
- Each Part will consist of 3 Questions.
- The sequence of the Questions will be determined by the Moderator.
- The paper pattern for Part–I and Part–II will be as follows:

Question 1:

This Question will carry 12 marks and consist of two sub-parts (A) and (B) as follows: **(12 Marks)**

(A) This Question will be based on Multiple Choice Questions.
There will be 3 MCQs, each carrying two marks.

(B) This Question will have 5 sub-questions, each carrying two marks.
Students will have to attempt any 3 out of the given 5 sub-questions.

Question 2:

This Question will carry 14 marks and consist of two sub-parts (A) and (B) as follows: **(14 Marks)**

(A) This Question will have 3 sub-questions, each carrying three marks.
Students will have to attempt any 2 out of the given 3 sub-questions.

(B) This Question will have 3 sub-questions, each carrying four marks.
Students will have to attempt any 2 out of the given 3 sub-questions.

Question 3:

This Question will carry 14 marks and consist of two sub-parts (A) and (B) as follows: **(14 Marks)**

(A) This Question will have 3 sub-questions, each carrying three marks.
Students will have to attempt any 2 out of the given 3 sub-questions.

(B) This Question will have 3 sub-questions, each carrying four marks.
Students will have to attempt any 2 out of the given 3 sub-questions.

Distribution of Marks According to Type of Questions

Type of Questions	Marks	Marks with option	Percentage (%)
Short Answers	24	32	30
Brief Answers	24	36	30
Detailed Answers	32	48	40
Total	**80**	**116**	**100**

Contents

In this book, we have deliberately included the Board Question Papers for March 2013 and October 2013 (Section II) although it follows the old pattern.

* marked questions in the above board papers are deleted from the new syllabus as compared to the earlier syllabus.

01 Continuity

Syllabus

1.0 Introduction (Revision)

1.1 Continuity of a Function at a Point

1.2 Discontinuity of a Function

1.3 Types of Discontinuity

1.4 Algebra of Continuous Functions

1.5 Continuity in an Interval

1.6 Continuity in the Domain of the Function

1.7 Continuity of some Standard Functions

Type of Problems	Exercise	Q. Nos.
Examine the Continuity of Function at a given point	1.1	Q.1
	Miscellaneous	Q.3 (iii,v)
Types of Discontinuity (Removable Discontinuity/Irremovable Discontinuity)	1.1	Q.3
	Miscellaneous	Q.1
Find the Value of Function at given point if it is Continuous	1.1	Q.4
Find Value of k/a/b/α/β if the Function is Continuous at a Given Point	1.1	Q.2,5
	Miscellaneous	Q.2, Q.3(ii)
Examine Continuity of a Function over given Domain/Find points of Discontinuity/Show that given Function is Continous	1.2	Q.1(v, vi, viii, ix), Q.2 (i, ii, iii, iv, v, vii, viii, ix, x)
	Miscellaneous	Q.3 (iv), Q.4(i, iii), Q.5
Find the Value of k/a/b/α/β if the Function is Continuous over a given Domain	1.2	Q.1(i, ii, iii, iv, vii), Q.2 (vi)
	Miscellaneous	Q.3(i), Q.4 (ii)

Introduction (Revision)

In this chapter, we will discuss continuity of a function which is closely related to the concept of limits. There are some functions for which graph is continuous while there are others for which this is not the case.

Limit of a function:

A function f(x) is said to have a limit l as x tends to 'a' if for every $\in > 0$, we can find a positive number δ such that,

$|f(x) - l| < \in$ whenever $0 < |x - a| < \delta$,

If $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x)$,

then the common value is $\lim\limits_{x\to a} f(x)$.

Algebra of limits:

If f(x) and g(x) are any two functions,

i. $\lim\limits_{x\to a} [f(x) + g(x)] = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$

ii. $\lim\limits_{x\to a} [f(x) - g(x)] = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$

iii. $\lim\limits_{x\to a} [f(x)\cdot g(x)] = \lim\limits_{x\to a} f(x)\cdot \lim\limits_{x\to a} g(x)$

iv. $\lim\limits_{x\to a}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$, where $\lim\limits_{x\to a} g(x) \neq 0$

v. $\lim\limits_{x\to a} [k.f(x)] = k\cdot \lim\limits_{x\to a} f(x)$, where k is a constant.

vi. $\lim\limits_{x\to a} [f(x)]^n = \left[\lim\limits_{x\to a} f(x)\right]^n$

vii. $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$

viii. $\lim\limits_{x\to a} \log[f(x)] = \log\left[\lim\limits_{x\to a} f(x)\right]$

ix. $\lim\limits_{x\to a} [f(x)]^{g(x)} = \left[\lim\limits_{x\to a} f(x)\right]^{\lim\limits_{x\to a} g(x)}$

Limits of Algebraic functions:

i. $\lim\limits_{x\to a} x = a$

ii. $\lim\limits_{x\to a} x^n = a^n$

iii. $\lim\limits_{x\to a} k = k$, where k is a constant.

iv. $\lim\limits_{x\to a} \sqrt[r]{x} = \sqrt[r]{a}$

v. If P(x) is a polynomial, then $\lim\limits_{x\to a} P(x) = P(a)$

vi. $\lim\limits_{x\to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$

Limits of Trigonometric functions:

i. $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1 = \lim\limits_{x\to 0} \dfrac{x}{\sin x}$

ii. $\lim\limits_{x\to 0} \dfrac{\tan x}{x} = 1 = \lim\limits_{x\to 0} \dfrac{x}{\tan x}$

iii. $\lim\limits_{x\to 0} \dfrac{\sin^{-1} x}{x} = 1 = \lim\limits_{x\to 0} \dfrac{x}{\sin^{-1} x}$

iv. $\lim\limits_{x\to 0} \dfrac{\tan^{-1} x}{x} = 1 = \lim\limits_{x\to 0} \dfrac{x}{\tan^{-1} x}$

v. $\lim\limits_{x\to 0} \dfrac{\sin x^\circ}{x} = \dfrac{\pi}{180}$

vi. $\lim\limits_{x\to 0} \cos x = 1$

vii. $\lim\limits_{x\to 0} \dfrac{\sin kx}{x} = k$

viii. $\lim\limits_{x\to \infty} \dfrac{\sin x}{x} = \lim\limits_{x\to \infty} \dfrac{\cos x}{x} = 0$

ix. $\lim\limits_{x\to \infty} \dfrac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = 1 = \lim\limits_{x\to \infty} \dfrac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}}$

x. $\lim\limits_{x\to a} \dfrac{\sin(x - a)}{x - a} = 1 = \lim\limits_{x\to a} \dfrac{\tan(x - a)}{x - a}$

Limits of Exponential functions:

i. $\lim\limits_{x\to 0} \frac{a^x - 1}{x} = \log a, (a > 0)$

ii. $\lim\limits_{x\to 0} \frac{e^x - 1}{x} = 1$

iii. $\lim\limits_{x\to 0} (1 + x)^{\frac{1}{x}} = e = \lim\limits_{x\to \infty} \left(1 + \frac{1}{x}\right)^x$

iv. $\lim\limits_{x\to 0} \frac{a^{mx} - 1}{x} = m \log a = \log a^m$

Limits of Logarithmic functions:

i. $\lim\limits_{x\to 0} \frac{\log(1+x)}{x} = 1$

ii. $\lim\limits_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a e, a > 0, a \neq 1$

iii. $\lim\limits_{x\to 0} \frac{\log(1-x)}{x} = -1$

Continuity of a function at a point

Left Hand Limit:

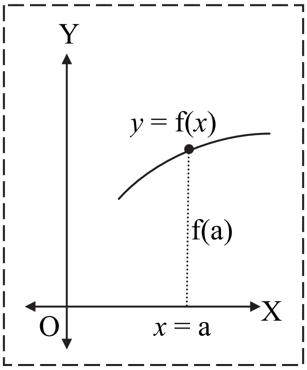
$\lim\limits_{x\to a^-} f(x)$ denotes the limit of f(x) when 'x' approaches to 'a' through values less than 'a'.

$\therefore$ L.H.L. $= \lim\limits_{x\to a^-} f(x) = \lim\limits_{\substack{x\to a\\ x<a}} f(x) = \lim\limits_{h\to 0} f(a - h), (h > 0)$ ….[Left hand limit]

Right Hand Limit:

$\lim\limits_{x\to a^+} f(x)$ denotes the limit of f(x) when 'x' approaches to 'a' through values greater than 'a'.

$\therefore$ R.H.L. $= \lim\limits_{x\to a^+} f(x) = \lim\limits_{\substack{x\to a\\ x>a}} f(x) = \lim\limits_{h\to 0} f(a + h), (h > 0)$ ....[Right hand limit]



$\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ are not always equal.

$\lim\limits_{x\to a} f(x)$ exists, if and only if $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x)$ i.e., L.H.L. = R.H.L.

Graphically, this can be shown as given in the adjoining figure.

Function f is said to be continuous at x = a, if:

i. f(a) exists

ii. $\lim\limits_{x\to a^+} f(x)$ exists

iii. $\lim\limits_{x\to a^-} f(x)$ exists

iv. $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x) = f(a)$

Discontinuity of a Function

f is said to be discontinuous at x = a, if it is not continuous at x = a.

The discontinuity may be due to any of the following reasons:

i. $\lim\limits_{x\to c^-} f(x)$ or $\lim\limits_{x\to c^+} f(x)$ or both may not exist.

ii. $\lim\limits_{x\to c^-} f(x)$ and $\lim\limits_{x\to c^+} f(x)$ both exist but are not equal.

iii. $\lim\limits_{x\to c^-} f(x)$ and $\lim\limits_{x\to c^+} f(x)$ exist and are equal but both may not be equal to f(c).

Consider the function defined by

f(x) = 1 for $1 \le x \le 2$ = 2 for $2 < x \le 3$

Here, f(x) is defined at every point in [1, 3].

Graph of this function is as shown adjacently.

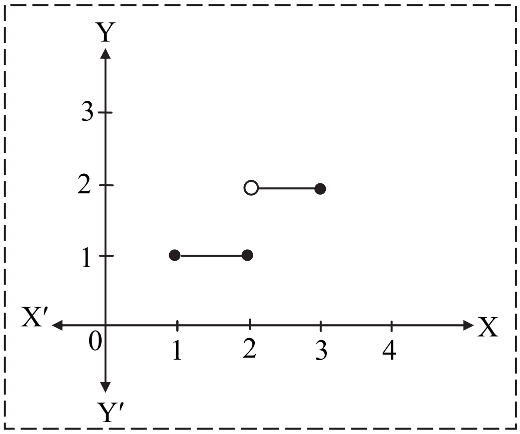
Left hand limit at $x = 2$ and value of f(x) at $x = 2$ are both equal to 1.

But right hand limit at $x = 2$ equals 2, which does not coincides with the common value of the left hand limit and f(2).

Again, we can not draw the continuous (without break) graph at $x = 2$.

Hence, we say that the function f(x) is not continuous at $x = 2$.

Here, we say that f(x) is discontinuous at $x = 2$ and $x = 2$ is the point of discontinuity.



Types of Discontinuity:

i. Removable discontinuity:

A real valued function f is said to have a removable discontinuity at x = c in its domain, if $\lim\limits_{x\to c} f(x)$ exists but

$\lim\limits_{x\to c} f(x) \ne f(c)$

i.e., if $\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x) \ne f(c)$

Eg.

Consider the following function,

$$f(x) = \frac{x^2-16}{x-4}, \quad x \ne 4$$
$$= 5 \quad , x = 4$$

Here f (4) = 5

$$\lim_{x\to 4} f(x) = \lim_{x\to 4} \frac{x^2-16}{x-4} = \lim_{x\to 4} \frac{(x-4)(x+4)}{(x-4)}$$
$$= \lim_{x\to 4} x + 4 = 4 + 4 = 8 \ne f(4)$$

$\therefore$ f is discontinuous at $x = 4$.

Now, let us find why f(x) is discontinuous at $x = 4$.

In the above function $\lim\limits_{x\to 4} f(x)$ exist, but is not equal to f(4) since f(4) = 5.

This value of f (4) is just arbitrarily defined.

Suppose, we redefine f(x) as follows:

$$f(x) = \frac{x^2-16}{x-4}, \quad x \ne 4$$
$$= 8 \quad , x = 4$$

Then f(x) becomes continuous at $x = 4$.

The discontinuity of f has been removed by redefining the function suitably. Note that we have not appreciably changed the function but redefined it by changing its value at one point only. Such a discontinuity is called a removable discontinuity.

This type of discontinuity can be removed by redefining function f(x) at x = c such that f(c) = $\lim\limits_{x\to c} f(x)$.

ii. Irremovable discontinuity:

A real valued function f is said to have an irremovable discontinuity at x = c in its domain, if $\lim\limits_{x\to c^-} f(x) \ne \lim\limits_{x\to c^+} f(x)$

i.e., if $\lim\limits_{x\to c} f(x)$ does not exist.

Such function cannot be redefined to make it continuous.

Exercise 1.1

1. Examine the continuity of the following functions at given points

i. $f(x) = \dfrac{e^{5x} - e^{2x}}{\sin 3x}$, for $x \neq 0$
$= 1$, for $x = 0$ } at $x = 0$

ii. $f(x) = \dfrac{\log 100 + \log(0.01 + x)}{3x}$, for $x \neq 0$
$= \dfrac{100}{3}$, for $x = 0$ } at $x = 0$

iii. $f(x) = \dfrac{x^n - 1}{x - 1}$, for $x \neq 1$
$= n^2$, for $x = 1$ } at $x = 1$

iv. $f(x) = \dfrac{\log x - \log 7}{x - 7}$, for $x \neq 7$
$= 7$, for $x = 7$ } at $x = 7$

v. $f(x) = (1 + 2x)^{\frac{1}{x}}$, for $x \neq 0$
$= e^2$, for $x = 0$ } at $x = 0$

vi. $f(x) = \dfrac{10^x + 7^x - 14^x - 5^x}{1 - \cos 4x}$, for $x \neq 0$
$= \dfrac{10}{7}$, for $x = 0$ } at $x = 0$ [Oct 13]

vii. $f(x) = \sin x - \cos x$, for $x \neq 0$
$= -1$, for $x = 0$ } at $x = 0$ [Mar 14]

viii. $f(x) = \dfrac{\log(2 + x) - \log(2 - x)}{\tan x}$, for $x \neq 0$
$= 1$, for $x = 0$ } at $x = 0$

ix. $f(x) = \dfrac{1 - \sin x}{\left(\dfrac{\pi}{2} - x\right)^2}$, for $x \neq \dfrac{\pi}{2}$
$= 3$, for $x = \dfrac{\pi}{2}$ } at $x = \dfrac{\pi}{2}$ [Oct 15]

x. $f(x) = \dfrac{x}{|x|}$, for $x \neq 0$
$= c$, for $x = 0$ } at $x = 0$ (where c is arbitrary constant)

xi. $f(x) = x$, for $0 \leq x < \dfrac{1}{2}$
$= 1 - x$, for $\dfrac{1}{2} \leq x < 1$ } at $x = \dfrac{1}{2}$

xii. $$f(x) = \begin{cases} \dfrac{x^2-9}{x-3}, & \text{for } 0 < x < 3 \\ = x+3, & \text{for } 3 \le x < 6 \\ = \dfrac{x^2-9}{x+3}, & \text{for } 6 \le x < 9 \end{cases} \quad \text{at } x = 3 \text{ and } x = 6$$

xiii. $$f(x) = \begin{cases} \dfrac{\sin 2x}{\sqrt{1-\cos 2x}}, & \text{for } 0 < x \le \dfrac{\pi}{2} \\ = \dfrac{\cos x}{\pi - 2x}, & \text{for } \dfrac{\pi}{2} < x < \pi \end{cases} \quad \text{at } x = \dfrac{\pi}{2}$$

xiv. $$f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}, & \text{for } x \ne 0 \\ = 1, & \text{for } x = 0 \end{cases} \quad \text{at } x = 0$$

xv. $$f(x) = \begin{cases} \dfrac{x+3x^2+5x^3+....+(2n-1)x^n-n^2}{(x-1)}, & \text{for } x \ne 1 \\ = \dfrac{n(n+1)(4n-1)}{6}, & \text{for } x = 1 \end{cases} \quad \text{at } x = 1$$

***Solution*:**

i. $f(0) = 1$ ….(given)

$$\lim_{x\to0} f(x) = \lim_{x\to0} \frac{e^{5x}-e^{2x}}{\sin 3x} = \lim_{x\to0} \frac{e^{2x}\left(e^{3x}-1\right)}{\sin 3x}$$

$$= \lim_{x\to0} \frac{e^{2x}\left(\dfrac{e^{3x}-1}{3x}\right)}{\dfrac{\sin 3x}{3x}} = e^{2(0)}\cdot\frac{1}{1} \qquad \ldots\left[\because \lim_{x\to0}\frac{e^x-1}{x}=1,\ \lim_{x\to0}\frac{\sin x}{x}=1\right]$$

$$= e^0 = 1 = f(0)$$

Since, $\lim\limits_{x\to0} f(x) = f(0)$, f is continuous at $x = 0$.

ii. $f(0) = \dfrac{100}{3}$ ….(given)

$$\lim_{x\to0} f(x) = \lim_{x\to0} \frac{\log 100 + \log(0.01+x)}{3x}$$

$$= \lim_{x\to0} \frac{\log(100\times0\cdot01+100x)}{3x} = \lim_{x\to0}\frac{\log(1+100x)}{3x}$$

$$= \frac{100}{3}\lim_{x\to0}\frac{\log(1+100x)}{100x}$$

$$= \frac{100}{3}(1) \qquad \ldots\left[\because \lim_{x\to0}\frac{\log(1+x)}{x}=1\right]$$

$$= \frac{100}{3} = f(0)$$

Since, $\lim\limits_{x\to0} f(x) = f(0)$, f is continuous at $x = 0$.

iii. $f(1) = n^2$ ....(given)

$$\lim_{x\to1} f(x) = \lim_{x\to1} \frac{x^n - 1}{x - 1}$$

$$= n(1)^{n-1} \qquad \ldots\left[\because \lim_{x\to a}\frac{x^n - a^n}{x - a} = n(a)^{n-1}\right]$$

$$= n$$

Since, $\lim_{x\to1} f(x) \neq f(1)$, f is discontinuous at $x = 1$.

iv. $f(7) = 7$ ....(given)

$$\lim_{x\to7} f(x) = \lim_{x\to7} \frac{\log x - \log 7}{x - 7}$$

Put $x - 7 = h$, then $x = 7 + h$, as $x \to 7$, $h \to 0$

$$\therefore \quad \lim_{x\to7} f(x) = \lim_{h\to0} \frac{\log(h+7) - \log 7}{h} = \lim_{h\to0} \frac{\log\left(\frac{h+7}{7}\right)}{h}$$

$$= \lim_{h\to0} \frac{\log\left(1+\frac{h}{7}\right)}{\left(\frac{h}{7}\right)} \times \frac{1}{7} = \frac{1}{7}\lim_{h\to0} \frac{\log\left(1+\frac{h}{7}\right)}{\left(\frac{h}{7}\right)} = \frac{1}{7}(1) \qquad \ldots\left[\because \lim_{x\to0}\frac{\log(1+x)}{x} = 1\right]$$

$$= \frac{1}{7} \neq f(7)$$

Since, $\lim_{x\to7} f(x) \neq f(7)$, f is discontinuous at $x = 7$.

v. $f(0) = e^2$ ....(given)

$$\lim_{x\to0} f(x) = \lim_{x\to0} (1+2x)^{\frac{1}{x}} = \lim_{x\to0}\left[(1+2x)^{\frac{1}{2x}}\right]^2$$

$$= \left[\lim_{x\to0}(1+2x)^{\frac{1}{2x}}\right]^2$$

$$= e^2 \qquad \ldots\left[\because \lim_{x\to0}(1+x)^{\frac{1}{x}} = e\right]$$

$$= f(0)$$

Since, $\lim_{x\to0} f(x) = f(0)$, f is continuous at $x = 0$.

vi. $f(0) = \frac{10}{7}$ ....(given)

$$\lim_{x\to0} f(x) = \lim_{x\to0} \frac{10^x + 7^x - 14^x - 5^x}{1 - \cos 4x} = \lim_{x\to0}\frac{2^x.5^x - 5^x - 7^x.2^x + 7^x}{2\sin^2 2x} = \lim_{x\to0}\frac{5^x(2^x - 1) - 7^x(2^x - 1)}{2\sin^2 2x}$$

$$= \lim_{x\to0}\frac{(2^x - 1)(5^x - 7^x)}{2\sin^2 2x} = \lim_{x\to0}\frac{\frac{(2^x - 1)(5^x - 7^x)}{x^2}}{\frac{2\sin^2 2x}{x^2}} = \lim_{x\to0}\frac{\frac{(2^x - 1)}{x}\left(\frac{5^x - 1}{x} - \frac{7^x - 1}{x}\right)}{2\times\frac{4}{4}\frac{\sin^2 2x}{x^2}}$$

$$= \frac{\left(\lim_{x\to0}\frac{2^x - 1}{x}\right)\left(\lim_{x\to0}\frac{5^x - 1}{x} - \lim_{x\to0}\frac{7^x - 1}{x}\right)}{8\lim_{x\to0}\left(\frac{\sin 2x}{2x}\right)^2} = \frac{\log 2(\log 5 - \log 7)}{8(1)^2} = \frac{\log 2\left(\log\frac{5}{7}\right)}{8} \neq f(0)$$

Since, $\lim_{x\to0} f(x) \neq f(0)$, f is discontinuous at $x = 0$.

vii. $f(0) = -1$ ….(given)

$\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} (\sin x - \cos x) = \lim\limits_{x\to 0} \sin x - \lim\limits_{x\to 0} \cos x = \sin 0 - \cos 0 = 0 - 1 = -1 = f(0)$

Since, $\lim\limits_{x\to 0} f(x) = f(0)$, f is continuous at $x = 0$.

viii. $f(0) = 1$ ….(given)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\log(2+x) - \log(2-x)}{\tan x}$$

$$= \lim_{x\to 0} \frac{\log\left(\frac{2+x}{2-x}\right)}{\tan x}$$

$$= \lim_{x\to 0} \frac{\log\left(\dfrac{1+\frac{x}{2}}{1-\frac{x}{2}}\right)}{\tan x}$$

$$= \lim_{x\to 0} \frac{\log\left(1+\frac{x}{2}\right) - \log\left(1-\frac{x}{2}\right)}{\tan x}$$

$$= \lim_{x\to 0} \left[\frac{\dfrac{\log\left(1+\frac{x}{2}\right) - \log\left(1-\frac{x}{2}\right)}{x}}{\dfrac{\tan x}{x}}\right]$$

$$= \frac{\lim\limits_{x\to 0}\dfrac{\log\left(1+\frac{x}{2}\right)}{\left(\frac{x}{2}\right)\times 2} - \lim\limits_{x\to 0}\dfrac{\log\left(1-\frac{x}{2}\right)}{\left(\frac{x}{-2}\right)\times(-2)}}{\lim\limits_{x\to 0}\dfrac{\tan x}{x}}$$

$$= \frac{\frac{1}{2}\lim\limits_{x\to 0}\dfrac{\log\left(1+\frac{x}{2}\right)}{\frac{x}{2}} + \frac{1}{2}\lim\limits_{x\to 0}\dfrac{\log\left(1-\frac{x}{2}\right)}{\left(\frac{-x}{2}\right)}}{1}$$

$$= \frac{1}{2}(1) + \frac{1}{2}(1) \qquad \ldots\left[\because \lim_{x\to 0}\frac{\log(1+x)}{x} = 1\right]$$

$= 1$

Since, $\lim\limits_{x\to 0} f(x) = f(0)$, f is continuous at $x = 0$.

ix. $f\left(\frac{\pi}{2}\right) = 3$ ….(given)

$$\lim_{x\to \frac{\pi}{2}} f(x) = \lim_{x\to \frac{\pi}{2}} \frac{1-\sin x}{\left(\frac{\pi}{2} - x\right)^2}$$

Put $\frac{\pi}{2} - x = \text{h}$, then as $x \to \frac{\pi}{2}$, $\text{h} \to 0$

$\therefore$ $\lim_{x\to\frac{\pi}{2}} \text{f}(x) = \lim_{\text{h}\to 0} \frac{1-\sin\left(\frac{\pi}{2}-\text{h}\right)}{\text{h}^2} = \lim_{\text{h}\to 0} \frac{1-\cos \text{h}}{\text{h}^2} = \lim_{\text{h}\to 0} \frac{2\sin^2\frac{\text{h}}{2}}{\frac{4}{4}\text{h}^2} = \lim_{\text{h}\to 0} \frac{1}{2}\left(\frac{\sin^2\frac{\text{h}}{2}}{\frac{\text{h}^2}{4}}\right) = \frac{1}{2}\left(\lim_{\text{h}\to 0} \frac{\sin\frac{\text{h}}{2}}{\frac{\text{h}}{2}}\right)^2 = \frac{1}{2}(1)^2$

$= \frac{1}{2} \neq \text{f}\left(\frac{\pi}{2}\right)$

Since, $\lim_{x\to\frac{\pi}{2}} \text{f}(x) \neq \text{f}\left(\frac{\pi}{2}\right)$, f is discontinuous at $x = \frac{\pi}{2}$.

x. $\text{f}(x) = \frac{x}{|x|}$ ….(given)

Thus, $|x| = x$, if $x \to 0^+$ $= -x$, if $x \to 0^-$

$\therefore$ $\text{f}(x) = \frac{x}{|x|} = 1$, if $x \to 0^+ = -1$, if $x \to 0^-$

Now, $\lim_{x\to 0^-} \text{f}(x) = \lim_{x\to 0^-} (-1) = -1$

and $\lim_{x\to 0^+} \text{f}(x) = \lim_{x\to 0^+} (1) = 1$

Since, $\lim_{x\to 0^-} \text{f}(x) \neq \lim_{x\to 0^+} \text{f}(x)$, f is discontinuous at $x = 0$.

xi. $\text{f}\left(\frac{1}{2}\right) = 1 - x = 1 - \frac{1}{2} = \frac{1}{2}$

$\lim_{x\to\frac{1}{2}^-} \text{f}(x) = \lim_{x\to\frac{1}{2}^-} x = \frac{1}{2}$

$\lim_{x\to\frac{1}{2}^+} \text{f}(x) = \lim_{x\to\frac{1}{2}^+} (1-x) = 1 - \frac{1}{2} = \frac{1}{2}$

Since, $\lim_{x\to\frac{1}{2}^-} \text{f}(x) = \lim_{x\to\frac{1}{2}^+} \text{f}(x) = \text{f}\left(\frac{1}{2}\right)$, f is continuous at $x = \frac{1}{2}$.

xii. **Case 1:**

When $x = 3$, $\text{f}(3) = x + 3 = 3 + 3 = 6$

$\lim_{x\to 3^-} \text{f}(x) = \lim_{x\to 3^-} \left(\frac{x^2-9}{x-3}\right) = \lim_{x\to 3^-} (x+3) = 6$

$\lim_{x\to 3^+} \text{f}(x) = \lim_{x\to 3^+} (x+3) = 6$

Since, $\lim_{x\to 3^-} \text{f}(x) = \lim_{x\to 3^+} \text{f}(x) = \text{f}(3)$, f is continuous at $x = 3$.

Case 2:

When $x = 6$, $\text{f}(6) = \frac{x^2-9}{x+3} = x - 3 = 6 - 3 = 3$

$\lim_{x\to 6^-} \text{f}(x) = \lim_{x\to 6^-} (x+3) = 6 + 3 = 9$

$\lim_{x\to 6^+} \text{f}(x) = \lim_{x\to 6^+} \left(\frac{x^2-9}{x+3}\right) = \lim_{x\to 6^+} (x-3) = 6 - 3 = 3$

Since, $\lim_{x\to 6^-} \text{f}(x) \neq \lim_{x\to 6^+} \text{f}(x) \neq \text{f}(6)$, f is discontinuous at $x = 6$.

xiii. $f\left(\frac{\pi}{2}\right) = \frac{\sin 2x}{\sqrt{1-\cos 2x}} = \frac{\sin 2\left(\frac{\pi}{2}\right)}{\sqrt{1-\cos 2\left(\frac{\pi}{2}\right)}} = \frac{\sin\pi}{\sqrt{1-\cos\pi}} = \frac{0}{\sqrt{1-(-1)}} = \frac{0}{\sqrt{2}} = 0$

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}} \frac{\sin 2x}{\sqrt{1-\cos 2x}} = \lim_{x\to\frac{\pi}{2}} \frac{2\sin x\cos x}{\sqrt{2\sin^2 x}} = \lim_{x\to\frac{\pi}{2}} \sqrt{2}\cos x = \sqrt{2}\,(0) = 0$$

$$\lim_{x\to\frac{\pi}{2}^+} f(x) = \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{\pi - 2x}$$

Put $x = \frac{\pi}{2} + h$, then as $x \to \frac{\pi}{2}$, $h \to 0$

$$\therefore \lim_{x\to\frac{\pi}{2}^+} f(x) = \lim_{h\to 0} \frac{\cos\left(\frac{\pi}{2}+h\right)}{\pi - 2\left(\frac{\pi}{2}+h\right)} = \lim_{h\to 0} \frac{-\sin h}{\pi-\pi-2h} = \lim_{h\to 0} \frac{1}{2}\left(\frac{\sin h}{h}\right) = \frac{1}{2}\left(\lim_{h\to 0}\frac{\sin h}{h}\right) = \frac{1}{2}(1) = \frac{1}{2}$$

Since, $\lim_{x\to\frac{\pi}{2}^-} f(x) \neq \lim_{x\to\frac{\pi}{2}^+} f(x) \neq f\left(\frac{\pi}{2}\right)$, f is discontinuous at $x = \frac{\pi}{2}$.

xiv. $f(0) = 1$ ....(given)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$$

As $x \to 0^-$, $\frac{1}{x} \to -\infty$, thus $e^{\frac{1}{x}} \to e^{-\infty} = \frac{1}{e^{\infty}}$ i.e., $e^{\frac{1}{x}} \to \frac{1}{\infty}$ i.e., $e^{\frac{1}{x}} \to 0$

$$\therefore \lim_{x\to 0^-} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} = \frac{0-1}{0+1} = -1$$

Also, as $x \to 0^+$, $\frac{1}{x} \to +\infty$, thus $e^{\frac{1}{x}} \to e^{\infty}$ i.e., $e^{\frac{1}{x}} \to \infty$ i.e., $\frac{1}{e^{\frac{1}{x}}} \to 0$

$$\therefore \lim_{x\to 0^+} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} = \lim_{\frac{1}{x}\to\infty} \frac{1-\left(\frac{1}{e^{\frac{1}{x}}}\right)}{1+\left(\frac{1}{e^{\frac{1}{x}}}\right)} = \frac{1-0}{1+0} = 1$$

Since, $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$, f is discontinuous at $x = 0$.

xv. $f(1) = \frac{n(n+1)(4n-1)}{6}$ ....(given)

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x+3x^2+5x^3+\ldots+(2n-1)x^n - n^2}{(x-1)}$$

$$= \lim_{x\to 1} \frac{\left[x+3x^2+5x^3+\ldots+(2n-1)x^n\right]-\left[1+3+5+\ldots+(2n-1)\right]}{(x-1)} \qquad \ldots\left[\because \sum_{r=1}^{n}(2r-1) = r^2\right]$$

$$= \lim_{x \to 1} \frac{\left[(x-1)+3(x^2-1)+5(x^3-1)+....+(2n-1)(x^n-1)\right]}{(x-1)}$$

$$= \lim_{x \to 1} \left[\frac{x-1}{x-1}+\frac{3(x^2-1)}{x-1}+\frac{5(x^3-1)}{x-1}+....+(2n-1)\frac{(x^n-1)}{x-1}\right]$$

$$= \lim_{x \to 1} 1+3\left(\lim_{x \to 1}\frac{x^2-1}{x-1}\right)+5\left(\lim_{x \to 1}\frac{x^3-1}{x-1}\right)+....+(2n-1)\left(\lim_{x \to 1}\frac{x^n-1}{x-1}\right)$$

$$= 1 + 3(2) + 5(3) + ... + (2n-1)(n) \qquad .... \left[\because \lim_{x \to 1}\frac{x^n-1}{x-1} = n\right]$$

$$= \sum_{r=1}^{n}(2r-1)r = \sum_{r=1}^{n}(2r^2-r) = 2\sum_{r=1}^{n}r^2 - \sum_{r=1}^{n}r = 2\cdot\frac{n}{6}(n+1)(2n+1) - \frac{n}{2}(n+1)$$

$$= \frac{n}{2}(n+1)\left[\frac{2}{3}(2n+1)-1\right] = \frac{n}{2}(n+1)\frac{(4n+2-3)}{3} = \frac{n}{6}(n+1)(4n-1)$$

Since, $\lim_{x \to 1} f(x) = f(1)$, f is continuous at $x = 1$.

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2. Find the value of k, so that the function f(x) is continuous at the indicated point

i. $f(x) = \dfrac{(e^{kx}-1)\sin kx}{x^2}$, for $x \neq 0$
$= 4$, for $x = 0$ } at $x = 0$

ii. $f(x) = \dfrac{3^x - 3^{-x}}{\sin x}$, for $x \neq 0$
$= k$, for $x = 0$ } at $x = 0$

iii. $f(x) = |x-3|$, for $x \neq 3$
$= k$, for $x = 3$ } at $x = 3$

iv. $f(x) = x^2 + 1$, for $x \geq 0$
$= 2\sqrt{x^2+1} + k$, for $x < 0$ } at $x = 0$

v. $f(x) = \dfrac{1-\cos 4x}{x^2}$, for $x < 0$
$= k$, for $x = 0$
$= \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}$, for $x > 0$ } at $x = 0$

vi. $f(x) = \dfrac{\log(1+kx)}{\sin x}$, for $x \neq 0$
$= 5$, for $x = 0$ } at $x = 0$

vii. $f(x) = \dfrac{8^x - 2^x}{k^x - 1}$, for $x \neq 0$
$= 2$, for $x = 0$ } at $x = 0$

viii. $f(x) = k(x^2 - 2)$, for $x \leq 0$
$= 4x + 1$, for $x > 0$ } at $x = 0$

ix. $f(x) = (\sec^2 x)^{\cot^2 x}$, for $x \neq 0$
$= k$, for $x = 0$ } at $x = 0$

x. $f(x) = \dfrac{\sqrt{3}-\tan x}{\pi - 3x}$, for $x \neq \dfrac{\pi}{3}$
$= k$, for $x = \dfrac{\pi}{3}$ } at $x = \dfrac{\pi}{3}$

***Solution*:**

i. $f(0) = 4$ ….(given)

Since, $f(x)$ is continuous at $x = 0$

$\therefore$ $f(0) = \lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{(e^{kx}-1)\sin kx}{x^2} = \lim\limits_{x\to 0}\left(\dfrac{e^{kx}-1}{kx}\right)k \times \left(\dfrac{\sin kx}{kx}\right)k = k^2\left(\lim\limits_{x\to 0}\dfrac{e^{kx}-1}{kx}\right)\left(\lim\limits_{x\to 0}\dfrac{\sin kx}{kx}\right)$

$= k^2(1)(1) = k^2$

$\therefore$ $f(0) = k^2$ $\therefore$ $4 = k^2$

$\therefore$ $k = \pm 2$

ii. $f(0) = k$ ….(given)

Since, $f(x)$ is continuous at $x = 0$

$\therefore$ $f(0) = \lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\dfrac{3^x - 3^{-x}}{\sin x} = \lim\limits_{x\to 0}\dfrac{3^x - 1 - 3^{-x} + 1}{\sin x} = \lim\limits_{x\to 0}\dfrac{\dfrac{(3^x-1)-(3^{-x}-1)}{x}}{\dfrac{\sin x}{x}}$

$= \lim\limits_{x\to 0}\dfrac{\left(\dfrac{3^x-1}{x}\right)+\left(\dfrac{3^{-x}-1}{-x}\right)}{\dfrac{\sin x}{x}} = \dfrac{\left(\lim\limits_{x\to 0}\dfrac{3^x-1}{x}\right)+\left(\lim\limits_{x\to 0}\dfrac{3^{-x}-1}{-x}\right)}{\left(\lim\limits_{x\to 0}\dfrac{\sin x}{x}\right)} = \dfrac{(\log 3)+(\log 3)}{1} = 2\log 3$

$\therefore$ $k = 2\log 3 = \log 3^2 = \log 9$

iii. $f(3) = k$ ….(given)

Since, $f(x)$ is continuous at $x = 3$

$\therefore$ $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} |x-3| = \lim\limits_{x\to 3^-} -(x-3) = \lim\limits_{x\to 3^-} -x + 3 = -3 + 3 = 0$

$\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} |x-3| = \lim\limits_{x\to 3^+} x - 3 = 3 - 3 = 0$

Since, $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x)$, f is continuous at $x = 3$.

Now, $f(3) = \lim\limits_{x\to 3} f(x) = 0$

$\therefore$ $k = 0$

iv. Since, $f(x)$ is continuous at $x = 0$

$\therefore$ $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x)$

$\therefore$ $\lim\limits_{x\to 0^-}\left(2\sqrt{x^2+1}+k\right) = \lim\limits_{x\to 0^+}(x^2+1)$

$\therefore$ $2\sqrt{0+1} + k = 0 + 1$

$\therefore$ $2 + k = 1$

$\therefore$ $k = -1$

v. $f(0) = k$ ….(given)

Since, $f(x)$ is continuous at $x = 0$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = f(0)$

Consider, $f(0) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1-\cos 4x}{x^2} = \lim_{x\to 0^-} \frac{2\sin^2 2x}{x^2}$

$$= \lim_{x\to 0^-} \frac{2\sin^2 2x}{\frac{4x^2}{4}} = \lim_{x\to 0^-} 8\,\frac{\sin^2 2x}{4x^2} = 8\left(\lim_{x\to 0^-}\frac{\sin 2x}{2x}\right)^2$$

∴ $f(0) = 8$

∴ $k = 8$

vi. $f(0) = 5$ ....(given)

Since, $f(x)$ is continuous at $x = 0$

∴ $f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\log(1+kx)}{\sin x} = \lim_{x\to 0}\frac{\frac{\log(1+kx)}{x}}{\frac{\sin x}{x}} = \frac{\lim_{x\to 0}\frac{k.\log(1+kx)}{kx}}{\lim_{x\to 0}\frac{\sin x}{x}} = \frac{k\left(\lim_{x\to 0}\frac{\log(1+kx)}{kx}\right)}{\left(\lim_{x\to 0}\frac{\sin x}{x}\right)} = k\left(\frac{1}{1}\right) = k$

∴ $f(0) = k$

∴ $k = 5$

vii. $f(0) = 2$ ....(given)

Since, $f(x)$ is continuous at $x = 0$

∴ $f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{8^x - 2^x}{k^x - 1} = \lim_{x\to 0}\frac{8^x - 1 - 2^x + 1}{k^x - 1} = \lim_{x\to 0}\frac{\frac{(8^x-1)-(2^x-1)}{x}}{\frac{k^x-1}{x}}$

$$= \lim_{x\to 0}\frac{\left(\frac{8^x-1}{x} - \frac{2^x-1}{x}\right)}{\left(\frac{k^x-1}{x}\right)} = \frac{\left(\lim_{x\to 0}\frac{8^x-1}{x}\right) - \left(\lim_{x\to 0}\frac{2^x-1}{x}\right)}{\left(\lim_{x\to 0}\frac{k^x-1}{x}\right)}$$

$$= \frac{\log 8 - \log 2}{\log k} = \frac{\log\left(\frac{8}{2}\right)}{\log k} = \frac{\log 4}{\log k} = \frac{\log 2^2}{\log k} = \frac{2\log 2}{\log k}$$

∴ $\frac{2\log 2}{\log k} = 2$

∴ $\log 2 = \log k$

∴ $k = 2$

viii. Since, f is continuous at $x = 0$

∴ $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

∴ $\lim_{x\to 0^-} k(x^2 - 2) = \lim_{x\to 0^+} 4x + 1$

∴ $k(0 - 2) = 4(0) + 1$

∴ $-2k = 1$

∴ $k = -\frac{1}{2}$

ix. $f(0) = k$ ....(given)

Since, $f(x)$ is continuous at $x = 0$

∴ $f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0}(\sec^2 x)^{\cot^2 x} = \lim_{x\to 0}(1+\tan^2 x)^{\frac{1}{\tan^2 x}} = e$ .... $\left[\because \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e\right]$

∴ $k = e$

x. $f\left(\frac{\pi}{3}\right) = k$ ....(given)

Since, f(x) is continuous at $x = \frac{\pi}{3}$

$\therefore$ $f\left(\frac{\pi}{3}\right) = \lim_{x\to\frac{\pi}{3}} f(x) = \lim_{x\to\frac{\pi}{3}} \frac{\sqrt{3}-\tan x}{\pi - 3x}$

Put $x = \frac{\pi}{3} + h$, then, as $x \to \frac{\pi}{3}$, $h \to 0$

$$= \lim_{h\to 0} \frac{\sqrt{3}-\tan\left(\frac{\pi}{3}+h\right)}{\pi - 3\left(\frac{\pi}{3}+h\right)} = \lim_{h\to 0} \frac{\sqrt{3}-\frac{\tan\frac{\pi}{3}+\tan h}{1-\tan\frac{\pi}{3}\tan h}}{\pi-\pi-3h} = \lim_{h\to 0} \frac{\sqrt{3}-\frac{\sqrt{3}+\tan h}{1-\sqrt{3}\tan h}}{-3h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3}\left(1-\sqrt{3}\tan h\right)-\left(\sqrt{3}+\tan h\right)}{-3h\left(1-\sqrt{3}\tan h\right)} = \lim_{h\to 0} \frac{\sqrt{3}-3\tan h-\sqrt{3}-\tan h}{-3h\left(1-\sqrt{3}\tan h\right)} = \lim_{h\to 0} \frac{-4\tan h}{-3h\left(1-\sqrt{3}\tan h\right)}$$

$$= \lim_{h\to 0} \frac{4}{3\left(1-\sqrt{3}\tan h\right)} \times \frac{\tan h}{h} = \frac{4}{3}\left(\lim_{h\to 0}\frac{1}{1-\sqrt{3}\tan h}\right)\left(\lim_{h\to 0}\frac{\tan h}{h}\right) = \frac{4}{3}\left[\frac{1}{1-\sqrt{3}(0)}\right](1) = \frac{4}{3}$$

$\therefore$ $k = \frac{4}{3}$

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3. Discuss the continuity of the following functions, which of these functions have a removable discontinuity? Redefine the function so as to remove the discontinuity.

i. $f(x) = \frac{\sin\left(x^2 - x\right)}{x}$, for $x \neq 0$
$= 2$, for $x = 0$ } at $x = 0$

ii. $f(x) = \frac{1-\cos 3x}{x\tan x}$, for $x \neq 0$
$= 9$, for $x = 0$ } at $x = 0$

iii. $f(x) = \frac{\left(e^{2x}-1\right)\tan x}{x\sin x}$, for $x \neq 0$
$= e^2$, for $x = 0$ } at $x = 0$

iv. $f(x) = \frac{1-\sin x}{(\pi - 2x)^2}$, for $x \neq \frac{\pi}{2}$
$= \frac{2}{7}$, for $x = \frac{\pi}{2}$ } at $x = \frac{\pi}{2}$

v. $f(x) = \frac{4^x - e^x}{6^x - 1}$, for $x \neq 0$
$= \log\left(\frac{2}{3}\right)$, for $x = 0$ } at $x = 0$

vi. $f(x) = \dfrac{\left(e^{3x} - 1\right)\sin x^\circ}{x^2}$, for $x \neq 0$
$= \dfrac{\pi}{60}$, for $x = 0$
} at $x = 0$

vii. $f(x) = \dfrac{(8^x - 1)^2}{\sin x \log\left(1 + \dfrac{x}{4}\right)}$, in $[-1, 1] - \{0\}$;

Define f(x) in [–1, 1] so that it becomes continuous at $x = 0$.

viii. $f(x) = x - 1$, for $1 \le x < 2$
$= 2x + 3$, for $2 \le x \le 3$
} at $x = 2$

***Solution*:**

i. $f(0) = 2$ ….(given)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\sin(x^2 - x)}{x} = \lim_{x\to 0}\frac{\sin(x^2 - x)}{x(x-1)} \times (x-1) = \lim_{x\to 0}\frac{\sin(x^2 - x)}{(x^2 - x)} \times (x-1) = 1 \times (0 - 1) = -1$$

$\therefore$ $\lim\limits_{x\to 0} f(x) \neq f(0)$

$\therefore$ f is discontinuous at $x = 0$.

The discontinuity of f is removable and it can be made continuous by redefining the function as

$$f(x) = \begin{cases} \dfrac{\sin\left(x^2 - x\right)}{x}, & \text{for } x \neq 0 \\ -1, & \text{for } x = 0 \end{cases} \text{ at } x = 0$$

ii. $f(0) = 9$ ….(given)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{1 - \cos 3x}{x\tan x} = \lim_{x\to 0}\frac{2\sin^2\frac{3x}{2}}{x\tan x} = \lim_{x\to 0}\frac{\dfrac{2\sin^2\frac{3x}{2}}{x^2}}{\dfrac{x\tan x}{x^2}}$$

$$= \lim_{x\to 0}\frac{\dfrac{2\sin^2\frac{3x}{2} \times \frac{9}{4}}{\frac{9}{4}x^2}}{\dfrac{\tan x}{x}} = \frac{\dfrac{9}{2}\left(\lim\limits_{x\to 0}\dfrac{\sin\frac{3x}{2}}{\frac{3x}{2}}\right)^2}{\lim\limits_{x\to 0}\dfrac{\tan x}{x}} = \frac{9}{2}\frac{(1)^2}{1} = \frac{9}{2}$$

$\therefore$ $\lim\limits_{x\to 0} f(x) \neq f(0)$

$\therefore$ f is discontinuous at $x = 0$.

The discontinuity of f is removable and it can be made continuous by redefining the function as

$$f(x) = \begin{cases} \dfrac{1 - \cos 3x}{x\tan x}, & \text{for } x \neq 0 \\ \dfrac{9}{2}, & \text{for } x = 0 \end{cases} \text{ at } x = 0$$

iii. $f(0) = e^2$ ….(given)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{(e^{2x}-1)\tan x}{x\sin x} = \lim_{x\to 0}\frac{\frac{(e^{2x}-1).\tan x}{x^2}}{\frac{x\sin x}{x^2}}$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\frac{2(e^{2x}-1)}{2x}\times\frac{\tan x}{x}}{\frac{x\sin x}{x^2}} = \frac{2\left(\lim_{x\to 0}\frac{e^{2x}-1}{2x}\right)\left(\lim_{x\to 0}\frac{\tan x}{x}\right)}{\left(\lim_{x\to 0}\frac{\sin x}{x}\right)} = \frac{2\times 1\times 1}{1} = 2$$

$\therefore$ $\lim_{x\to 0} f(x) \neq f(0)$

$\therefore$ f is discontinuous at $x = 0$.

The discontinuity of f is removable and it can be made continuous by redefining the function as

$$\left.\begin{aligned} f(x) &= \frac{\left(e^{2x}-1\right)\tan x}{x\sin x} && \text{for } x \neq 0 \\ &= 2 && \text{for } x = 0 \end{aligned}\right\} \text{at } x = 0$$

iv. $f\left(\frac{\pi}{2}\right) = \frac{2}{7}$ ….(given)

$$\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{(\pi-2x)^2} = \lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{\left[2\left(\frac{\pi}{2}-x\right)\right]^2}$$

Put $\frac{\pi}{2} - x = h$, then $x = \frac{\pi}{2} - h$

As $x \to \frac{\pi}{2}$, $h \to 0$

$$\therefore \lim_{x\to\frac{\pi}{2}} f(x) = \lim_{h\to 0}\frac{1-\sin\left(\frac{\pi}{2}-h\right)}{(2h)^2} = \lim_{h\to 0}\frac{1-\cos h}{4h^2} = \lim_{h\to 0}\frac{1-\cos h}{4h^2}\times\frac{1+\cos h}{1+\cos h} = \lim_{h\to 0}\frac{1-\cos^2 h}{4h^2(1+\cos h)}$$

$$= \lim_{h\to 0}\frac{\sin^2 h}{4h^2(1+\cos h)} = \frac{1}{4}\left(\lim_{h\to 0}\frac{\sin h}{h}\right)^2\times\frac{1}{\lim_{h\to 0}(1+\cos h)} = \frac{1}{4}\times(1)^2\times\frac{1}{1+1} = \frac{1}{8}$$

$\therefore$ $\lim_{x\to\frac{\pi}{2}} f(x) \neq f\left(\frac{\pi}{2}\right)$

$\therefore$ f is discontinuous at $x = \frac{\pi}{2}$.

The discontinuity of f is removable and it can be made continuous by redefining the function as

$$\left.\begin{aligned} f(x) &= \frac{1-\sin x}{(\pi-2x)^2}, && \text{for } x \neq \frac{\pi}{2} \\ &= \frac{1}{8}, && \text{for } x = \frac{\pi}{2} \end{aligned}\right\} \text{at } x = \frac{\pi}{2}$$

v. $f(0) = \log\left(\frac{2}{3}\right)$ ....(given)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{4^x - e^x}{6^x - 1} = \lim_{x\to 0} \frac{4^x - 1 - e^x + 1}{6^x - 1} = \lim_{x\to 0} \frac{\frac{(4^x - 1) - (e^x - 1)}{x}}{\frac{6^x - 1}{x}}$$

$$= \lim_{x\to 0} \frac{\frac{4^x - 1}{x} - \frac{e^x - 1}{x}}{\frac{6^x - 1}{x}} = \frac{\lim_{x\to 0}\frac{4^x - 1}{x} - \lim_{x\to 0}\frac{e^x - 1}{x}}{\lim_{x\to 0}\frac{6^x - 1}{x}}$$

$$= \frac{\log 4 - \log e}{\log 6} = \frac{\log\left(\frac{4}{e}\right)}{\log 6}$$

$\therefore$ $\lim_{x\to 0} f(x) \neq f(0)$

$\therefore$ f is discontinuous at $x = 0$.

The discontinuity of f is removable and it can be made continuous by redefining the function as

$$f(x) = \left.\begin{array}{ll} \dfrac{4^x - e^x}{6^x - 1}, & \text{for } x \neq 0 \\[2ex] \dfrac{\log\left(\frac{4}{e}\right)}{\log 6}, & \text{for } x = 0 \end{array}\right\} \text{at } x = 0$$

vi. $f(0) = \frac{\pi}{60}$ ....(given)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\left(e^{3x} - 1\right)\sin x^\circ}{x^2} = \lim_{x\to 0} \left(\frac{e^{3x} - 1}{x}\right)\left(\frac{\sin x^\circ}{x}\right)$$

$$= \lim_{x\to 0} 3\left(\frac{e^{3x} - 1}{3x}\right).\left(\frac{\sin\frac{\pi x}{180}}{\frac{\pi x}{180}}\right)\times\frac{\pi}{180} = 3\left(\lim_{x\to 0}\frac{e^{3x} - 1}{3x}\right)\times\frac{\pi}{180}.\left(\lim_{x\to 0}\frac{\sin\frac{\pi x}{180}}{\frac{\pi x}{180}}\right) = 3\log e \times \frac{\pi}{180}(1)$$

$\therefore$ $\lim_{x\to 0} f(x) = 3 \times 1 \times \frac{\pi}{180} = \frac{\pi}{60}$

Since, $\lim_{x\to 0} f(x) = f(0)$, f is continuous at $x = 0$.

vii. $f(x)$ will be continuous at all points in $[-1, 1]$ except where denominator is zero.

i.e., $\sin x . \log\left(1 + \frac{x}{4}\right) = 0$

$\therefore$ $\sin x = 0$ or $\log\left(1 + \frac{x}{4}\right) = 0$

$\therefore$ $x = 0$ or $1 + \frac{x}{4} = 1$

$\therefore$ $x = 0$ or $\frac{x}{4} = 0$

$\therefore$ $x = 0$

$\therefore$ f(x) is continuous in $[-1, 1] - \{0\}$

Now, for f(x) to be continuous in $[-1, 1]$ including $x = 0$,

$f(0) = \lim\limits_{x\to 0} f(x)$

$$\therefore \quad f(0) = \lim_{x\to 0}\frac{\left(8^x-1\right)^2}{\sin x.\log\left(1+\frac{x}{4}\right)} = \lim_{x\to 0}\frac{\frac{(8^x-1)^2}{x^2}}{\frac{\sin x.\log\left(1+\frac{x}{4}\right)}{x^2}}$$

$$= \frac{\lim\limits_{x\to 0}\left(\frac{8^x-1}{x}\right)^2}{\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right).\lim\limits_{x\to 0}\frac{\log\left(1+\frac{x}{4}\right)}{x}} = \frac{\left(\lim\limits_{x\to 0}\frac{8^x-1}{x}\right)^2}{\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)\lim\limits_{x\to 0}\frac{\log\left(1+\frac{x}{4}\right)}{4\times\frac{x}{4}}} = \frac{(\log 8)^2}{(1).\frac{1}{4}.(1)}$$

$\therefore$ $f(0) = 4 (\log 8)^2$

f(x) can be made continuous in $[-1,1]$ including $x = 0$, as

$$f(x) = \frac{(8^x-1)^2}{\sin x.\log\left(1+\frac{x}{4}\right)}, \quad \text{for } x \in [-1, 1]-\{0\}$$

$$= 4 (\log 8)^2, \quad \text{for } x = 0$$

viii. $f(2) = x - 1 = 2 - 1 = 1$

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} (x - 1) = 2 - 1 = 1$

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (2x + 3) = 2(2) + 3 = 7$

Since $\lim\limits_{x\to 2^-} f(x) \neq \lim\limits_{x\to 2^+} f(x)$, $\lim\limits_{x\to 2} f(x)$ does not exist.

$\therefore$ f is discontinuous at $x = 2$ and this discontinuity is irremovable.

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4. i. If f(x) = $\dfrac{e^{x^2}-\cos x}{x^2}$, for $x \neq 0$ is continuous at $x = 0$, find f(0).

ii. If f(x) = $\dfrac{1-\cos[7(x-\pi)]}{5(x-\pi)^2}$, for $x \neq \pi$ is continuous at $x = \pi$, find f(π).

iii. If f(x) = $\dfrac{\sqrt{2}-\sqrt{1+\sin x}}{\cos^2 x}$, for $x \neq \dfrac{\pi}{2}$ is continuous at $x = \dfrac{\pi}{2}$, find $f\left(\dfrac{\pi}{2}\right)$.

iv. If the function f(x) = $\dfrac{\left(4^{\sin x}-1\right)^2}{x\log(1+2x)}$, for $x \neq 0$ is continuous at $x = 0$, find f(0).

v. If f(x) = $\dfrac{1-\sin x}{(\pi-2x)^2}$, for $x \neq \dfrac{\pi}{2}$ is continuous at $x = \dfrac{\pi}{2}$, find $f\left(\dfrac{\pi}{2}\right)$.

vi. If f(x) = $\dfrac{1-\tan x}{1-\sqrt{2}\sin x}$, for $x \neq \dfrac{\pi}{4}$ is continuous at $x = \dfrac{\pi}{4}$, find $f\left(\dfrac{\pi}{4}\right)$.

vii. If f(x) = $\dfrac{4^x-2^{x+1}+1}{1-\cos x}$, for $x \neq 0$ is continuous at $x = 0$, find f(0).

viii. If f(x) = $\dfrac{1-\sqrt{3}\tan x}{\pi-6x}$, for $x \neq \dfrac{\pi}{6}$ is continuous at $x = \dfrac{\pi}{6}$, find $f\left(\dfrac{\pi}{6}\right)$.

***Solution*:**

i. f is continuous at $x = 0$.

$\therefore$ $$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^{x^2} - \cos x}{x^2} = \lim_{x\to 0} \frac{e^{x^2} - 1 - \cos x + 1}{x^2} = \lim_{x\to 0}\left(\frac{e^{x^2}-1}{x^2} + \frac{1-\cos x}{x^2}\right)$$

$$= \lim_{x\to 0}\frac{e^{x^2}-1}{x^2} + \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{\frac{4}{4}x^2} = \log e + \frac{1}{2}\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = \log e + \frac{1}{2}(1)^2 = 1 + \frac{1}{2}$$

$\therefore$ $f(0) = \dfrac{3}{2}$

ii. f is continuous at $x = \pi$.

$\therefore$ $$f(\pi) = \lim_{x\to\pi} f(x) = \lim_{x\to\pi}\frac{1-\cos[7(x-\pi)]}{5(x-\pi)^2}$$

Put $x - \pi = h$, as $x \to \pi$, $h \to 0$

$\therefore$ $$f(\pi) = \lim_{h\to 0}\frac{1-\cos 7h}{5h^2} = \lim_{h\to 0}\frac{2\sin^2\left(\frac{7h}{2}\right)}{5h^2} = \frac{2}{5}\lim_{h\to 0}\frac{\sin^2\left(\frac{7h}{2}\right)}{\left(\frac{7h}{2}\right)^2}\times\left(\frac{7}{2}\right)^2 = \frac{2}{5}\left[\lim_{h\to 0}\frac{\sin\left(\frac{7h}{2}\right)}{\left(\frac{7h}{2}\right)}\right]^2\times\frac{49}{4} = \frac{2}{5}\times(1)^2\times\frac{49}{4}$$

$\therefore$ $f(\pi) = \dfrac{49}{10}$

iii. f is continuous at $x = \dfrac{\pi}{2}$.

$\therefore$ $$f\left(\frac{\pi}{2}\right) = \lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}}\frac{\sqrt{2}-\sqrt{1+\sin x}}{\cos^2 x} = \lim_{x\to\frac{\pi}{2}}\frac{\sqrt{2}-\sqrt{1+\sin x}}{\cos^2 x}\times\frac{\sqrt{2}+\sqrt{1+\sin x}}{\sqrt{2}+\sqrt{1+\sin x}}$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{2-(1+\sin x)}{\cos^2 x}\times\frac{1}{\left[\sqrt{2}+\sqrt{1+\sin x}\right]}$$

Put $\dfrac{\pi}{2} - x = h$, then $x = \dfrac{\pi}{2} - h$

As $x \to \dfrac{\pi}{2}$, $h \to 0$

$\therefore$ $$f\left(\frac{\pi}{2}\right) = \lim_{h\to 0}\frac{1-\sin\left(\frac{\pi}{2}-h\right)}{\cos^2\left(\frac{\pi}{2}-h\right)}\times\frac{1}{\left[\sqrt{2}+\sqrt{1+\sin\left(\frac{\pi}{2}-h\right)}\right]}$$

$$= \lim_{h\to 0}\frac{1-\cos h}{\sin^2 h}\times\frac{1}{\left[\sqrt{2}+\sqrt{1+\cos h}\right]} = \lim_{h\to 0}\frac{1-\cos h}{1-\cos^2 h}\times\frac{1}{\left[\sqrt{2}+\sqrt{1+\cos h}\right]}$$

$$= \lim_{h\to 0}\frac{1-\cos h}{(1-\cos h)(1+\cos h)\left[\sqrt{2}+\sqrt{1+\cos h}\right]} = \frac{1}{(1+\cos 0)\left[\sqrt{2}+\sqrt{1+\cos 0}\right]} = \frac{1}{(1+1)\left[\sqrt{2}+\sqrt{2}\right]}$$

$\therefore$ $f\left(\dfrac{\pi}{2}\right) = \dfrac{1}{4\sqrt{2}}$

iv. f is continuous at $x = 0$.

$\therefore$ $$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\left(4^{\sin x}-1\right)^2}{x\log(1+2x)} = \lim_{x\to 0}\frac{\frac{\left(4^{\sin x}-1\right)^2}{x^2}}{\frac{x\cdot\log(1+2x)}{x^2}} = \lim_{x\to 0}\frac{\left(\frac{4^{\sin x}-1}{\sin x}\right)^2\times\frac{\sin^2 x}{x^2}}{\frac{\log(1+2x)}{2x}\times 2} = \frac{\left(\lim_{x\to 0}\frac{4^{\sin x}-1}{\sin x}\times\lim_{x\to 0}\frac{\sin x}{x}\right)^2}{2\lim_{x\to 0}\frac{\log(1+2x)}{2x}}$$

$\therefore$ $$f(0) = \frac{(\log 4)^2}{2}$$

v. f is continuous at $x = \frac{\pi}{2}$.

$\therefore$ $$f\left(\frac{\pi}{2}\right) = \lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{(\pi-2x)^2} = \lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{\left[2\left(\frac{\pi}{2}-x\right)\right]^2}$$

Put $\frac{\pi}{2} - x = h$, then $x = \frac{\pi}{2} - h$

As $x \to \frac{\pi}{2}$, $h \to 0$

$\therefore$ $$f\left(\frac{\pi}{2}\right) = \lim_{h\to 0}\frac{1-\sin\left(\frac{\pi}{2}-h\right)}{(2h)^2} = \lim_{h\to 0}\frac{1-\cos h}{4h^2} = \lim_{h\to 0}\frac{1-\cos h}{4h^2}\times\frac{1+\cos h}{1+\cos h} = \lim_{h\to 0}\frac{1-\cos^2 h}{4h^2(1+\cos h)}$$

$$= \lim_{h\to 0}\frac{\sin^2 h}{4h^2(1+\cos h)} = \frac{1}{4}\left(\lim_{h\to 0}\frac{\sin h}{h}\right)^2\times\frac{1}{\lim_{h\to 0}(1+\cos h)} = \frac{1}{4}\times 1\times\frac{1}{1+1} = \frac{1}{4}\times\frac{1}{2}$$

$\therefore$ $$f\left(\frac{\pi}{2}\right) = \frac{1}{8}$$

vi. f is continuous at $x = \frac{\pi}{4}$.

$\therefore$ $$f\left(\frac{\pi}{4}\right) = \lim_{x\to\frac{\pi}{4}} f(x) = \lim_{x\to\frac{\pi}{4}}\frac{1-\tan x}{1-\sqrt{2}\sin x} = \lim_{x\to\frac{\pi}{4}}\frac{1-\frac{\sin x}{\cos x}}{1-\sqrt{2}\sin x} = \lim_{x\to\frac{\pi}{4}}\frac{\frac{\cos x-\sin x}{\cos x}}{1-\sqrt{2}\sin x}\times\frac{1+\sqrt{2}\sin x}{1+\sqrt{2}\sin x}$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{(\cos x-\sin x)\left(1+\sqrt{2}\sin x\right)}{(\cos x)\left(1-2\sin^2 x\right)} = \lim_{x\to\frac{\pi}{4}}\frac{(\cos x-\sin x)\left(1+\sqrt{2}\sin x\right)}{(\cos x)\left(\cos^2 x+\sin^2 x-2\sin^2 x\right)}$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{(\cos x-\sin x)\left(1+\sqrt{2}\sin x\right)}{(\cos x)\left(\cos^2 x-\sin^2 x\right)} = \lim_{x\to\frac{\pi}{4}}\frac{(\cos x-\sin x)\left(1+\sqrt{2}\sin x\right)}{(\cos x)(\cos x+\sin x)(\cos x-\sin x)}$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{\left(1+\sqrt{2}\sin x\right)}{(\cos x)(\cos x+\sin x)} \quad \ldots\left[\begin{array}{l}\because x\to\frac{\pi}{4}, \cos x-\sin x\to 0\\ \therefore \cos x-\sin x\neq 0\end{array}\right]$$

$$= \frac{\lim_{x\to\frac{\pi}{4}}\left(1+\sqrt{2}\sin x\right)}{\lim_{x\to\frac{\pi}{4}}(\cos x)(\cos x+\sin x)} = \frac{1+\sqrt{2}\times\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)} = \frac{1+1}{\frac{1}{\sqrt{2}}\cdot\left(\frac{2}{\sqrt{2}}\right)}$$

$\therefore$ $$f\left(\frac{\pi}{4}\right) = 2$$

vii. f is continuous at $x = 0$.

$\therefore$ $$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{4^x - 2^{x+1} + 1}{1-\cos x} = \lim_{x\to 0}\frac{\left(2^x\right)^2 - 2.2^x + 1}{1-\cos x} = \lim_{x\to 0}\frac{\left(2^x - 1\right)^2}{2\sin^2\frac{x}{2}}$$

.... [$\because$ $4^x = (2^2)^x = (2^x)^2$ interchanging of exponential power]

$$= \lim_{x\to 0}\frac{\frac{(2^x-1)^2}{x^2}}{\frac{2\sin^2\frac{x}{2}}{x^2}} = \frac{\left(\lim_{x\to 0}\frac{2^x-1}{x}\right)^2}{\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{\frac{4x^2}{4}}} = \frac{\left(\lim_{x\to 0}\frac{2^x-1}{x}\right)^2}{\frac{1}{2}\left(\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2} = \frac{(\log 2)^2}{\frac{1}{2}.(1)^2}$$

$\therefore$ $f(0) = 2(\log 2)^2$

viii. f is continuous at $x = \frac{\pi}{6}$.

$\therefore$ $$f\left(\frac{\pi}{6}\right) = \lim_{x\to\frac{\pi}{6}} f(x) = \lim_{x\to\frac{\pi}{6}}\frac{1-\sqrt{3}\tan x}{\pi - 6x}$$

Put $x = \frac{\pi}{6} + h$, as $x \to \frac{\pi}{6}$, $h \to 0$

$\therefore$ $$\lim_{x\to\frac{\pi}{6}} f(x) = \lim_{h\to 0}\frac{1-\sqrt{3}\tan\left(\frac{\pi}{6}+h\right)}{\pi - 6\left(\frac{\pi}{6}+h\right)}$$

$$= \lim_{h\to 0}\frac{1-\sqrt{3}\left(\frac{\tan\frac{\pi}{6}+\tan h}{1-\tan\frac{\pi}{6}\tan h}\right)}{\pi-\pi-6h} = \lim_{h\to 0}\frac{1-\sqrt{3}\left(\frac{\frac{1}{\sqrt{3}}+\tan h}{1-\frac{1}{\sqrt{3}}\tan h}\right)}{-6h}$$

$$= \lim_{h\to 0}\frac{1-\frac{1}{\sqrt{3}}\tan h - 1 - \sqrt{3}\tan h}{-6h\left(1-\frac{1}{\sqrt{3}}\tan h\right)} = \lim_{h\to 0}\frac{\left(-\frac{1}{\sqrt{3}}-\sqrt{3}\right)\tan h}{-6h\left(1-\frac{1}{\sqrt{3}}\tan h\right)} = \lim_{h\to 0}\frac{-\frac{4}{\sqrt{3}}\tan h}{-6h\left(1-\frac{1}{\sqrt{3}}\tan h\right)}$$

$$= \lim_{h\to 0}\frac{2}{3\sqrt{3}}\left(\frac{\tan h}{h}\right)\times\frac{1}{1-\frac{1}{\sqrt{3}}\tan h} = \frac{2}{3\sqrt{3}}\left(\lim_{h\to 0}\frac{\tan h}{h}\right)\times\frac{1}{\lim_{h\to 0}\left(1-\frac{1}{\sqrt{3}}\tan h\right)}$$

$$= \frac{2}{3\sqrt{3}}(1)\left[\frac{1}{1-\frac{1}{\sqrt{3}}(0)}\right]$$

$\therefore$ $$f\left(\frac{\pi}{6}\right) = \frac{2}{3\sqrt{3}}$$

5. i. If $f(x) = x^2 + \alpha$, for $x \geq 0$

$= 2\sqrt{x^2+1} + \beta$, for $x < 0$

and $f\left(\frac{1}{2}\right) = 2$, is continuous at $x = 0$, find α and β. **[Oct 15]**

ii. If $f(x) = \dfrac{\sin 4x}{5x} + a$, for $x > 0$

$= x + 4 - b$, for $x < 0$

$= 1$, for $x = 0$

is continuous at $x = 0$, find a and b.

iii. If $f(x) = \dfrac{\sin \pi x}{x-1} + a$, for $x < 1$

$= 2\pi$, for $x = 1$

$= \dfrac{1+\cos \pi x}{\pi(1-x)^2} + b$, for $x > 1$

is continuous at $x = 1$, find a and b.

iv. If $f(x) = \dfrac{x^2-9}{x-3} + \alpha$, for $x > 3$

$= 5$, for $x = 3$

$= 2x^2 + 3x + \beta$, for $x < 3$

is continuous at $x = 3$, then find α and β.

v. If f(x) is defined by

$f(x) = \sin 2x$, if $x \leq \frac{\pi}{6}$

$= ax + b$, if $x > \frac{\pi}{6}$

Find the values of a and b, if f(x) and $f'(x)$ are continuous at $x = \frac{\pi}{6}$.

vi. Find the value of a and b such that the function defined by

$f(x) = 5$, if $x \leq 2$

$= ax + b$, if $2 < x < 10$

$= 21$, if $x \geq 10$

is continuous on at $x = 2$ as well as $x = 10$.

vii. Find k, so that the function f(x) is continuous at $x = 1$, where

$f(x) = kx^2$, for $x \geq 1$

$= 4$, for $x < 1$

viii. Determine the values of a, b, c for which the function defined by

$f(x) = \dfrac{\sin(a+1)x + \sin x}{x}$, for $x < 0$

$= c$, for $x = 0$

$= \dfrac{\left(x+bx^2\right)^{\frac{1}{2}} - x^{\frac{1}{2}}}{bx^{\frac{1}{2}}}$, for $x > 0$

is continuous at $x = 0$.

***Solution*:**

i. $f\left(\frac{1}{2}\right) = 2$ ....(given)

and $f(x) = x^2 + \alpha$, for $x \geq 0$

$\therefore$ $f\left(\frac{1}{2}\right) = \lim\limits_{x \to \frac{1}{2}} f(x) = \lim\limits_{x \to \frac{1}{2}} x^2 + \alpha = \frac{1}{4} + \alpha$

$\therefore$ $2 = \frac{1}{4} + \alpha$

$\therefore$ $\alpha = \frac{8-1}{4}$

$\therefore$ $\alpha = \frac{7}{4}$

Also, $f(x)$ is continuous at $x = 0$ ....(given)

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = f(0)$

$\therefore$ $\lim\limits_{x \to 0^-} 2\sqrt{x^2+1} + \beta = \lim\limits_{x \to 0^+} x^2 + \alpha$

$\therefore$ $2\sqrt{0+1} + \beta = 0 + \frac{7}{4}$

$\therefore$ $\beta = -\frac{1}{4}$

$\therefore$ $\alpha = \frac{7}{4}$ and $\beta = -\frac{1}{4}$

ii. $f(x)$ is continuous at $x = 0$. ....(given)

$\therefore$ $\lim\limits_{x \to 0^-} f(x) = f(0)$

$\therefore$ $\lim\limits_{x \to 0^-} x + 4 - b = 1$

$\therefore$ $0 + 4 - b = 1$

$\therefore$ $b = 3$

Also, $\lim\limits_{x \to 0^+} f(x) = f(0)$

$\therefore$ $\lim\limits_{x \to 0^+} \frac{\sin 4x}{5x} + a = 1$

$\therefore$ $\lim\limits_{x \to 0^+} \frac{4}{5}\left(\frac{\sin 4x}{4x}\right) + a = 1$

$\therefore$ $\frac{4}{5} \lim\limits_{x \to 0^+} \left(\frac{\sin 4x}{4x}\right) + a = 1$

$\therefore$ $\frac{4}{5} + a = 1$

$\therefore$ $a = 1 - \frac{4}{5} = \frac{1}{5}$

$\therefore$ $a = \frac{1}{5}$ and $b = 3$

iii. $f(1) = 2\pi$ ....(given)

As function is continuous at $x = 1$

$\therefore \lim_{x\to 1^-} f(x) = f(1)$

$\therefore \lim_{x\to 1^-} \left(\frac{\sin \pi x}{x-1} + a\right) = 2\pi$

Put $x - 1 = h$, then $x = 1 + h$

As $x \to 1$, $h \to 0$

$\therefore \lim_{h\to 0} \frac{\sin \pi(1+h)}{h} + a = 2\pi$

$\therefore \lim_{h\to 0} \frac{\sin(\pi + \pi h)}{h} + a = 2\pi$

$\therefore \lim_{h\to 0} \pi\left[\frac{-\sin \pi h}{\pi h}\right] + a = 2\pi$

$\therefore -\pi + a = 2\pi$

$\therefore a = 3\pi$

Similarly, $\lim_{x\to 1^+} f(x) = f(1)$

$\therefore \lim_{x\to 1^+} \left(\frac{1+\cos \pi x}{\pi(x-1)^2} + b\right) = 2\pi$

Put $x - 1 = h$, then $x = h + 1$

As $x \to 1$, $h \to 0$

$\therefore \lim_{h\to 0} \frac{1+\cos \pi(1+h)}{\pi h^2} + b = 2\pi$

$\therefore \lim_{h\to 0} \frac{1+\cos(\pi + \pi h)}{\pi h^2} + b = 2\pi$

$\therefore \lim_{h\to 0} \left[\left(\frac{1-\cos \pi h}{\pi h^2}\right) + b\right] = 2\pi$

$\therefore \lim_{h\to 0} \left[\frac{1-\cos \pi h}{\pi h^2} \times \frac{1+\cos \pi h}{1+\cos \pi h} + b\right] = 2\pi$

$\therefore \lim_{h\to 0} \left[\frac{1-\cos^2 \pi h}{\pi h^2(1+\cos \pi h)} + b\right] = 2\pi$

$\therefore \lim_{h\to 0} \left[\frac{\sin^2 \pi h}{\pi h^2(1+\cos \pi h)} + b\right] = 2\pi$

$\therefore \lim_{h\to 0} \left[\left(\frac{\sin \pi h}{\pi h}\right)^2 \times \frac{\pi}{1+\cos \pi h} + b\right] = 2\pi$

$\therefore \pi \lim_{h\to 0} \left(\frac{\sin \pi h}{\pi h}\right)^2 \times \frac{1}{\lim_{h\to 0}(1+\cos \pi h)} + b = 2\pi$

$\therefore \pi(1)^2 \times \frac{1}{1+1} + b = 2\pi$

$\therefore \frac{\pi}{2} + b = 2\pi$

$\therefore b = 2\pi - \frac{\pi}{2}$

$\therefore$ $b = \dfrac{3\pi}{2}$

$\therefore$ $a = 3\pi$ and $b = \dfrac{3\pi}{2}$

iv. f(x) is continuous at $x = 3$ ….(given)

$\therefore$ $f(3) = \lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} 2x^2 + 3x + \beta$

$\therefore$ $5 = 2(3)^2 + 3(3) + \beta = 18 + 9 + \beta$

$\therefore$ $\beta = -22$

$f(3) = \lim\limits_{x\to 3^+} f(x)$

$\therefore$ $5 = \lim\limits_{x\to 3^+} \dfrac{x^2-9}{x-3} + \alpha = \lim\limits_{x\to 3^+} \dfrac{(x+3)(x-3)}{(x-3)} + \alpha = \lim\limits_{x\to 3^+} (x+3) + \alpha = (3+3) + \alpha$

$\therefore$ $5 = 6 + \alpha$

$\therefore$ $\alpha = -1$

$\therefore$ $\alpha = -1, \beta = -22$

v. f (x) is continuous at $x = \dfrac{\pi}{6}$ ….(given)

$\therefore$ $\lim\limits_{x\to \frac{\pi}{6}^-} f(x) = \lim\limits_{x\to \frac{\pi}{6}^+} f(x) = f\left(\dfrac{\pi}{6}\right)$ ….(i)

Now, $\lim\limits_{x\to \frac{\pi}{6}^-} f(x) = \lim\limits_{x\to \frac{\pi}{6}} \sin 2x = \sin \dfrac{2\pi}{6} = \sin \dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2} = f\left(\dfrac{\pi}{6}\right)$

and $\lim\limits_{x\to \frac{\pi}{6}^+} f(x) = \lim\limits_{x\to \frac{\pi}{6}} (ax + b) = a\left(\dfrac{\pi}{6}\right) + b$

$\therefore$ from (i), $a\left(\dfrac{\pi}{6}\right) + b = \dfrac{\sqrt{3}}{2}$ ….(ii)

We have, $f'(x) = 2\cos 2x$, if $x \le \dfrac{\pi}{6}$ $= a(1) + 0$, if $x > \dfrac{\pi}{6}$

Now, $f'(x)$ is continuous at $x = \dfrac{\pi}{6}$ ….(given)

$\therefore$ $\lim\limits_{x\to \frac{\pi}{6}^-} f'(x) = \lim\limits_{x\to \frac{\pi}{6}^+} f'(x) = f'\left(\dfrac{\pi}{6}\right)$ ….(iii)

But, $\lim\limits_{x\to \frac{\pi}{6}^-} f'(x) = \lim\limits_{x\to \frac{\pi}{6}} 2\cos 2x = 2\cos \dfrac{2\pi}{6} = 2\cos \dfrac{\pi}{3} = 2\left(\dfrac{1}{2}\right) = 1$

and $\lim\limits_{x\to \frac{\pi}{6}^+} f'(x) = \lim\limits_{x\to \frac{\pi}{6}} a = a$

$\therefore$ from (iii), 1 = a i.e., a = 1

Now, from (ii), $\dfrac{\pi}{6}(1) + b = \dfrac{\sqrt{3}}{2}$

$\therefore$ $b = \dfrac{\sqrt{3}}{2} - \dfrac{\pi}{6}$ $\therefore$ $a = 1$ and $b = \dfrac{\sqrt{3}}{2} - \dfrac{\pi}{6}$

vi. f(x) is continuous at $x = 2$ ….(given)

$\therefore \lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x)$

$\therefore \lim\limits_{x\to 2^-} 5 = \lim\limits_{x\to 2^+} ax + b$

$\therefore$ $2a + b = 5$ ….(i)

f(x) is continuous at $x = 10$ ….(given)

$\therefore \lim\limits_{x\to 10^-} f(x) = \lim\limits_{x\to 10^+} f(x)$

$\therefore \lim\limits_{x\to 10^-} ax + b = \lim\limits_{x\to 10^+} 21$

$\therefore$ $10a + b = 21$ ….(ii)

Subtracting (i) from (ii), we get

a = 2

Putting a = 2 in (i), we get

b = 1

$\therefore$ a = 2 and b = 1

vii. f(x) is continuous at $x = 1$ ….(given)

$\therefore \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x)$

$\therefore 4 = \lim\limits_{x\to 1^+} kx^2$

$\therefore 4 = k(1)^2$

$\therefore$ k = 4

viii. f(x) is continuous at $x = 0$ ….(given)

$\therefore \lim\limits_{x\to 0^-} f(x) = f(0)$

$$\therefore \lim_{x\to 0^-} \frac{\sin(a+1)x + \sin x}{x} = c$$

$$\therefore \lim_{x\to 0^-} \frac{\sin(a+1)x}{x} + \lim_{x\to 0^-} \frac{\sin x}{x} = c$$

$$\therefore (a+1) \lim_{x\to 0^-} \frac{\sin(a+1)x}{(a+1)x} + \lim_{x\to 0^-} \frac{\sin x}{x} = c$$

$\therefore$ $a + 1 + 1 = c$

$\therefore$ $a + 2 = c$ ….(i)

Also, $\lim\limits_{x\to 0^+} f(x) = f(0)$

$$\therefore \lim_{x\to 0^+} \frac{(x+bx^2)^{\frac{1}{2}} - (x)^{\frac{1}{2}}}{bx^{\frac{1}{2}}} = c$$

$$\therefore \lim_{x\to 0^+} \frac{x^{\frac{1}{2}}\left[(1+bx)^{\frac{1}{2}} - 1\right]}{bx^{\frac{1}{2}}} = c \qquad \therefore \frac{\sqrt{1+b(0)}-1}{b} = c$$

$$\therefore \frac{0}{b} = c$$

$\therefore$ c = 0 and b $\neq$ 0

Putting the value of c in (i), we get

a = – 2

Hence, a = –2, b $\neq$ 0, c = 0

Algebra of Continuous Functions

If f(x) and g(x) are two real valued functions continuous at x = c, then

i. The function k·f(x) is continuous at x = c, where k $\in$ R.
ii. The function f(x) + g(x) is continuous at x = c.
iii. The function f(x) – g(x) is continuous at x = c
iv. The function f(x)·g(x) is continuous at x = c.
v. The function $\frac{f(x)}{g(x)}$ is continuous at x = c, (where g(c) $\neq$ 0)
vi. Composition of two continuous functions is always continuous function.

Continuity in an Interval

A real valued function 'f' is said to be continuous in an interval if it is continuous at every point of the interval. A function which is continuous at the entire real line (– ∞, ∞) is said to be continuous every where.

Continuity in the domain of the function

A real valued function f: D → R is said to be a continuous function if it is continuous at every point in the domain D of the function f.

Eg.

The functions f(x) = sin x and g(x) = cos x are continuous in every domain D, where D $\subseteq$ R.

Continuity of some standard functions

i. Constant function:

If f(x) = k, $x \in$ R and k be a fixed real number (constant), then f is called the constant function. This constant function f is continuous in its domain R.

ii. Polynomial function:

If $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where n$\in$W and $a_0, a_1, a_2, \ldots, a_n \in$ R, $x \in$ R, then f is said to be a polynomial function. f is continuous in its domain R.

iii. Rational function:

If f(x) and g(x) are two polynomial functions, then $\frac{f(x)}{g(x)}$, g(x) $\neq$ 0 is called a rational function. This rational function is continuous for all $x \in$ R except for which g(x) = 0.

iv. Trigonometric function:

a. sin x and cos x are continuous for all $x \in$ R.
b. Tangent, cotangent, secant and cosecant functions are continuous on their respective domains.

v. Exponential function:

If f(x) = a^x, a > 0, a $\neq$ 1 and $x \in$ R then f is called as an exponential function.

This function is continuous for all $x \in$ R.

vi. Logarithmic function:

If f(x) = $\log_a x$, a > 0, a $\neq$ 1 and $x \in$ R then f is called logarithmic function.

This function is continuous at every positive real number i.e., for all $x \in R^+$, where R^+ = set of positive real numbers.

Exercise 1.2

1. i. If f(x) is continuous on [0, 8], defined as

$f(x) = x^2 + ax + 6,$	for $0 \le x < 2$
$= 3x + 2,$	for $2 \le x \le 4$
$= 2ax + 5b,$	for $4 < x \le 8$

Find a and b.

ii. If f(x) is continuous on [0, π], where

$f(x) = x + a\sqrt{2}\sin x,$	for $0 \le x < \frac{\pi}{4}$
$= 2x\cot x + b,$	for $\frac{\pi}{4} \le x \le \frac{\pi}{2}$
$= a\cos 2x - b\sin x$	for $\frac{\pi}{2} < x \le \pi$

Find a and b.

iii. Find α and β, so that the function f(x) defined by

$f(x) = -2\sin x,$	for $-\pi \le x \le -\frac{\pi}{2}$
$= \alpha\sin x + \beta,$	for $-\frac{\pi}{2} < x < \frac{\pi}{2}$
$= \cos x,$	for $\frac{\pi}{2} \le x \le \pi$

is continuous on [−π, π].

iv. If the function f(x), is continuous in [0, 3] where

$f(x) = 3x - 4,$	for $0 \le x \le 2$
$= 2x + k,$	for $2 < x \le 3$

Find the value of constant k.

v. If $f(x) = \dfrac{x^3 + 3x + 5}{x^3 - 3x + 2}$ **. Discuss the continuity of f(x) on [0, 5].**

vi. Discuss the continuity of the function $\log_c x$ where $c > 0$, $x > 0$.

vii. If function f(x) is continuous in interval [−2, 2], find the value of (a + b) where

$f(x) = \dfrac{\sin ax}{x} - 2,$	for $-2 \le x < 0$	
$= 2x + 1,$	for $0 \le x \le 1$	
$= 2b\sqrt{x^2 + 3} - 1,$	for $1 < x \le 2$	**[Mar 14]**

viii. Test the continuity of function $f(x) = \dfrac{x+1}{(x-2)(x-5)}$ **in the interval [0, 1] and [4, 6].**

ix. Discuss the continuity of the function f(x) in its domain if f(x) is defined by

$f(x) = x,$	for $x \ge 0$
$= x^2,$	for $x < 0$

***Solution*:**

i. As function is continuous on [0, 8], it is continuous at $x = 2$

$\therefore \quad \lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x)$

$\therefore \quad \lim\limits_{x\to 2^-} (x^2 + ax + 6) = \lim\limits_{x\to 2^+} (3x + 2)$

$\therefore \quad (2)^2 + 2a + 6 = 3(2) + 2$

$\therefore$ $4 + 2a + 6 = 8$

$\therefore$ $2a = -2$

$\therefore$ $a = -1$

Also function is continuous at $x = 4$

$\therefore$ $\lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x)$

$\therefore$ $\lim_{x \to 4^-} (3x + 2) = \lim_{x \to 4^+} (2ax + 5b)$

$\therefore$ $3(4) + 2 = 2(4)a + 5b$

$\therefore$ $8a + 5b = 14$ ….(i)

Putting the value of a in (i), we get

$b = \frac{22}{5}$

$\therefore$ $a = -1$ and $b = \frac{22}{5}$

ii. As function is continuous on $[0, \pi]$, it is continuous at $x = \frac{\pi}{4}$

$\therefore$ $\lim_{x \to \frac{\pi}{4}^-} f(x) = \lim_{x \to \frac{\pi}{4}^+} f(x) = f\left(\frac{\pi}{4}\right)$

$\therefore$ $\lim_{x \to \frac{\pi}{4}^-} (x + a\sqrt{2}\sin x) = \lim_{x \to \frac{\pi}{4}^+} (2x\cot x + b)$

$\therefore$ $\frac{\pi}{4} + a\sqrt{2}\sin\left(\frac{\pi}{4}\right) = 2\left(\frac{\pi}{4}\right)\cot\left(\frac{\pi}{4}\right) + b$

$\therefore$ $\frac{\pi}{4} + a\sqrt{2} \times \frac{1}{\sqrt{2}} = 2\left(\frac{\pi}{4}\right)(1) + b$

$\therefore$ $\frac{\pi}{4} + a = \frac{\pi}{2} + b$

$\therefore$ $a - b = \frac{\pi}{4}$ ….(i)

Also function is continuous at $x = \frac{\pi}{2}$

$\therefore$ $\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^+} f(x) = f\left(\frac{\pi}{2}\right)$

$\therefore$ $\lim_{x \to \frac{\pi}{2}^-} (2x\cot x + b) = \lim_{x \to \frac{\pi}{2}^+} (a\cos 2x - b\sin x)$

$\therefore$ $2\left(\frac{\pi}{2}\right)\cot\left(\frac{\pi}{2}\right) + b = a\cos 2\left(\frac{\pi}{2}\right) - b\sin\left(\frac{\pi}{2}\right)$

$\therefore$ $\pi(0) + b = a\cos\pi - b\sin\left(\frac{\pi}{2}\right)$

$\therefore$ $0 + b = (-1)a - b(1)$

$\therefore$ $a = -2b$ ….(ii)

Solving (i) and (ii), we get

$b = -\frac{\pi}{12}$

Putting the value of b in (ii), we get

$a = \frac{\pi}{6}$

$\therefore$ $a = \frac{\pi}{6}$ and $b = -\frac{\pi}{12}$

iii. As function is continuous on $[-\pi, \pi]$, it is continuous at $x = -\frac{\pi}{2}$

$\therefore$ $\lim_{x \to -\frac{\pi}{2}^-} f(x) = \lim_{x \to -\frac{\pi}{2}^+} f(x)$

$\therefore$ $\lim_{x \to -\frac{\pi}{2}^-} (-2\sin x) = \lim_{x \to -\frac{\pi}{2}^+} (\alpha \sin x + \beta)$

$\therefore$ $-2\sin\left(\frac{-\pi}{2}\right) = \alpha \sin\left(\frac{-\pi}{2}\right) + \beta$

$\therefore$ $-2(-1) = \alpha(-1) + \beta$

$\therefore$ $-\alpha + \beta = 2$ ....(i)

Also function is continuous at $x = \frac{\pi}{2}$

$\therefore$ $\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^+} f(x)$

$\therefore$ $\lim_{x \to \frac{\pi}{2}^-} (\alpha \sin x + \beta) = \lim_{x \to \frac{\pi}{2}^+} \cos x$

$\therefore$ $\alpha \sin\left(\frac{\pi}{2}\right) + \beta = \cos\frac{\pi}{2}$

$\therefore$ $\alpha + \beta = 0$ ....(ii)

Solving (i) and (ii), we get

$\alpha = -1$ and $\beta = 1$

iv. As function is continuous on $[0, 3]$, it is continuous at $x = 2$

$\therefore$ $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$

$\therefore$ $\lim_{x \to 2^-} 3x - 4 = \lim_{x \to 2^+} 2x + k$

$\therefore$ $3(2) - 4 = 2(2) + k$

$\therefore$ $6 - 4 = 4 + k$

$\therefore$ $k = -2$

v. $f(x)$ is a rational polynomial.

$\therefore$ $f(x)$ is continuous for all real values of x, except when its denominator becomes zero.

i.e., $x^3 - 3x + 2 = 0$

Since, $x = 1$ satisfies the above equation, $(x - 1)$ is its factor.

$\therefore$ By synthetic division, we get

$\therefore$ $(x - 1)(x^2 + x - 2) = 0$

$\therefore$ $(x - 1)(x + 2)(x - 1) = 0$

$\therefore$ $(x - 1)^2(x + 2) = 0$

$\therefore$ $x = 1$ or $x = -2$

But $-2 \notin [0, 5]$

$\therefore$ $f(x)$ is continuous for all real values of x in $[0, 5]$, except at $x = 1$.

Thus, $f(x)$ is discontinuous at $x = 1$.

vi. Let $f(x) = \log_c x$

Let a be any positive real number, then $f(a) = \log_c a$

Let $L = \lim_{x\to a} [f(x) - f(a)]$

Put $x = a + h$, then as $x \to a$, $h \to 0$

$\therefore$ $L = \lim_{h\to 0} [f(a+h) - f(a)] = \lim_{h\to 0} \left[\log_c(a+h) - \log_c a\right]$

$$= \lim_{h\to 0}\left[\log_c\left(\frac{a+h}{a}\right)\right] = \lim_{h\to 0}\frac{1}{h}\left[\log_c\left(\frac{a+h}{a}\right)\right]\times h$$

$$= \lim_{h\to 0}\left[\log_c\left(1+\frac{h}{a}\right)^{\frac{1}{h}}\right]\times h = \lim_{h\to 0}\left[\log_c\left(1+\frac{h}{a}\right)^{\frac{a}{h}}\right]^{\frac{1}{a}}\times h$$

$$= \log_c\left[\lim_{h\to 0}\left(1+\frac{h}{a}\right)^{\frac{a}{h}}\right]^{\frac{1}{a}}\times\left(\lim_{h\to 0} h\right) = \left(\log_c e^{\frac{1}{a}}\right)(0)$$

$= 0$

Thus, $\lim_{x\to a} [f(x) - f(a)] = 0$

$\therefore$ f is continuous at $x = a$.

But, a is any positive real number.

$\therefore$ f is continuous at all positive real number.

Thus, $\log_c x$ where $c > 0$, $c \neq 1$, $x > 0$ is continuous.

vii. As $f(x)$ is continuous on $[-2, 2]$, it is continuous at $x = 0$

$\therefore$ $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

$\therefore$ $\lim_{x\to 0^-}\left(\frac{\sin ax}{x} - 2\right) = \lim_{x\to 0^+} (2x + 1)$

$\therefore$ $\left(\lim_{x\to 0^-}\frac{\sin ax}{ax}\right)\times a - 2 = 2(0) + 1$

$\therefore$ $a - 2 = 1$

$\therefore$ $a = 3$ …(i)

Also function is continuous at $x = 1$.

$\therefore$ $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$

$\therefore$ $\lim_{x\to 1^-} 2x + 1 = \lim_{x\to 1^+}\left(2b\sqrt{x^2+3} - 1\right)$

$\therefore$ $2(1) + 1 = 2b\sqrt{1+3} - 1$

$\therefore$ $3 = 2b(2) - 1$

$\therefore$ $b = 1$ ….(ii)

From (i) and (ii), we get

$a + b = 3 + 1 = 4$

viii. Here, f is a rational function, which is the ratio of two polynomials.

$\therefore$ f is continuous for all $x \in R$ in $[0,1]$ except at those points where the denominator is zero.

Now, $(x-2)(x-5) = 0$ when $x = 2$ or $x = 5$

But, $2 \notin [0, 1]$ and $5 \notin [0,1]$.

$\therefore$ $x = 2$ and $x = 5$ are not points of discontinuity.

$\therefore$ f is continuous in [0,1].
However, $5 \in [4, 6]$
$\therefore$ $x = 5$ is a point of discontinuity in [4, 6].
Hence, f is continuous in [4, 6] except at $x = 5$

ix. $f(x) = x$ for $x > 0$.
Since f is a linear function, f is continuous for all $x > 0$.
Also $f(x) = x^2$ for $x < 0$
Since f is a quadratic function, f is continuous for all $x < 0$.
Now, $f(0) = 0$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0} x^2 = 0$$

and $$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0$$

$$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

Hence, f is continuous at $x = 0$.
Thus, f is continuous in its domain R.

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2. i. Discuss the continuity of the following functions in its domain, where

$$\begin{aligned} f(x) &= x^2 - 4, && \text{for } 0 \le x \le 2 \\ &= 2x + 3, && \text{for } 2 < x \le 4 \\ &= x^2 - 5, && \text{for } 4 < x \le 6 \end{aligned}$$

ii. Examine the continuity of f(x) on its domain where,

$$\begin{aligned} f(x) &= \frac{1}{x+1}, && \text{for } 2 \le x \le 4 \\ &= \frac{x+1}{x-3}, && \text{for } 4 < x \le 6 \end{aligned}$$

iii. Discuss the continuity of function f(x) in its respective domain defined by

$$\begin{aligned} f(x) &= 3, && \text{if } 0 \le x \le 1 \\ &= 4, && \text{if } 1 < x < 3 \\ &= 5, && \text{if } 3 \le x \le 10 \end{aligned}$$

Justify the answer with the help of graph.

iv. Discuss the continuity of f(x) in its domain

$$\begin{aligned} f(x) &= -2, && \text{if } x \le -1 \\ &= 2x, && \text{if } -1 < x \le 1 \\ &= 2, && \text{if } x > 1 \end{aligned}$$

v. Examine the continuity of f(x) on its domain, where

$$\begin{aligned} f(x) &= x + 3, && \text{for } x \le -3 \\ &= -2x, && \text{for } -3 < x < 3 \\ &= 6x + 2, && \text{for } x \ge 3 \end{aligned}$$

vi. Find the value of a and b such that the function defined by

$$\begin{aligned} f(x) &= 5, && \text{for } x \le 2 \\ &= ax + b, && \text{for } 2 < x < 10 \\ &= 21, && \text{for } x \ge 10 \end{aligned}$$

is continuous in its domain.

vii. Show that the function defined by $f(x) = \sin(x^2)$ is a continuous function.

viii. Show that $f(x) = |(1 + x) + |x||$ is continuous for all $x \in R$.

ix. Prove that the exponential function, a^x is continuous at every point (where $a > 0$)

x. Prove that sine function is continuous at every real number.

***Solution*:**

i. The domain of f is [0, 6].
For $0 \le x \le 2$, $f(x) = x^2 - 4$.
Since f is a polynomial function it is continuous in [0, 2].
For $2 < x \le 4$, $f(x) = 2x + 3$.
Since f is a polynomial function it is continuous in (2, 4].
For $4 < x \le 6$, $f(x) = x^2 - 5$
Since f is a polynomial function it is continuous in (4, 6].

$\therefore$ f is continuous at every point in $[0, 2] \cup [2, 4] \cup (4, 6]$
For $x = 2$, $f(x) = x^2 - 4$
$f(2) = (2)^2 - 4 = 0$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} x^2 - 4 = (2)^2 - 4 = 0$ $\qquad$ $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} 2x + 3 = 2(2) + 3 = 7$

$\therefore$ $\lim\limits_{x\to 0^-} f(x) \ne \lim\limits_{x\to 0^+} f(x)$

$\therefore$ f is discontinuous at $x = 2$
For $x = 4$, $f(x) = 2x + 3$
$f(4) = 2(4) + 3 = 11$

$\lim\limits_{x\to 4^-} f(x) = \lim\limits_{x\to 4^-} 2x + 3 = 2(4) + 3 = 11$ $\qquad$ $\lim\limits_{x\to 4^+} f(x) = \lim\limits_{x\to 4^+} x^2 - 5 = (4)^2 - 5 = 11$

$\therefore$ $\lim\limits_{x\to 4^-} f(x) = \lim\limits_{x\to 4^+} f(x) = f(4)$

$\therefore$ f is continuous at $x = 4$.
Hence, f is continuous at $x = 4$ and discontinuous at $x = 2$.

ii. The domain of f is [2, 6].
For $2 \le x \le 4$, $f(x) = \dfrac{1}{x+1}$
Since, f is a rational function it is continuous for all $x \in R$ in [2, 4], except at those points where denominator is zero.
Now, $x + 1 = 0$ when $x = -1$
Since, $-1 \notin [2, 4]$, function f is continuous in [2, 4].
For $4 < x \le 6$, $f(x) = \dfrac{x+1}{x-3}$
Since, f is a rational function it is continuous for all $x \in R$ in (4, 6], except at those points where denominator is zero.
Now, $x - 3 = 0$ when $x = 3$
Since, $3 \notin (4, 6]$, function f is continuous in (4, 6]
For $x = 4$, $f(x) = \dfrac{1}{x+1}$

$\therefore$ $f(4) = \dfrac{1}{(4)+1} = \dfrac{1}{5}$

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \frac{1}{x+1} = \frac{1}{(4)+1} = \frac{1}{5}$$

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \frac{x+1}{x-3} = \frac{(4)+1}{(4)-3} = 5$$

$\therefore$ $\lim\limits_{x\to 4^-} f(x) \ne \lim\limits_{x\to 4^+} f(x)$

$\therefore$ f is discontinuous at $x = 4$
Hence, f is continuous on [2, 6] except at $x = 4$.

iii. The domain of f is [0, 10].

For $0 \le x \le 1$, $f(x) = 3$

Since f is a constant function it is continuous in [0, 1]

For $1 < x < 3$, $f(x) = 4$

Since f is a constant function it is continuous in (1, 3)

For $3 \le x \le 10$, $f(x) = 5$

Since f is a constant function it is continuous in [3, 10]

$\therefore$ f is continuous at every point in $[0, 1] \cup (1, 3) \cup [3, 10]$

For $x = 1$, $f(x) = 3$

$\therefore$ $f(1) = 3$

$\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^-} 3 = 3$

$\lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1^+} 4 = 4$

$\therefore$ $\lim\limits_{x\to1^-} f(x) \ne \lim\limits_{x\to1^+} f(x)$

$\therefore$ f is discontinuous at $x = 1$

For $x = 3$, $f(x) = 5$

$\therefore$ $f(3) = 5$

$\lim\limits_{x\to3^-} f(x) = \lim\limits_{x\to3^-} 4 = 4$

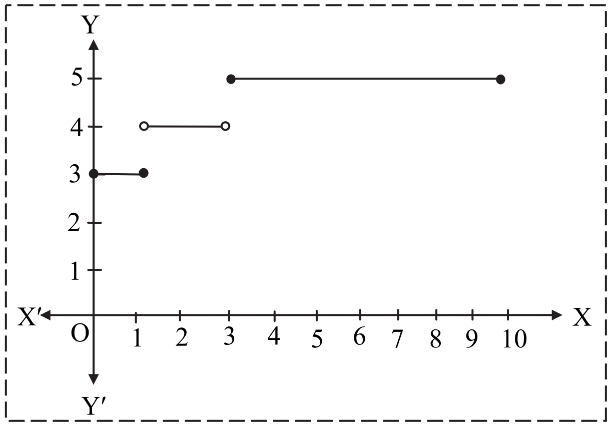
$\lim\limits_{x\to3^+} f(x) = \lim\limits_{x\to3^+} 5 = 5$

$\therefore$ $\lim\limits_{x\to3^-} f(x) \ne \lim\limits_{x\to3^+} f(x)$

$\therefore$ f is discontinuous at $x = 3$

$\therefore$ $f(x)$ is discontinuous at $x = 1$ and $x = 3$ in its domain [0,10]

$f(x)$ can be represented on a graph as follows. The graph is broken at $x = 1$ and $x = 3$. This proves that the function $f(x)$ is discontinuous at these two points.



iv. The domain of f is $[-\infty, \infty]$ such that $x \in R$.

For $x \le -1$, $f(x) = -2$.

Since f is a constant function it is continuous in $[-\infty, -1]$

For $-1 < x \le 1$, $f(x) = 2x$.

Since f is a polynomial function it is continuous in $(-1, 1]$

For $x > 1$, $f(x) = 2$

Since f is a polynomial function it is continuous in $(1, \infty]$

For $x = -1$, $f(x) = -2$

$\therefore$ $f(-1) = -2$

$\lim\limits_{x\to-1^-} f(x) = \lim\limits_{x\to-1^-} -2 = -2$

$\lim\limits_{x\to-1^+} f(x) = \lim\limits_{x\to-1^+} 2x = 2(-1) = -2$

$\therefore$ $\lim\limits_{x\to-1^-} f(x) = \lim\limits_{x\to-1^+} f(x) = f(-1)$

$\therefore$ f is continuous at $x = -1$

For $x = 1$, $f(x) = 2x$

$\therefore$ $f(1) = 2(1) = 2$

$\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^-} 2x = 2(1) = 2$

$\lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1^+} 2 = 2$

$\therefore$ $\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^+} f(x) = f(1)$

$\therefore$ f is continuous at $x = 1$

Hence, f is continuous for all $x \in R$.

v. The domain of f is $[-\infty, \infty]$

Such that $x \in R$.

For $x \le -3$, $f(x) = x + 3$.

Since f is a polynomial function it is continuous in $[-\infty,-3]$

For $-3 < x < 3$, $f(x) = -2x$

Since f is a polynomial function it is continuous in $(-3, 3)$

For $x \ge 3$, $f(x) = 6x + 2$

Since f is a polynomial function it is continuous in $[3, \infty]$

For $x = -3$; $f(x) = x + 3$

$\therefore$ $f(-3) = (-3) + 3 = 0$

$$\lim_{x\to-3^-} f(x) = \lim_{x\to-3^-} x+3 = (-3)+3 = 0$$

$$\lim_{x\to-3^+} f(x) \lim_{x\to-3^+} -2x = -2(-3) = 6$$

$\therefore$ $\lim_{x\to-3^-} f(x) \ne \lim_{x\to-3^+} f(x)$

$\therefore$ f is discontinuous at $x = -3$

For $x = 3$; $f(x) = 6x + 2$

$\therefore$ $f(3) = 6(3) + 2 = 20$

$$\lim_{x\to3^-} f(x) = \lim_{x\to3^-} -2x = -2(3) = -6$$

$$\lim_{x\to3^+} f(x) = \lim_{x\to3^+} 6x+2 = 6(3)+2 = 20$$

$$\lim_{x\to3^-} f(x) \ne \lim_{x\to3^+} f(x)$$

$\therefore$ f is discontinuous at $x = 3$

Hence, f is discontinuous at $x = -3$ and $x = 3$ on its domain.

vi. The function is continuous on $[2, 10]$.

At $x = 2$, $\lim_{x\to2^-} f(x) = \lim_{x\to2^+} f(x) = f(2)$

$f(2) = 5$

$\lim_{x\to2^-} f(x) = 5$

$\lim_{x\to2^+} f(x) = \lim_{x\to2^+} ax + b = 2a + b$

$\therefore$ $2a + b = 5$ ....(i)

At $x = 10$, $\lim_{x\to10^-} f(x) = \lim_{x\to10^+} f(x) = f(10)$

$f(10) = 21$ $\lim_{x\to10^-} f(x) = \lim_{x\to10^-} ax + b = 10a + b$

$\therefore$ $10a + b = 21$ ....(ii)

Solving (i) and (ii), we get

$a = 2$ and $b = 1$

vii. Let $g(x) = \sin x$ and $h(x) = x^2$ be functions defined on R.

Then, $(goh)(x) = g[h(x)]$

$= g(x^2)$

$= \sin(x^2)$

$= f(x)$, for all $x \in R$

$g(x) = \sin x$, being a sine function, is continuous for all $x \in R$.

Also, $h(x) = x^2$, being a polynomial function, is continuous for all $x \in R$.

Since, both g and h are continuous functions.

$\therefore$ f is a continuous function.

Hence, $f(x) = \sin(x^2)$ is a continuous function.

viii. Let $g(x) = (1+x) + |x|$ and $h(x) = |x|$ for all $x \in R$

Then, $(hog)(x) = h[g(x)]$

$= |g(x)|$ ... [$\because h(x) = |x|$]

$= |(1+x)+|x||$

$= f(x)$

Now, $h(x) = |x|$ is continuous for all $x \in R$,

$g(x) = (1+x) + |x|$ is the sum of a polynomial function and modulus function. Hence it is continuous for all $x \in R$.

Thus, f is a composite function of two continuous functions g and h.

Hence, the function f is continuous for all $x \in R$.

ix. We know that $f(x) = a^x$ (where $a > 0$) is defined for every rational number x.

Let c be a real number.

Put $x = c + h$, then as $x \to c$, $h \to 0$

$\therefore$ $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} a^x = \lim\limits_{h\to 0} a^{c+h} = \lim\limits_{h\to 0} a^c \cdot a^h = \left(\lim\limits_{h\to 0} a^c\right)\left(\lim\limits_{h\to 0} a^h\right) = (a^c)(a^0) = a^c(1) = a^c$

$= f(c)$

Thus, $\lim\limits_{x\to c} f(x) = f(c)$ (where $c > 0$)

Hence, $f(x) = a^x$ (where $a > 0$) is continuous for $x \in R$.

x. We know that $f(x) = \sin x$ is defined for every real number x.

Let c be a real number.

Put $x = c + h$, then as $x \to c$, $h \to 0$

$\therefore$ $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} \sin x = \lim\limits_{h\to 0} \sin(c+h) = \lim\limits_{h\to 0}(\sin c \cos h + \cos c \sin h)$

$= \lim\limits_{h\to 0}(\sin c \cdot \cos h) + \lim\limits_{h\to 0}(\cos c \cdot \sin h) = \sin c(1) + \cos c(0) = \sin c = f(c)$

Thus, $\lim\limits_{x\to c} f(x) = f(c)$

$\therefore$ sine function is continuous on R.

Miscellaneous Exercise – 1

1. Discuss the continuity of the following functions. Which of these functions have removable discontinuity? Redefine such a function at the given point so as to remove discontinuity.

i. $f(x) = \dfrac{(3^{\sin x} - 1)^2}{x \cdot \log(1+x)}$, for $x \neq 0$
$= 2\log 3$, for $x = 0$ } at $x = 0$

ii. $f(x) = (1 + \cos 2x)^{4\sec 2x}$, for $x \neq \dfrac{\pi}{4}$
$= e^4$, for $x = \dfrac{\pi}{4}$ } at $x = \dfrac{\pi}{4}$

iii. $f(x) = \dfrac{|x|}{x}$, for $x \neq 0$
$= 1$, for $x = 0$ } at origin

iv. $f(x) = \dfrac{\sin(a+x) + \sin(a-x) - 2\sin a}{x \sin x}$, for $x \neq a$
$= \sin a$, for $x = a$ } at $x = a$

v. $f(x) = \dfrac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2+1}-1}$, for $x \neq 0$
$= -4$, for $x = 0$ } at $x = 0$

***Solution*:**

i. $f(0) = 2\log 3 = \log 3^2$ ….(given)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{(3^{\sin x}-1)^2}{x\cdot\log(1+x)} = \lim_{x\to 0}\frac{\frac{\left(3^{\sin x}-1\right)^2}{x^2}}{\frac{x\cdot\log(1+x)}{x^2}} = \lim_{x\to 0}\frac{\frac{\left(3^{\sin x}-1\right)^2}{\sin^2 x}\times\frac{\sin^2 x}{x^2}}{\frac{x\cdot\log(1+x)}{x^2}}$$

$$= \lim_{x\to 0}\frac{\left(\frac{3^{\sin x}-1}{\sin x}\times\frac{\sin x}{x}\right)^2}{\frac{\log(1+x)}{x}} = \frac{\left(\lim\limits_{x\to 0}\frac{3^{\sin x}-1}{\sin x}\times\lim\limits_{x\to 0}\frac{\sin x}{x}\right)^2}{\lim\limits_{x\to 0}\frac{\log(1+x)}{x}}$$

$\therefore$ $\lim\limits_{x\to 0} f(x) = \dfrac{(\log 3\times 1)^2}{1} = (\log 3)^2$

$\therefore$ $\lim\limits_{x\to 0} f(x) \neq f(0)$

$\therefore$ f is discontinuous at $x = 0$.

The discontinuity of f is removable and it can be made continuous by redefining the function as

$f(x) = \dfrac{\left(3^{\sin x}-1\right)^2}{x\cdot\log(1+x)}$, for $x \neq 0$
$= (\log 3)^2$, for $x = 0$ } at $x = 0$

ii. $f\left(\dfrac{\pi}{4}\right) = e^4$ ….(given)

$\lim\limits_{x\to\frac{\pi}{4}} f(x) = \lim\limits_{x\to\frac{\pi}{4}} (1+\cos 2x)^{4\sec 2x}$

Put $\dfrac{\pi}{4} - x = h$, then $x = \dfrac{\pi}{4} - h$

As $x \to \dfrac{\pi}{4}$, $h \to 0$

$$\therefore \lim_{x\to\frac{\pi}{4}} f(x) = \lim_{h\to 0}\left[1+\cos 2\left(\frac{\pi}{4}-h\right)\right]^{4\sec\left[2\left(\frac{\pi}{4}-h\right)\right]} = \lim_{h\to 0}\left[1+\cos\left(\frac{\pi}{2}-2h\right)\right]^{4\sec\left[\left(\frac{\pi}{2}-2h\right)\right]} = \lim_{h\to 0}\left[1+\sin 2h\right]^{4\operatorname{cosec} 2h}$$

$$= \lim_{h\to 0}\left[(1+\sin 2h)^{\frac{1}{\sin 2h}}\right]^4 = e^4$$

$\therefore$ $\lim\limits_{x\to\frac{\pi}{4}} f(x) = f\left(\dfrac{\pi}{4}\right)$

$\therefore$ f is continuous at $x = \dfrac{\pi}{4}$.

iii. $f(0) = 1$ ….(given)

Now, $|x| = -x$ $\forall\ x < 0$

$\therefore$ $\lim\limits_{x\to 0^-}\dfrac{|x|}{x} = \dfrac{-x}{x} = -1$

Also, $|x| = x$ $\forall\ x > 0$

$\therefore \quad \lim_{x\to 0^+} \dfrac{|x|}{x} = \dfrac{x}{x} = 1$

$\therefore \quad \lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

$\therefore$ f is discontinuous at $x = 0$.

The discontinuity is irremovable as f(x) is defined for all $x \neq 0$.

iv. f(a) = sin a ….(given)

$$\lim_{x\to a} f(x) = \lim_{x\to a} \frac{\sin(a+x)+\sin(a-x)-2\sin a}{x\sin x} = \lim_{x\to a} \frac{2\sin a\cos x - 2\sin a}{x\sin x} = \lim_{x\to a} \frac{2\sin a(\cos x - 1)}{x\sin x}$$

$$= \lim_{x\to a} \frac{2\sin a\left(-2\sin^2\frac{x}{2}\right)}{x\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)} = \lim_{x\to a} \frac{-2\sin a\sin\frac{x}{2}}{x\cos\frac{x}{2}} = \lim_{x\to a} (-2\sin a)\ \frac{\tan\left(\frac{x}{2}\right)}{x}$$

$$= -2\sin a \lim_{x\to a} \frac{\tan\left(\frac{x}{2}\right)}{\frac{x}{2}\times 2} = -2\sin a\ \frac{1}{2}\left[\lim_{x\to a}\frac{\tan\left(\frac{x}{2}\right)}{\frac{x}{2}}\right]$$

$$= -2\sin a\left(\frac{1}{2}\right)$$

$$= -\sin a$$

$\therefore \quad \lim_{x\to a} f(x) \neq f(a)$

$\therefore$ f is discontinuous at x = a.

The discontinuity of f is removable and it can be made continuous by redefining the function as

$$f(x) = \left.\begin{array}{ll} \dfrac{\sin(a+x)+\sin(a-x)-2\sin a}{x\sin x}, & \text{for } x \neq a \\ = -\sin a, & \text{for } x = a \end{array}\right\} \text{at } x = a$$

v. f(0) = – 4 ….(given)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2+1}-1} = \lim_{x\to 0} \frac{-\left(1-\cos^2 x\right)-\sin^2 x}{\sqrt{x^2+1}-1} \times \frac{\left(\sqrt{x^2+1}+1\right)}{\left(\sqrt{x^2+1}+1\right)}$$

$$= \lim_{x\to 0} \frac{\left(-\sin^2 x - \sin^2 x\right)\left(\sqrt{x^2+1}+1\right)}{\left(\sqrt{x^2+1}-1\right)\left(\sqrt{x^2+1}+1\right)} = \lim_{x\to 0} \frac{\left(-2\sin^2 x\right)\left(\sqrt{x^2+1}+1\right)}{x^2+1-1}$$

$$= \lim_{x\to 0} \frac{\left(-2\sin^2 x\right)\left(\sqrt{x^2+1}+1\right)}{x^2} = -2\lim_{x\to 0} \frac{\sin^2 x}{x^2}\left(\sqrt{x^2+1}+1\right)$$

$$= -2\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2 \times \lim_{x\to 0}\left(\sqrt{x^2+1}+1\right)$$

$$= -2 \times (1)^2 \times \left(\sqrt{0+1}+1\right) = -2 \times (1+1)$$

$$= -2 \times 2$$

$\therefore \quad \lim_{x\to 0} f(x) = -4$

$\therefore \quad \lim_{x\to 0} f(x) = f(0)$

$\therefore$ f is continuous at $x = 0$.

2. Find the value of k, if the functions are continuous at the points indicated.

i. $f(x) = \log_{(1-2x)}(1+2x)$, for $x \neq 0$; $= k$, for $x = 0$ } at $x = 0$

ii. $f(x) = \dfrac{k\cos x}{\pi - 2x}$, for $x \neq \dfrac{\pi}{2}$; $= 3$, for $x = \dfrac{\pi}{2}$ } at $x = \dfrac{\pi}{2}$

iii. $f(\theta) = \dfrac{1-\tan\theta}{1-\sqrt{2}\sin\theta}$, for $\theta \neq \dfrac{\pi}{4}$; $= \dfrac{k}{2}$, for $\theta = \dfrac{\pi}{4}$ } at $\theta = \dfrac{\pi}{4}$

iv. $f(x) = \left[\tan\left(\dfrac{\pi}{4} + x\right)\right]^{\frac{1}{x}}$, for $x \neq 0$; $= k$, for $x = 0$ } at $x = 0$

***Solution*:**

i. $f(0) = k$ ....(given)

Since $f(x)$ is continuous at $x = 0$

$\therefore$ $f(0) = \lim\limits_{x \to 0} f(x)$

$$\therefore \quad k = \lim_{x\to 0} \log_{(1-2x)}(1+2x) = \lim_{x\to 0} \frac{\log(1+2x)}{\log(1-2x)} = \frac{\lim\limits_{x\to 0}\left[\dfrac{\log(1+2x)}{2x}\right]}{-\lim\limits_{x\to 0}\left[\dfrac{\log(1-2x)}{-2x}\right]} = \frac{1}{-1} = -1$$

$\therefore$ $k = -1$

ii. $f\left(\dfrac{\pi}{2}\right) = 3$ ....(given)

Since $f(x)$ is continuous at $x = \dfrac{\pi}{2}$

$\therefore$ $f\left(\dfrac{\pi}{2}\right) = \lim\limits_{x \to \frac{\pi}{2}} f(x)$

$$\therefore \quad 3 = \lim_{x\to\frac{\pi}{2}} \frac{k\cos x}{\pi - 2x}$$

Put $\dfrac{\pi}{2} - x = h$, then $x = \dfrac{\pi}{2} - h$

As $x \to \dfrac{\pi}{2}$, $h \to 0$

$$\therefore \quad 3 = \lim_{h\to 0} \frac{k\cos\left(\dfrac{\pi}{2} - h\right)}{\pi - 2\left(\dfrac{\pi}{2} - h\right)}$$

$$\therefore \quad 3 = k\lim_{h\to 0} \frac{\cos\left(\dfrac{\pi}{2} - h\right)}{2h} = \frac{k}{2}\lim_{h\to 0}\frac{\sinh}{h} = \frac{k}{2}$$

$\therefore$ $k = 6$

iii. $f\left(\frac{\pi}{4}\right) = \frac{k}{2}$ ….(given)

Since f(x) is continuous at $\theta = \frac{\pi}{4}$

$\therefore$ $f\left(\frac{\pi}{4}\right) = \lim\limits_{\theta\to\frac{\pi}{4}} f(\theta)$

$$\therefore \frac{k}{2} = \lim_{\theta\to\frac{\pi}{4}} \frac{1-\tan\theta}{1-\sqrt{2}\sin\theta} = \lim_{\theta\to\frac{\pi}{4}} \frac{1-\frac{\sin\theta}{\cos\theta}}{1-\sqrt{2}\sin\theta} = \lim_{\theta\to\frac{\pi}{4}} \frac{\cos\theta-\sin\theta}{\cos\theta\left(1-\sqrt{2}\sin\theta\right)}$$

$$= \lim_{\theta\to\frac{\pi}{4}} \frac{\cos\theta-\sin\theta}{\cos\theta\left(1-\sqrt{2}\sin\theta\right)} \times \frac{1+\sqrt{2}\sin\theta}{1+\sqrt{2}\sin\theta} = \lim_{\theta\to\frac{\pi}{4}} \frac{(\cos\theta-\sin\theta)\left(1+\sqrt{2}\sin\theta\right)}{\cos\theta\left(1-2\sin^2\theta\right)}$$

$$= \lim_{\theta\to\frac{\pi}{4}} \frac{(\cos\theta-\sin\theta)(1+\sqrt{2}\sin\theta)}{\cos\theta(\cos^2\theta+\sin^2\theta-2\sin^2\theta)} = \lim_{\theta\to\frac{\pi}{4}} \frac{(\cos\theta-\sin\theta)\left(1+\sqrt{2}\sin\theta\right)}{\cos\theta(\cos^2\theta-\sin^2\theta)}$$

$$= \lim_{\theta\to\frac{\pi}{4}} \frac{(\cos\theta-\sin\theta)\left(1+\sqrt{2}\sin\theta\right)}{\cos\theta(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}$$

$$= \lim_{\theta\to\frac{\pi}{4}} \frac{\left(1+\sqrt{2}\sin\theta\right)}{\cos\theta(\cos\theta+\sin\theta)} \quad \ldots\left[\because \theta\to\frac{\pi}{4}, \cos\theta\to\frac{1}{\sqrt{2}} \text{ and } \sin\theta\to\frac{1}{\sqrt{2}}, \cos\theta-\sin\theta\neq 0\right]$$

$$= \frac{1+\sqrt{2}\sin\left(\frac{\pi}{4}\right)}{\cos\frac{\pi}{4}\left(\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\right)} = \frac{1+\sqrt{2}\times\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)}$$

$$= \frac{1+1}{\frac{1}{\sqrt{2}}\times\frac{2}{\sqrt{2}}} = \frac{2}{1} = 2$$

$\therefore$ $\frac{k}{2} = 2$

$\therefore$ $k = 4$

iv. $f(0) = k$ ….(given)

Since f(x) is continuous at $x = 0$

$$\therefore f(0) = \lim_{x\to0} f(x) = \lim_{x\to0}\left[\tan\left(\frac{\pi}{4}+x\right)\right]^{\frac{1}{x}} = \lim_{x\to0}\left(\frac{1+\tan x}{1-\tan x}\right)^{\frac{1}{x}} = \lim_{x\to0}\frac{(1+\tan x)^{\frac{1}{x}}}{(1-\tan x)^{\frac{1}{x}}}$$

$$= \lim_{x\to0}\frac{\left((1+\tan x)^{\frac{1}{\tan x}}\right)^{\frac{\tan x}{x}}}{\left((1-\tan x)^{\frac{1}{-\tan x}}\right)^{\frac{-\tan x}{x}}} = \frac{\left(\lim\limits_{x\to0}(1+\tan x)^{\frac{1}{\tan x}}\right)^{\lim\limits_{x\to0}\frac{\tan x}{x}}}{\left(\lim\limits_{x\to0}(1-\tan x)^{\frac{1}{-\tan x}}\right)^{-\lim\limits_{x\to0}\frac{\tan x}{x}}}$$

$$= \frac{e^1}{e^{-1}} = e^2$$

$\therefore$ $k = e^2$

3. i. If f (x) is continuous on [–4, 2] defined as

$f(x) = 6b - 3ax$, for $-4 \le x < -2$

$= 4x + 1$, for $-2 \le x \le 2$,

Find the value of a + b.

ii. Find the relationship between a and b, so that the function f(x) defined by

$$f(x) = \begin{cases} ax+1, & \text{if } x \le 3 \\ bx+3, & \text{if } x > 3 \end{cases}$$

is continuous at $x = 3$

iii. A function f(x) is defined as $f(x) = \begin{cases} \dfrac{1}{1+e^{\frac{1}{x}}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

Is the function continuous at $x = 0$?

iv. Prove that every rational function is continuous.

v. Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$?

***Solution*:**

i. Since f is continuous on [–4, 2],

$\therefore$ f is continuous on $x = -2$

$\therefore$ $\lim\limits_{x\to -2^-} f(x) = \lim\limits_{x\to -2^+} f(x)$

$\therefore$ $\lim\limits_{x\to -2^-} 6b - 3ax = \lim\limits_{x\to -2^+} 4x + 1$

$\therefore$ $6b - 3a(-2) = 4(-2) + 1$

$\therefore$ $6b + 6a = -7$

$\therefore$ $6(a + b) = -7$

$\therefore$ $a + b = \dfrac{-7}{6}$

ii. Since f is continuous at $x = 3$

$\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x)$

$\therefore$ $\lim\limits_{x\to 3^-} ax + 1 = \lim\limits_{x\to 3^+} bx + 3$

$\therefore$ $a(3) + 1 = b(3) + 3$

$\therefore$ $3a - 3b = 2$

$\therefore$ $3(a - b) = 2$

$\therefore$ $a - b = \dfrac{2}{3}$

$\therefore$ $a = \dfrac{2}{3} + b$

iii. $f(0) = 0$ ....(given)

$\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{1}{1+e^{\frac{1}{x}}}$

As $x \to 0^+$, $\dfrac{1}{x} \to +\infty$, $e^{\frac{1}{x}} \to e^{\infty}$ i.e., $e^{\frac{1}{x}} \to +\infty$

$\therefore$ $\lim\limits_{x\to 0^+} f(x) \lim\limits_{x\to 0^+} \dfrac{1}{1+e^{\frac{1}{x}}} = \dfrac{1}{1+\infty} = \dfrac{1}{\infty} = 0$

And as $x \to 0^-$, $\dfrac{1}{x} \to -\infty$, $e^{\frac{1}{x}} \to e^{-\infty} = \dfrac{1}{e^{\infty}}$ i.e., $e^{\frac{1}{x}} \to 0$

$\therefore$ $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{1+e^{\frac{1}{x}}} = \frac{1}{1+0} = 1$

$\therefore$ $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

$\therefore$ f is discontinuous at $x = 0$.

iv. Let $f(x) = \frac{g(x)}{h(x)}$, [$h(x) \neq 0$] be a rational function where g and h are polynomial functions.

The domain of f is the set of all real numbers except those points at which h is zero. Since polynomial functions are continuous, $f(x)$ is continuous.

v. $f(x) = x^2 - \sin x + 5$

$\therefore$ $f(\pi) = \pi^2 - \sin \pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$

Also, $\lim_{x\to\pi} f(x) = \lim_{x\to\pi} (x^2 - \sin x + 5) = \pi^2 - \sin \pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$

Since, $\lim_{x\to\pi} f(x) = f(\pi)$, f is continuous at $x = \pi$.

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4. i. Discuss the continuity of the function $f(x) = |x| + |x - 1|$ in interval [–1, 2].

ii. A function defined by

$$f(x) = \begin{cases} x + a, & \text{for } x < 0 \\ x, & \text{for } 0 \le x < 1 \\ b - x, & \text{for } x \ge 1 \end{cases}$$

is continuous in [–2, 2]. Show that (a + b) is even. **[Mar 13]**

iii. Show that $f(x) = \begin{cases} \frac{\sin x}{x}, & x < 0 \\ x+1, & x \ge 0 \end{cases}$ is a continuous function.

***Solution*:**

i. Let $g(x) = |x|$.

Then, $g(x) = x$, for all $x \ge 0$

$= -x$, for all $x < 0$

Now, $g(0) = 0$

Also, $\lim_{x\to 0^-} g(x) = \lim_{x\to 0} (-x) = 0$ and $\lim_{x\to 0^+} g(x) = \lim_{x\to 0} x = 0$

Thus, $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^+} g(x) = g(0)$

$\therefore$ $\lim_{x\to 0} g(x)$ exists and $\lim_{x\to 0} g(x) = 0$.

$\therefore$ $\lim_{x\to 0} g(x) = g(0)$

Hence, $g(x) = |x|$ is continuous at $x = 0$.

Let $x \in [-1, 2] - \{0\}$

Since, $g(x)$ is a polynomial in [–1, 0) and (0, 2], it is continuous in these intervals.

$\therefore$ $g(x)$ is continuous in [–1, 2].

Let $h(x) = |x - 1|$.

Then, $h(x) = x - 1$, for all $x \ge 1$

$= -(x - 1)$, for all $x < 1$

Now, $h(1) = 1 - 1 = 0$

Also, $\lim_{x\to 1^-} h(x) = \lim_{x\to 1} -(x - 1) = -(1 - 1) = 0$ and $\lim_{x\to 1^+} h(x) = \lim_{x\to 1} (x - 1) = 1 - 1 = 0$.

Thus, $\lim_{x\to 1^-} h(x) = \lim_{x\to 1^+} h(x) = h(1)$

$\therefore$ $\lim_{x\to 1} h(x)$ exists and $\lim_{x\to 1} h(x) = h(1)$

Hence, $h(x) = |x - 1|$ is continuous at $x = 1$.

Let $x \in [-1, 2] - \{1\}$. Since h(x) is a polynomial in [–1, 1) and (1, 2], it is continuous in these intervals.

∴ h(x) = $|x - 1|$ is continuous in [– 1, 2].

Now, f(x) = g(x) + h(x) and g(x) as well as h(x) are continuous on [–1, 2].

The sum of two continuous functions is a continuous function.

Hence, f(x) is continuous in [– 1, 2].

ii. f(x) is continuous in [– 2, 2].

∴ f(x) is continuous at x = 0.

∴ $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x)$

∴ $\lim\limits_{x\to 0^-} (x + a) = \lim\limits_{x\to 0^+} x$

∴ 0 + a = 0

∴ a = 0

Also, f(x) is continuous at x = 1.

∴ $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x)$

∴ $\lim\limits_{x\to 1^-} x = \lim\limits_{x\to 1^+} (b - x)$

∴ 1 = b – 1

∴ b = 2

∴ a + b = 0 + 2 = 2

Hence, (a + b) is even.

iii. f(0) = (0) + 1 = 1

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x + 1 = (0) + 1 = 1$$

∴ $\lim\limits_{x\to 0^-} \dfrac{\sin x}{x} = 0 + 1 = \lim\limits_{x\to 0^+} (x + 1)$

∴ 1 = 1 = 1

Hence, f(x) is continuous at x = 0.

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5. i. Examine that |f(x)| is a continuous function on its domain D, where f(x) is continuous on its domain D.

ii. Prove that composition of two continuous functions is continuous.

iii. Show that the function defined by f(x) = |cos x| is continuous functions.

***Solution*:**

i. Let x = a be the point of the domain D of function f.

We have, $|f(x)| = -f(x)$ for f(x) < a (say D_1)

$= f(x)$ for f(x) ≥ a (say D_2)

Since f(x) is continuous on its domain, say D, –f(x) is continuous on its domain, say D_1.

Further, f(x) is continuous on its domain, say D_2, such that $D = D_1 \cup D_2$

∴ $|f(x)|$ is continuous on its domain D.

ii. If f and g are two real valued functions, then (gof) (x) = g(f(x)) is defined (i.e., exists) whenever the range of f is a subset of domain of g.

Let f and g be two real valued functions such that gof is defined at x = a (‘a’ is an element of domain of f)

Then, g(f(a)) exists.

Let f be continuous at x = a.

∴ $\lim\limits_{x\to a} f(x) = f(a) = b$ (say) ….(i)

Let g be continuous at x = b.

∴ $\lim\limits_{x\to b} g(x) = g(b) = g(f(a))$ ….(ii)

Now, $\lim\limits_{x\to a} g(f(x)) = g\left(\lim\limits_{x\to a} f(x)\right) = g(b)$ ....[From (i)]

$= g(f(a))$ ....[From (ii)]

Hence, gof is continuous at $x = a$.

But, $x = a$ is an arbitrary point on the domain of f.

$\therefore$ the composition of two continuous functions is continuous.

iii. First, we need to verify that cosine function is continuous.

Now, $\lim\limits_{x\to 0} \cos x = 1$ and $\lim\limits_{x\to 0} \sin x = 0$

We know that, $f(x) = \cos x$ is defined for every real number.

Let c be a real number.

Put $x = c + h$, then as $x \to c$, $h \to 0$

$\therefore$ $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} \cos x = \lim\limits_{h\to 0} \cos(c + h) = \lim\limits_{h\to 0} (\cos c \cdot \cos h - \sin c \cdot \sin h)$

$= \lim\limits_{h\to 0} (\cos c \cdot \cos h) - \lim\limits_{h\to 0} (\sin c \cdot \sin h) = \cos c(1) - \sin c(0) = \cos c = f(c)$

Thus, $\lim\limits_{x\to c} f(x) = f(c)$

Hence, cosine function is a continuous function.

Let $g(x) = |\cos x|$

Then, $g(x) = -\cos x$ for $\cos x < 0$

$= \cos x$ for $\cos x \geq 0$

Since $\cos x$ is continuous, $-\cos x$ (i.e., $k \cos x$, where $k = -1$) is also continuous.

$\therefore$ $g(x) = |\cos x|$ is a continuous function.

Multiple Choice Questions

1. The function $f(x) = \dfrac{4 - x^2}{4x - x^3}$ is
   (A) discontinuous at only one point.
   (B) discontinuous exactly at two points.
   (C) discontinuous exactly at three points.
   (D) discontinuous at many points.

2. If $f(x) = \begin{cases} \dfrac{|x|}{x}, \text{for } x \neq 0 \\ 0, \text{for } x = 0 \end{cases}$, then f(x) is
   (A) continuous at $x = 0$.
   (B) discontinuous at $x = 0$.
   (C) discontinuous everywhere.
   (D) continuous everywhere.

3. If $f(x) = \begin{cases} kx + 5, \text{if } x < 2 \\ x - 1, \text{if } x > 2 \end{cases}$ and $\lim\limits_{x\to 2} f(x)$ exist, then the value of k is
   (A) -2 (B) 2 (C) $\dfrac{1}{2}$ (D) $\dfrac{-1}{2}$

4. If $f(x) = \begin{cases} 2 &, 0 \leq x < 1 \\ c - 2x &, 1 \leq x \leq 2 \end{cases}$ is continuous at $x = 1$, then c =
   (A) 2 (B) 4
   (C) 0 (D) 1

5. If $f(x) = \begin{cases} 1 &, \text{if } x \leq 3 \\ ax + b &, \text{if } 3 < x < 5 \\ 7 &, \text{if } 5 \leq x \end{cases}$ is continuous, then the value of a and b is
   (A) 3, 8 (B) -3, 8
   (C) 3, -8 (D) -3, -8

6. For what value of k the function $f(x) = \begin{cases} \dfrac{\sqrt{5x+2} - \sqrt{4x+4}}{x - 2}, & \text{if } x \neq 2 \\ k &, \text{if } x = 2 \end{cases}$ is continuous at $x = 2$?
   (A) $\dfrac{-1}{4\sqrt{3}}$ (B) $\dfrac{1}{2\sqrt{3}}$
   (C) $\dfrac{1}{4\sqrt{3}}$ (D) $\dfrac{-1}{2\sqrt{3}}$

7. The sum of two discontinuous functions
   (A) is always discontinuous.
   (B) may be continuous.
   (C) is always continuous.
   (D) may be discontinuous.

8. If $f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$ is continuous at $x = 0$, then the value of k =
   (A) 1 (B) -1 (C) 0 (D) 2

9. $f(x) = \begin{cases} 1 & , \text{ when } x \in Q \\ -1 & , \text{ when } x \notin Q \end{cases}$ is
(A) continuous everywhere.
(B) discontinuous everywhere.
(C) continuous only at $x = 0$.
(D) continuous at every rational number.

10. If $f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x}, & x<0 \\ \dfrac{1}{2} & , x=0 \\ \dfrac{x^{3\sqrt{2}}+1}{2} & , x>0 \end{cases}$
is continuous at $x = 0$, then the value of a is
(A) $\dfrac{1}{2}$ (B) $-\dfrac{1}{2}$
(C) $\dfrac{3}{2}$ (D) $-\dfrac{3}{2}$

11. Function $f(x) = \begin{cases} \dfrac{x^2-4x+3}{x^2-1}, & x \neq 1 \\ 2 & , x=1 \end{cases}$, is
(A) continuous at $x = 1$.
(B) continuous at $x = -1$.
(C) continuous at $x = 1$ and $x = -1$.
(D) discontinuous at $x = 1$.

12. Determine the value of constant k so that the function $f(x) = \begin{cases} \dfrac{1-\cos kx}{x\sin x}, & x \neq 0 \\ 2 & , x=0 \end{cases}$ is continuous at $x = 0$
(A) $k = \pm 2$ (B) $k = \pm 4$
(C) $k = \pm 1$ (D) $k = \pm 3$

13. If function $f(x) = \begin{cases} 5x-4 & , 0<x \le 1 \\ 4x^2+3bx & , 1<x<2 \end{cases}$
is continuous at every point of its domain, then b is equal to
(A) 0 (B) 1 (C) -1 (D) 3

14. If $f(x) = \begin{cases} \dfrac{1-\sin x}{\pi-2x}, & x \neq \dfrac{\pi}{2} \\ k & , x=\dfrac{\pi}{2} \end{cases}$ is continuous at $x = \dfrac{\pi}{2}$, then k =
(A) 0 (B) 1 (C) -1 (D) 3

15. If $f(x) = \begin{cases} \dfrac{\sin 3x}{\sin x}, & x \neq 0 \\ k & , x=0 \end{cases}$ is a continuous function, then k =
(A) 1 (B) 3 (C) $\dfrac{1}{3}$ (D) 0

16. The function
$f(x) = \begin{cases} 3x-5, & \text{for } x<3 \\ x+1, & \text{for } x>3 \\ c, & \text{for } x=3 \end{cases}$
is continuous at $x = 3$, if c is equal to
(A) 4 (B) 3 (C) 1 (D) 2

17. The function $y = 3\sqrt{x} - |x-1|$ is continuous at
(A) $x = 0$ (B) $x > 0$
(C) $0 \le x \le 1$ (D) $x \ge 1$

18. Let $f(x) = \begin{cases} 1, & x \le -1 \\ |x|, & -1<x<1 \\ 0, & x \ge 1 \end{cases}$, then
(A) f is continuous at $x = -1$.
(B) f is differentiable at $x = -1$.
(C) f is continuous everywhere.
(D) f is differentiable for all x.

19. $f(x) = \begin{cases} \left(\dfrac{3}{x^2}\right)\sin 2x^2, & \text{if } x<0 \\ \dfrac{x^2+2x+c}{1-3x^2}, & \text{if } x \ge 0, x \neq \dfrac{1}{\sqrt{3}} \\ 0, & \text{if } x = \dfrac{1}{\sqrt{3}} \end{cases}$
Then in order that f be continuous at $x = 0$, the value of c =
(A) 2 (B) 4 (C) 6 (D) 8

20. The function $f(x) = \dfrac{\log(1+ax)-\log(1-bx)}{x}$ is not defined at $x = 0$. The value which should be assigned to f at $x = 0$ so that it is continuous at $x = 0$, is
(A) a – b (B) a + b
(C) log a + log b (D) log a – log b

Answers to Multiple Choice Questions

1.	(C)	2.	(B)	3.	(A)	4.	(B)
5.	(C)	6.	(C)	7.	(B)	8.	(C)
9.	(B)	10.	(D)	11.	(D)	12.	(A)
13.	(C)	14.	(A)	15.	(B)	16.	(A)
17.	(B)	18.	(A)	19.	(C)	20.	(B)