MBA

(DISTANCE MODE)

DBA 1701

APPLIED OPERATIONAL RESEARCH FOR MANAGEMENT

III SEMESTER
COURSE MATERIAL

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FOREWORD

Let me at the outset express my hearty congratulations to all the students on the eve of themselves being promoted to higher semester.

Anna University Chennai, one of the world's leading Technological Universities has launched the distance mode of education with the prime objective of providing education to the deserving aspirants, who have been deprived of enhancing their professional competencies.

Response to distance education programme of Anna University Chennai is very encouraging from all stake holders – industries, individuals, professional bodies and others. This is evident from the number of applicants who belong to diverse clusters that include Information Technology, Manufacturing, Marketing, Medical, Administrative Services, Engineering and Consultancy etc.

In this background, the course materials have been prepared by experts, focusing on the theoretical and conceptual underpinning of the concerned subject matters along with practical exposure to the extent needed.

The Indian industries witness brand new challenges in terms of frequent new entrants from both within and outside the nation, technology upgradation at a faster rate, shorter product life cycle, increase in the customer expectations and the like.

This scenario demands our professionals to be empowered with world class managerial and technical inputs, so that they can render their best in the profession of their choice. The course materials prepared shall provide a strong platform to enable professionals to meet the challenges effectively.

Further keeping in mind the constraints of time and other resources of the enrolled students, the course materials prepared are intended to serve as ready reckoner to meet the academic requirements in a rewarding manner.

This endeavour of Anna University Chennai is to make abundant availability of value embedded human capital to the nation at large and there by enable India to be globally strengthened on multiple facets.

I wish the students all the very best.

(D. VISWANATHAN)

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- Pannerselvam R, Operations Research, Prentice Hall of India, New Delhi.
- Hamdy A. Taha, Operations Research – An Introduction, Sixth Edition, Prentice Hall of India, New Delhi.
- Gupta P.K and Man Mohan, Problems in Operations Research (Methods & Solutions), Sultan Chand & Sons, New Delhi.
- Kalavathy.S. Operations Research, Vikas Publishing House Pvt.Ltd

Inspite of at most care taken to prepare the list of references any omission in the list is only accidental and not purposeful.

Mr.Jayanth Jacob
Author

DBA 1701 APPLIED OPERATIONAL RESEARCH FOR MANAGEMENT

UNIT I – INTRODUCTION TO LINEAR PROGRAMMING (LP)

Introduction to applications of operations research in functional areas of management. Linear programming-Formulation, Solution by graphical and simplex methods (primal –penalty, two phase), Special cases, Dual simplex method, Principles of duality, Sensitivity analysis.

UNIT II – LINEAR PROGRAMMING EXTENSIONS

Transportation models (minimizing and maximizing cases) –Balanced and unbalanced cases –Initial basic feasible solution by N-W corner rule, least cost and Vogel's approximation methods. Check for optimality. Solution by MODI /Stepping stone method. Cases of degeneracy. Transhipment models.

UNIT III – INTEGER LINEAR PROGRAMMING AND GAME THEORY

Solution to pure and mixed integer programming problem by Branch and bound and cutting plane algorithms. Game theory-Two person zero sum games-saddle point, Dominance Rule, Convex Linear combination (averages), methods of matrices, graphical and LP solutions.

UNIT IV – DYNAMIC PROGRAMMING, SIMULATION AND DECISION THEORY

Dynamic programming (DP) – Deterministic cases – Maximizing and minimizing problems. DP techniques for LP problems. Decision making under risk – Decision trees – Decision making under uncertainty. Application of simulation techniques for decision making.

UNIT V – QUEUING THEORY AND REPLACEMENT MODELS

Queuing theory – single and multi-channel models – Infinite number of customers and infinite calling source. Replacement models –Individual replacement models (with and without time value of money) –Group replacement models.

REFERENCES

1. Paneerselvam R., Operations Research, Prentice Hall of India, Fourth print, August 2003.
2. TulsianP.C, Vishal Pandey, Quantitative Techniques (Theory and problems), Pearson Education (Asia), First Indian Reprint, 2002

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UNIT I

INTRODUCTION TO LINEAR PROGRAMMING (LP)

INTRODUCTION

Operations Research (OR) (a term coined by McClosky and Trefthen in 1940) was a technique that evolved during World War II to effectively use the limited military resources and yet achieve the best possible results in military operations. In essence you can state that OR is a technique that helps achieve best (optimum) results under the given set of limited resources. Over the years, OR has been adapted and used very much in the manufacturing sector towards optimization of resources. That is to use minimum resources to achieve maximum output or profit or revenue.

Learning Objectives

The learning objectives in this unit are

1. To formulate a Linear programming problem (LPP) from set of statements.
2. To solve the LPP using graphical method (For 2 variables)
3. To solve the LPP using primal simplex method (For > 2 variables and all <= constraints)
4. To solve the LPP using Big-M and Two-Phase methods(For > 2 variables and all mixed constraints)
5. To solve the LPP using dual simplex method (For > 2 variables and the solution is infeasible)
6. We also have a look at the effect of changing the values of parameters on the decision variables (Sensitivity Analysis)

Applications of Operations Research in functional areas of Management

The models of OR can be used by economists, statisticians, administrators and technicians both in defence and in business. In general OR can be applied in all the functional areas of Management namely., Accounting, Construction, Facilities Planning, Marketing, Finance, Personnel, Research and Development and Production.

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Accounting

Assigning audit teams effectively
Credit policy analysis
Cash flow planning Developing standard costs
Establishing costs for byproducts
Planning of delinquent account strategy

Construction

Project scheduling,
Monitoring and control
Determination of proper work force
Deployment of work force
Allocation of resources to projects

Facilities Planning

Factory location and size decision
Estimation of number of facilities Hospital planning
International logistic system design Transportation loading and unloading Warehouse location decision

Finance

Building cash management models
Allocating capital among various alternatives
Building financial planning models Investment analysis
Portfolio analysis
Dividend policy making

Production

Inventory control
Marketing balance projection Production scheduling
Production smoothing

Marketing

Advertising budget allocation
Product introduction timing
Selection of Product mix
Deciding most effective packaging alternative

Organizational Behavior / Human Resources

Personnel planning
Recruitment of employees
Skill balancing
Training program scheduling
Designing organizational structure more effectively

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Purchasing

Optimal buying
Optimal reordering
Materials transfer

Research and Development

R & D Projects control
R & D Budget allocation
Planning of Product introduction

In general solution to an OR problem will have the following stages.

1. Formulating the problem
2. Constructing a mathematical model based on the formulation
3. Solving the problem based on the model.
4. Checking whether the solution is optimal and feasible
5. Iterate till optimal and feasible solution is reached.

We have two new terms “optimal” and “feasible”. Optimal means the best possible solution under the given conditions and feasible means a practical solution. Therefore optimal and feasible solution means that the suggested solution must be both practical to implement and the best one under the given conditions.

All types of problems in OR can be categorized as either MINIMIZING or MAXIMIZING type.

We will focus on MINIMIZING costs, time and distances while we will be interested in MAXIMIZING revenue, profits and returns. So you must be very careful in identifying the type of problem while deciding upon the choice of algorithm to solve the given problem.

With this brief introduction I think it is time we get going with the real content. As I had mentioned earlier the first stage in solving a problem through OR models is to formulate the given problem statement into an Linear Programming Problem or Model (LPP)

1.1 FORMULATION OF LINEAR PROGRAMMING PROBLEM (LPP)

Example 1.1

Elixir paints produces both interior and exterior paints from two raw materials M1 and M2. The following table provides the data

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Table 1.1

Raw Material	Tons of Raw materials per ton of		Maximum daily availability (In tons)
	Exterior Paint	Interior Paint	
M1	6	4	24
M2	1	2	6
Profits in thousands of Rupees per ton	5	4	

The market survey restricts the market daily demand of interior paints to 2 tons. Additionally the daily demand for interior paints cannot exceed that of exterior paints by more than 1 ton. Formulate the LPP.

Solution:

The general procedure for formulation of an LPP is as follows

1. To identify and name the decision variables.

That is to find out the variables of significance and what is to be ultimately determined. In this example the quantity (In tons) of exterior (EP) and interior paints (IP) to be produced under the given constraints to maximize the total profit is to be found. Therefore the EP and IP are the decision variables. The decision variables EP and IP can be assigned name such as EP = x_1 and IP =x_2.

2. To the frame the objective equation.

Our objective is to maximize the profits by producing the right quantity of EP and IP. For every ton of EP produced a profit Rs 5000/- and for every ton of IP a profit of Rs 4000/- is made. This is indicated as 5 and 4 in table 1.1 as profit in thousands of rupees. We can use the values of 5 and 4 in our objective equation and later multiply by a factor 1000 in the final answer.

Therefore the objective equation is framed as

Max (Z) = $5\,x_1 + 4\,x_2$

Where x_1 and x_2 represent the quantities (in tons) of EP and IP to be produced.

3. To identify the constraints and frame the constraint equations

In the problem statement there are constraints relating to the raw materials used and there are constraints relating to the demand for the exterior and interior paints. Let us first examine the raw material constraints. There are two types of raw materials used namely

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M1 and M2. The maximum availability of M1 every day is given as 24 tons and the problem (refer table 1.1) states that 6 tons of M1 is required for producing 1 ton of exterior paint. Now the quantity of exterior paint to be produced is denoted as $x_{1,}$ so if 6 tons of M1 is required for producing 1 ton of exterior paint, to produce x_1 tons of exterior paint (6 * $x_{1)}$ tons of M1 is required. Similarly the problem states that 4 tons of M1 is required for producing 1 ton of interior paint. Now the quantity of interior paint to be produced is denoted as $x_{2,}$ so if 4 tons of M1 is required for producing 1 ton of interior paint, to produce x_2 tons of exterior paint (4 * $x_{2)}$ tons of M1 is required. But the total quantity of M1 used for producing x_1 quantity of exterior paint and x_2 quantity of interior paint cannot exceed 24 tons (since that is the maximum availability). Therefore the constraint equation can be written as

$6 x_1 + 4 x_2 <= 24$

In the same way the constraint equation for the raw material M2 can be framed. At this point I would suggest that you must try to frame the constraint equation for raw material M2 on your own and then look into the equation given in the text. To encourage you to frame this equation on your own I am not exposing the equation now but am showing this equation in the consolidated solution to the problem.

Well now that you have become confident by framing the second constraint equation correctly (I am sure you have), let us now look to frame the demand constraints for the problem. The problem states that the daily demand for interior paints is restricted to 2 tons. In other words a maximum of 2 tons of interior paint can be sold per day. If not more than 2 tons of interior paints can be sold in a day, it advisable to limit the production of interior paints also to a maximum of 2 tons per day (I am sure you agree with me). Since the quantity of interior paints produced is denoted by x_2, the constraint is now written as

$x_2 <= 2$

Now let us look into the other demand constraint. The problem states that the daily demand for interior paints cannot exceed that of exterior paints by more than 1 ton. This constraint has to be understood and interpreted carefully. Read the statement carefully and understand that the daily demand for interior paints can be greater than the demand for exterior paints but that difference cannot be more than 1 ton. Again we can conclude that based on demand it is advisable that if we produce interior paints more than exterior paints, that difference in tons of production cannot exceed 1 ton. By now you are familiar that the quantities of exterior paint and interior paint produced are denoted by x_1 and x_2 respectively. Therefore let us frame the constraint equation as the difference in the quantities of paints produced.

$x_2 - x_1 <= 1$

In addition to the constraints derived from the statements mentioned in the problem,

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there is one more standard constraint known as the non-negativity constraint. The rationale behind this constraint is that the quantities of exterior and interior paints produced can never be less than zero. That is it is not possible to produce negative quantity of any commodity. Therefore x_1 and x_2 must take values of greater than or equal to zero. This constraint is now written as

$x_{1,}$ x_2 are > = 0

Thus we have formulated (that is written in the form of equations) the given statement problem.

The consolidated formulation is given below

The objective equation is Max (Z) = 5 x_1 + 4 x_2

Subject to the constraints: 6 x_1 +4 x_2 < = 24

x_1 +2 x_2 < = 6

x_2 <= 2

x_2 - x_1 <=1

$x_{1,}$ x_2 are > = 0 (Non-Negativity constraint)

We have successfully formulated the given statement problem and I am sure you are looking forward to know the quantities of external and internal paints that must be produced in-order to maximize the profits. Well we eventually will do that in the next chapter and in the meanwhile let try to formulate another problem.

Example 1.2:

John must work at-least 20 hours a week to supplement his income while attending school. He has the opportunity to work in 2 retail stores. In store 1, John can work between 5 to 12 hours a week and in store 2, he can work for 6 to 10 hours. Both the retail stores pay the same hourly wages. John thus wants to base his decision about how many hours to work in each store based on the stress factor. He estimates the stress factors as 8 and 6 at stores 1 and 2 respectively on a 10 point rating scale. Because stress mounts by the hour he presumes that the total stress at the end of the week is proportional to the number of hours he works in the store. Formulate the problem to find the number of hours John must work in each store with minimal stress.

Solution

I trust you must have read through the problem and have understood that John has to

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work in the two retail stores with minimum stress levels.

Can you identify the decision variables? Yes, you definitely can. Since the number of hours that John must work in the two retail stores is to be found, the decision variables can be identified and named like this.

Let x_1 be the number of hours that John must work in store 1 and let x_2 be the number of hours John must work in store 2. The stress levels on a 10-point rating scale, by working in store 1 and store 2 are mentioned as 8 and 6 respectively in the problem. That is, the stress induced by working in store 1 is more than store 2 on a uniform rating scale. So John wants to minimize the stress levels while working specific hours in the two retail stores. The problem also states that stress is proportional to the number of hours worked. Therefore if John works for one hour in store 1 his stress level is 8 points. So if he works x_1 hours then his stress levels by working in store 1 will be $8x_1$. Similarly if John works for 1 hour in store 2 his stress level is 6 points. Therefore if he works x_2 hours in store 2 than his stress levels by working in store 2 will be $6x_2$.

John wants to cumulatively minimize the stress levels. This is his objective. The objective equation is thus written as

Minimize $(Z) = 8x_1 + 6x_2$

Typically John can decide never to work in store 1 since the stress levels in store 1 is greater than store 2 for the same hours of work. But it is not possible to completely stay away from store 1 since a minimum time range of work in store 1 and store 2 have been mentioned in the problem statement. The problem states that John can work between 5 to 12 hours a week in store 1 and between 6 to 10 hours in store 2. Since John can work a maximum of 10 hours in store 2, he cannot avoid working in store 1 (since he must work a minimum of 20 hours).

The constraint equations with respect to store 1 can be written as

$5 <= x_1 <= 12$

That is John can work between 5 and 12 hours in store 1.

Similarly for store 2 the constraint equation can be written as

$6 <= x_2 <= 10$

That is John can work between 6 to 10 hours in store 2.

Moreover John must work for a minimum of 20 hours in both stores put together. So the constraint equation is written as

$x_1 + x_2 >= 20$

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As in the previous example the non-negativity constraints have to be included. It is written as

$x_{1,}\ x_2$ are > = 0

We have now framed the objective equation and the constraint equations for the problem. It can be summarized as follows.

Minimize (Z) = $8x_1 + 6x_2$

Subject to the constraints:

$5 <= x_1 <= 12$

$6 <= x_2 <= 10$

$x_1 + x_2 >= 20$

$x_{1,}\ x_2$ are > = 0

I trust that you have now got more than a fair idea of how to formulate a problem statement. I shall give you a couple of problems for your trial and practice in end exercises.

1.2. SOLUTION TO LPP THOUGHT GRA[HICAL METHOD

We have successfully formulated at-least two problems and I am sure you are eager to solve the formulated problems. We will first solve the problems graphically and then go ahead to the simplex methods. One has to keep in mind that to solve a LPP graphically it must not have more than two variables. You will understand the procedure as it is explained below through an example.

Example 1.3:

Let us solve graphically, the problem Max (Z) = $5\ x_1 + 4\ x_2$

Subject to the constraints: $6\ x_1 + 4\ x_2 <= 24$

$x_1 + 2\ x_2 <= 6$

$x_2 <= 2$

$x_2 - x_1 <= 1$

$x_{1,}\ x_2$ are > = 0 (Non-Negativity constraint)

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Solution:

Do you recognize the problem? It is just the problem that we formulated from example 1.1.

Let us replace all the in-equalities in the constraint equations into equalities. Therefore the constraint equations: $6\ x_1 + 4\ x_2 < = 24$

$x_1 + 2\ x_2 < = 6$

$x_2 <= 2$

$x_2 - x_1 <= 1$

are replaced by : $6\ x_1 + 4\ x_2 = 24$

$x_1 + 2\ x_2 = 6$

$x_2 = 2$

$x_2 - x_1 = 1$

In-order to solve the given LPP through the graphical method, we have to represent each of the constraint equations as straight lines in the graph. To represent each constraint equation as a straight line we must have at-least two coordinates for each constraint equation.

In the first constraint equation $6\ x_1 + 4\ x_2 = 24$, assume $x_1 = 0$, then $x_2 = 6$. Therefore one coordinate in (0,6). Next assume $x_2 = 0$ and then $x_1 = 4$. The other coordinate is (4,0). Now we have two coordinates (0,6) and (4,0) that represent the constraint line $6\ x_1 + 4\ x_2 = 24$.

Similarly we can get the coordinates for the other constraint lines.

Table 1.2

Constraint Line Number in the graph	Constraint Equation	Coordinates that define the line
1	$6\ x_1 + 4\ x_2 = 24$	(0,6) and (4,0)
2	$x_1 + 2\ x_2 = 6$	(0,3) and (6,0)
3	$x_2 = 2$	(0,2)
4	$x_2 - x_1 = 1$	(0,1) and (-1,0)

With the coordinates we can plot the graph as shown in chart 1.1.

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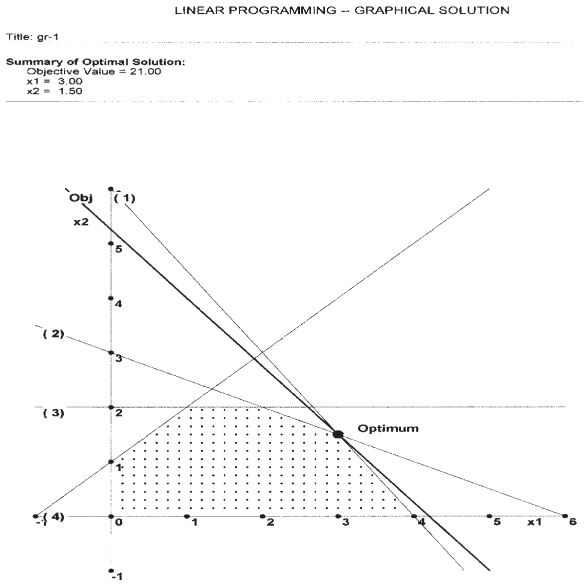

Chart 1.1

What we have done so far is just a bit of analytical geometry. We shall now try to identify a feasible solution region which will satisfy all the constraint equations.

In-order to identify the feasible region, the following points are to be borne in mind.

1. The feasible solution region for all < = constraints will lie towards the origin.
2. The feasible solution region for all >= constraints will lie away from the origin.
3. The feasible region for = constraints will lie on the line.

In this problem all the constraints are of < = type. Therefore for all the constraint equations, the feasible solution regions lie towards the origin. We now identify and shade the common feasible region (as shown in the graph). The shaded region shows the feasible solution region. The coordinates in the boundary show the possible feasible solutions.

Starting from the origin (in chart 1.1) and moving in the clockwise direction, we shall identify the coordinates of $x_{1,}$ x_2 and the corresponding objective Z equation value.

We find that the best or the optimal solution among the feasible ones is the point where $\mathbf{x_1}$ = 3 and $\mathbf{x_2}$ = 1.5. This yields an objective Z value of 21 (by substitution in the Objective equation).

Table 1.3

Point Number	Coordinates		Objective Z Value
	x_1	x_2	
1	0	0	0
2	0	1	4
3	1	2	13
4	2	2	18
5	**3**	**1.5**	**21**
6	4	0	20

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Alternatively we can identify the optimal point in the feasible region by using the iso-profit line. To construct the iso-profit line, assume a value for Z in the objective equation which is easily divisible by the coefficients of the variables in the objective equation namely $\mathbf{x_1}$ and $\mathbf{x_2}$. The coefficients of $\mathbf{x_1}$ and $\mathbf{x_2}$ are 5 and 4 respectively. So the number that can easily be divisible by 5 and 4 is 20. we now re-write the objective Z equation as $5\,\mathbf{x_1} + 4\,\mathbf{x_2} = 20$. This equation can now be represented as a straight line in the graph by two coordinates (0,5) and (4,0) using the procedure adopted for plotting the constraint lines. Now the iso-profit line can be moved away from the origin and the last point that it touches on the solution space is the maximum point. In this problem the iso-profit line shown as 'obj' is touching the maximum point.

Important Points to be remembered regarding the iso-profit line.

- The coordinates and position of the iso-profit line may vary depending upon the chosen value of Z. In this problem, since the value of Z = 20 (assumed) the coordinates were (0, 5) and (4, 0). If the assumed value of Z was 40, then the coordinates would have been (0, 10) and (8, 0). However the slope of the iso-profit will not change, whatever be the assumed value of Z.
- In this problem for an assumed value of 20 for Z and the coordinates (0, 5) and (4, 0), the iso-profit line passes through the shaded feasible region. But an iso-profit line with coordinates (0, 10) and (8, 0) for an assumed value of Z = 40 would be above the shaded region.
- If the iso-profit line passes through the shaded feasible region, then for a maximizing problem, the line would have to be moved away from the origin (without altering the slope) and the last point (coordinate) that it touches in the feasible region would be maximizing point. For a minimizing problem, the iso-profit line would be moved towards the origin and the lowest point it touches in the shaded feasible region will be the minimizing point.
- If the iso-profit line lies above the shaded feasible region, then for a maximizing problem, the line would have to be moved towards the origin (without altering the slope) and the first point (coordinate) that it touches in the feasible region would be maximizing point. For a minimizing problem, the iso-profit line would be moved towards the origin and the lowest point it touches in the shaded feasible region will be the minimizing point.
- If the iso-profit line lies below the origin and the shaded feasible region, then for a minimizing problem, the line would have to be moved towards the origin, and beyond if required, (without altering the slope) and the first point (coordinate) that it touches in the feasible region would be minimizing point. For a maximizing problem, the iso-profit line would be moved towards the origin and beyond if required and the last (highest) point it touches in the shaded feasible region will be the maximizing point.

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- If the iso-profit line lies below the shaded feasible region but above the origin, then for a minimizing problem, the line would have to be moved away from the origin (without altering the slope) and the first point (coordinate) that it touches in the feasible region would be minimizing point. For a maximizing problem, the iso-profit line would be moved away from the origin and the last (highest) point it touches in the shaded feasible region will be the maximizing point.
- In general it can be presumed that in a shaded feasible region, the lowest point is the minimizing point and the highest point is the maximizing point. Hence the iso-profit line would have to be moved accordingly depending upon its position without altering its slope to identify the maximum or minimum point.
- With these points in mind, in this problem, if you move the iso-profit line without altering the slope then the maximum point or the last point that it touches in the shaded feasible region is $x_1 = 3$ and $x_2 = 1.5$. This yields a Z value of 21.

The interpretation is that the company must produce 3 tons of exterior paint and 1.5 tons of interior paint and this would yield a maximum profit of Rs.21, 000.

I am sure you now appreciate the merit of solving a formulated LPP through the graphical method.

Let us now solve another problem (a minimizing case) by the graphical method.

Example 1.4: Solve graphically

Minimize (Z) = $8x_1 + 6x_2$

Subject to the constraints:

$5 <= x_1 <= 12$

$6 <= x_2 <= 10$

$x_2 >= 20$

x_2 are $>= 0$

Solution

his problem is the second case that we formulated in example 1.2 (John works in two stores). We shall solve this problem graphically to find out how many hours John must work in store 1 and store 2, in-order to minimize his levels of stress.

Just as we did in the previous problem, let us convert all inequalities in the constraint equations into equalities. We have to be a little careful in the first and second constraint equation. Each of these equations represents a range and hence will be shown as two lines in the graph.

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The constraint equation with respect to store 1 ($5 <= x_1 <= 12$) is split into two equalities. That is $x_1 = 5$ (coordinates are (5, 0), shown as line 2 in the graph)) and $x_1 = 12$ (coordinates are (12, 0), shown as line 3 in the graph). Similarly the constraint equation with respect to store 2 ($6 <= x_2 <= 10$) is also split into equalities. That is $x_2 = 6$ (coordinates are (0, 6), shown as line 4 in the graph) and $x_2 = 10$ (coordinates are (0, 10), shown as line 5 in the graph). Hence you will see that each constraint represents a range (with upper and lower limits) and is shown as two lines in the graph. The third constraint relating to the minimum total hours John must work in both stores put together ($x_1 + x_2 >= 20$) will be shown as a straight line with the coordinates (0, 20) and (20, 0). This constraint is shown as line 1 in the graph.

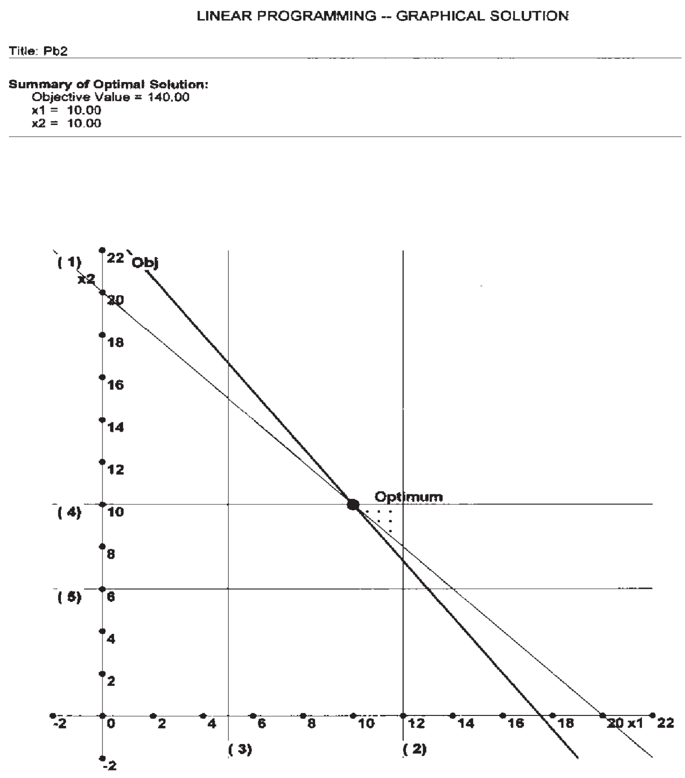

Chart 1.2

Just as we identified the feasible solution region in the previous problem, let look for the feasible region here also. With respect to the first constraint ($5 <= x_1 <= 12$), the lines are 2 and 3 in the graph. Line 2 represents $5 <= x_1$, which means the feasible region will lie away from the origin. Line 3 represents $x_1 <= 12$, and this means that the feasible region will lie towards the origin for this line. Similarly, for line 4 the feasible region will lie away from the origin and for line 5 the feasible region will lie towards the origin. For the third constraint line shown as line 1 ($x_1 + x_2 >= 20$) the feasible region will lie away from the origin. (I hope you remember that these regions have been identified based on the rule that

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whenever the decision variable has to be greater than or equal to the given constant then the feasible region lies away from the origin and vice-versa. For a purely equal to constraint, the solution will lie on the line.)

The identified feasible region is shown as a dotted portion in chart 1.2. The optimal point can be identified either by using the coordinates in the objective equation and then choosing the minimal value or by using the iso-profit line. Both these procedures have been elaborated in the previous problem. So I trust you would easily determine the minimum point from chart 1.2. The minimum point corresponds to x_1 =10 and x_2 = 10. That is John must work 10 hours in each store to minimize his stress. If he does so, then his total stress value would be 20 (Based on the objective Z equation)

Special Cases

Case 1: Solution area is bounded and there is infinite number of solutions

Example 1.4:

Max $Z = 6 x_1 + 15 x_2$

Subject to: $5 x_1 + 3 x_2 <= 15$

$2 x_1 + 5 x_2 <= 10$

$x_1, x_2 >= 0$

Case 2: Solution area is unbounded and there is no solution.

Example 1.5:

Max $Z = 3 x_1 + 2 x_2$

Subject to: $x_1 - x_2 <= 1$

$x_1 + x_2 >= 3$

$x_1, x_2 >= 0$

Case 3: Solution area is unbounded yet there is a solution.

Example 1.6:

Max $Z = 6 x_2 - 2 x_1$

Subject to: $2x_2 - x_1 <= 2$

$x_2 <= 3$

$x_1, x_2 >= 0$

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Case 4: There is no solution space and hence there is no solution.

Example 1.6:

Max Z = 3 x_1 - 2 x_2

Subject to: $x_1 + x_2 <= 1$

$3x_1 + 3x_2 >= 6$

$x_1, x_2 >= 0$

All the four special cases have been shown as charts 1.3, 1.4, 1.5 and 1,6 respectively. I would urge to first try to draw the charts on your own, then compare it with the charts shown in the text and then see whether it matches with the titles mentioned in the case heading.

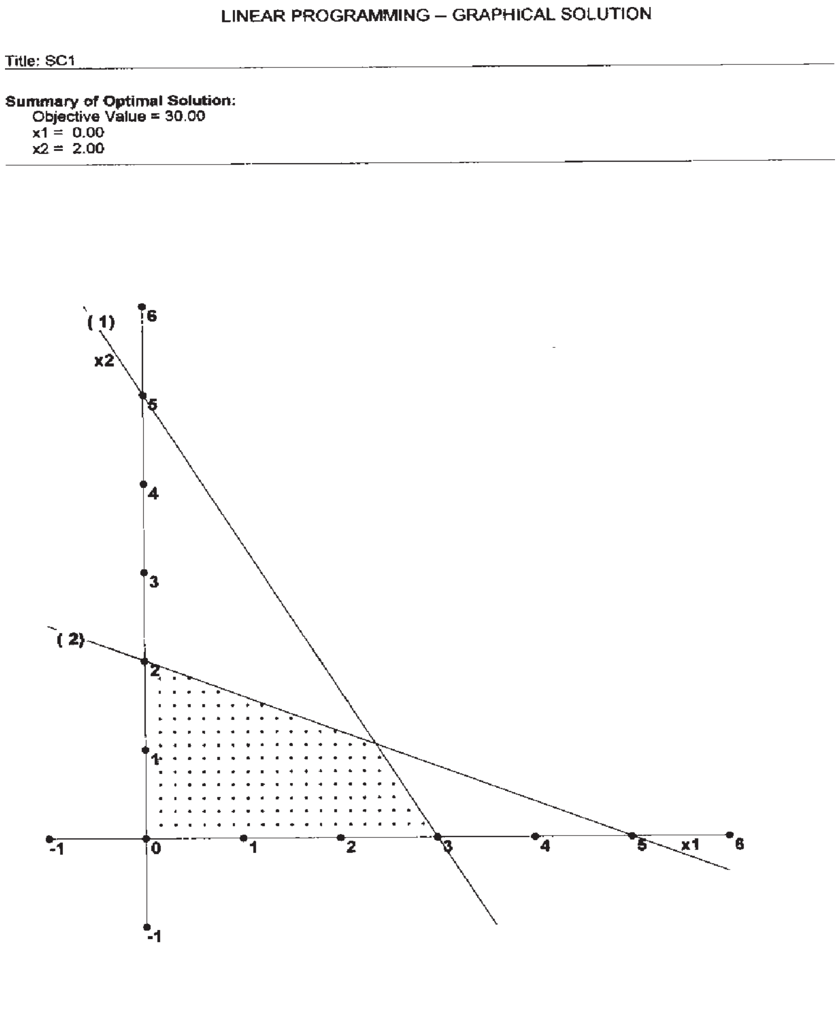

Chart 1.3

NOTES

LINEAR PROGRAMMING – GRAPHICAL SOLUTION

Title: sc3

Summary of Optimal Solution:
Objective Value = 999999.00
x1 = 0.00
x2 = 0.00

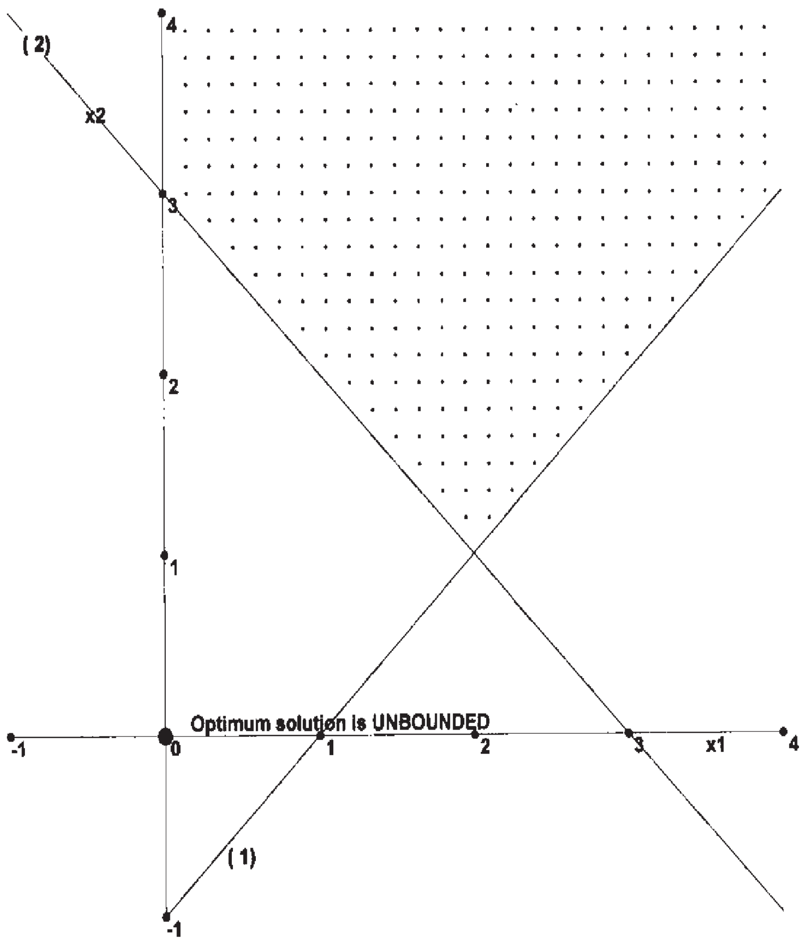

Chart 1.4

NOTES

LINEAR PROGRAMMING – GRAPHICAL SOLUTION

Title: sc4

Summary of Optimal Solution:
Objective Value = 10.00
x1 = 4.00
x2 = 3.00

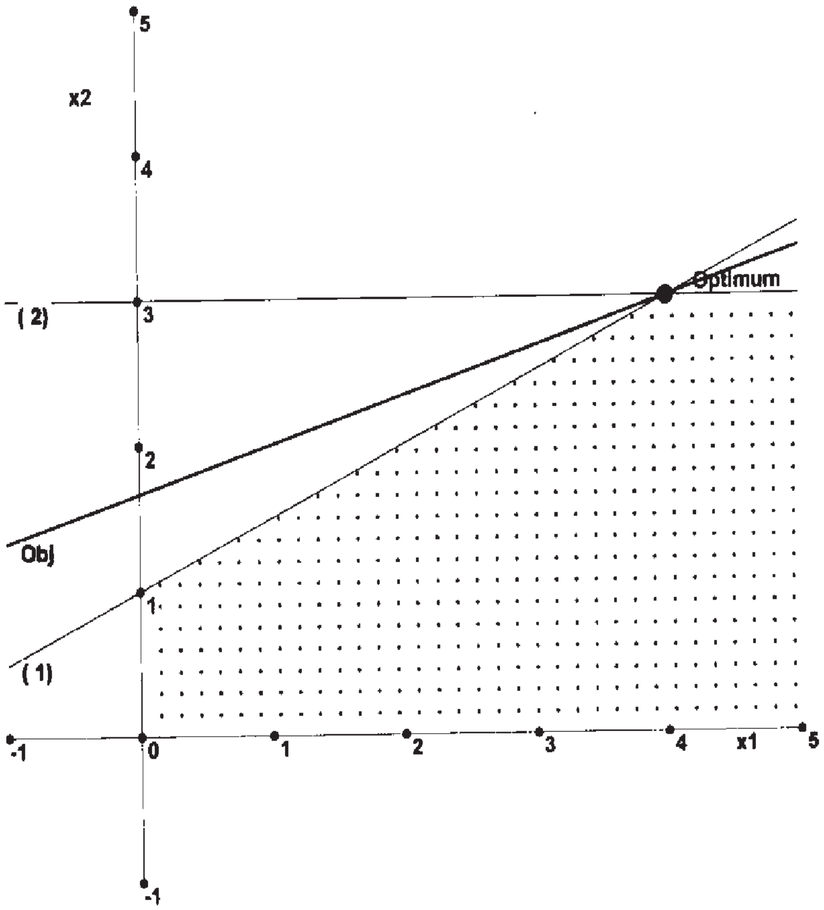

Chart 1.5

NOTES

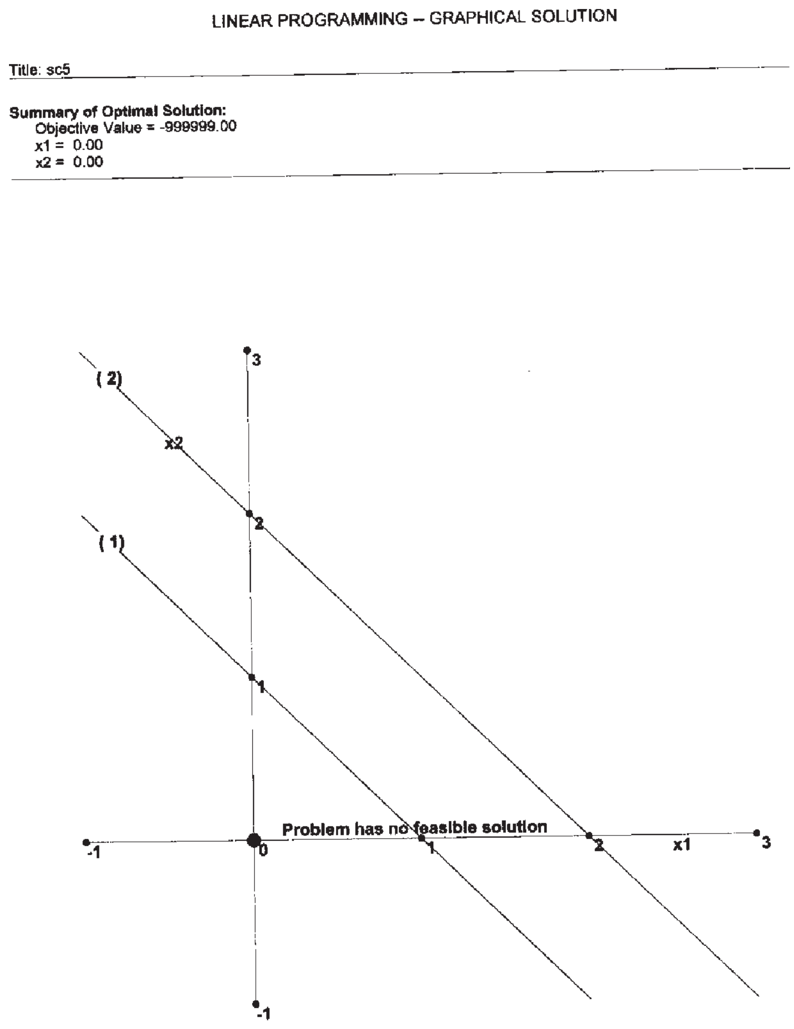

Chart 1.6

Now that you have become experts in solving 2 variable problems through the graphical method we shall learn to solve through the simplex method.

1.3 INTRODUCTION TO SOLUTIONS TO LPP THROUGH SIMPLEX METHOD.

The LP problems that involve more than 2 variables cannot be solved by graphical method but will have to make use of the simplex algorithm. Brace yourselves and fasten your seat belts since the simplex method is going to call for the best of concentration from your end.

We will first solve a 2 variable problem through simplex method and then go on to solve a three variable problem.

At this point I would like to introduce four types of simplex algorithms or methods

NOTES

which we will familiarize ourselves with shortly. The methods are

1. Primal Simplex
2. Big – M (or) Penalty
3. Two Phase and
4. Dual Simplex method

The first three methods that is, Primal simplex, Big – M and Two phase methods are adopted to solve when there is a initial feasible solution formulated and then the algorithms work towards achieving optimality in the consequent iterations. That is these three methods start from a feasible but non-optimal solution and iterate towards feasible and optimal solution. But if there exists an optimal but infeasible solution, then the dual simplex method would be adopted to achieve feasibility while still maintaining optimality. That is the dual simplex method starts from optimal but infeasible solution and proceeds to achieve an optimal and feasible solution.

Well the above information is just an introduction so don't get too disturbed if you are not able to immediately understand all that is said about the above methods, you are soon going to be masters in these methods as we progress through the course. However I would urge you to read a little more carefully from this point.

The primal simplex method is adopted when all constraints are <= type. But the Big-M and Two phase methods are adopted when there are mixed constraints. That is there are constraints of <=, > = and = types.

1.3.1 Primal Simplex Method

If you had read the previous line it would be to obvious to you that to adopt the primal simple method it is required that all the constraints are to be < = type.

Also two conditions called the Optimality condition and feasibility condition will help us solve LPP through the simplex method. It is very important that you completely understand the meaning of these conditions. I shall mention the conditions now and you may read it once now. As we solve a problem you will keep coming back to read these conditions in-order to understand the full impact of these conditions.

Optimality Condition: The entering variable in a maximizing (minimizing) problem is the non-basic variable with the most negative (positive) coefficient in the objective Z equation. A tie may be broken arbitrarily. The optimum is reached when all the non-basic coefficients in the Z equation are non-negative (non-positive).

Feasibility Condition: For both maximizing and minimizing problems, the leaving variable is the current basic variable having the smallest intercept (a minimum ratio with strictly positive denominator) in the direction of the entering variable. A tie may be broken arbitrarily.

NOTES

As I said earlier you may not be able to understand the impact of the condition, but don't worry, I shall explain the applicability of these condition as we solve a problem.

Let us take a simple problem for our example.

Example: 1.5

Maximize (Z) = $5\ x_1 + 4\ x_2$

Subject to: $6x_1 + 4x_2 <= 24$

$x_1 + 2x_2 <= 6$

$x_2 <= 2$

$x_1 + x_2 <= 1$

$x_1, x_2 >= 0$

Do you recognize this problem? This is the same problem that we first formulated and then solved graphically. We shall now solve the same problem through primal simplex method for you to appreciate (or not to) the same.

Solution

The first step here is to convert all the inequalities in the constraint equations into equalities. On examining the constraint equations you will notice that they are all of < = type. That is, the left hand side (LHS) of the equation is less than the right hand side (RHS). So we need to add a quantity (as variables) to the LHS in each of the equations that will help balance both sides and replace the inequalities into equalities.

You will agree that x_1 and x_2 are variables in the problem. Additionally let us introduce slack variables (s) to each of the equations to balance and introduce the equality sign.

Considering the first constraint, $6x_1 + 4x_2 <= 24$

Let us introduce a slack variable (s_1) to the LHS and balance the equation. Hence the constraint equation can now be written as

$6x_1 + 4x_2 + s_1 = 24$

Similarly we can introduce slack variables s_2, s_3, s_4 for the constraints 2, 3 and 4 respectively. Therefore the constraints are now written as

$x_1 + 2x_2 + s_2 <= 6$

$x_2 + s_3 <= 2$

$x_1 + x_2 + s_4 <= 1$

NOTES

We now have six variables (two original variables and four slack variables introduced by us). We must understand that the slack variable introduced by us have no weight-age in the objective Z equation. Therefore the coefficients of the slack variables in the objective Z equation will be 0 (Zero).

The number of variables (n) = 6

The number of constraint equations (m) = 4

Therefore the number of non-basic variables in the problem will be n-m = 6 – 4 = 2 If 2 out of 6 variables are non-basic variables then the number of basic variables = 4 We have to now identify the basic and the non-basic variables. In our earlier discussion we had stated that in the primal simplex method we will start with a feasible solution and then iterate towards optimality. A sure feasible solution to the problem is with the assumption that $x_1 = 0$ and $x_2 = 0$. Yes, this is a feasible but not optimal solution. We start with this assumption and then determine better values for x_1 and x_2 that will give us the optimal solution. If we substitute the values of $x_1 = 0$ and $x_2 = 0$ in the four constraint equations, then we get $s_1 = 24$, $s_2 = 6$, $s_3 = 2$, $s_4 = 1$.

Therefore $s_1 = 24$, $s_2 = 6$, $s_3 = 2$, $s_4 = 1$ are going to be starting points of our solution and are known as the basic variables. The non basic variables are $x_1 = 0$ and $x_2 = 0$.

All the variables in the Z equation are brought to the LHS and are written like this

$Z - 5x_1 - 4x_2 - 0s_1 - 0s_2 - 0s_3 - 0s_4 = 0$

We have already mentioned that the coefficients of the slack variables are zero in the objective equation.

We must now frame the initial simplex table as given under.

Name the columns like this : Basis, x_1, x_2, s_1, s_2, s_3, s_4, Solution and Ratio.

That is the first column must always be named as the basis, then all the participating variables must be listed, then the solution column and finally the ratio column. Under the basis column, the Z variable and the basic variables are listed. Please remember the basic variables will change with every iteration (as variables enter or leave the basis) and the variables that are not under the basis column as all non-basic variables. In this problem, since the initial basic feasible solution has started with s_1, s_2, s_3, s_4 as the basic variables they are appearing in the basis in the initial simplex table. This table represents the initial basic feasible solution.

NOTES

Table 1.4 - Initial simplex table

Basis	x_1	x_2	s_1	s_2	s_3	s_4	Solution	Ratio
Z	-5	-4	0	0	0	0	0	
s_1	6	4	1	0	0	0	24	
s_2	1	2	0	1	0	0	6	
s_3	0	1	0	0	1	0	2	
s_4	-1	1	0	0	0	1	1	

On close observation you will understand that the values in the Z row represent the coefficients of the existing variables in the objective Z equation where the RHS =0. Similarly the values of s_1 are the coefficients of the first constraint equation where the solution or RHS = 24. In similar fashion, the values in the rows s_2, s_3, and s_4 represent the 2^{nd}, 3^{rd} and 4^{th} constraint equations respectively.

Having framed the initial simplex table, let us now find the variable that will enter the basis (know as entering variable and abbreviated as EV). Only the variables that are not part of the basis (That is the non-basic variables) can qualify to be an EV. Now to choose and EV from the non-basic variables (we have two non-basic variables x_1 and x_2 in this problem) I suggest you read the optimality condition once again and then return to this page. Trusting that you have read the optimality condition, let me refresh the statement for you in a simple form. The optimality condition says that the entering variable in a maximizing problem is the variable with the most negative coefficient in the objective Z equation. Going by this condition and observing the objective Z equation in the initial simplex table, you find that -5 is the most negative coefficient. Therefore x_1 will be the entering variable.

In-order to identify the leaving variable; let us refresh our memory with the feasibility condition. The feasibility condition states that "for both maximizing and minimizing problems, the leaving variable is the current basic variable having the smallest intercept (a minimum ratio with strictly positive denominator) in the direction of the entering variable. A tie may be broken arbitrarily". Examine the EV column and the solution in table 1.4. There are four basis variables in this stage s_1, s_2, s_3, and s_4. Therefore one of them is going to be the leaving variable. The values in the EV column (also known as the pivotal column) for the four basic variables will be the denominators. Hence if there are any negative values in the EV column then such basic variables need not be considered for leaving the basis. The corresponding solution column will be the numerator. Now select the least ratio by dividing the solution column by the EV column. That is for s_1 it is 24/6, for s_2 it is 6/1, for s_3 it is 2/0 and we need not consider s_4 since it has a negative denominator. These values are filled and shown in table 1.5 below

NOTES

Table 1.5 - Initial simplex table with EV and LV identified

Basis	X_1	x_2	s_1	s_2	s_3	s_4	Solution	Ratio
Z	-5	-4	0	0	0	0	0	
s_1	*6*	*4*	*1*	*0*	*0*	*0*	*24*	*24/6 = 4*
s_2	*1*	2	0	1	0	0	6	6/1 = 6
s_3	*0*	1	0	0	1	0	2	2 / 0 = infinity
s_4	*-1*	1	0	0	0	1	1	Negative denominator

Based on the feasibility condition where the least ratio with a strictly positive denominator is to be chosen, we choose s_1 to be the leaving variable (LV). The leaving variable row is also known as the pivotal row. The Pivotal row (LV row) and the Pivotal Column (EV column) are shown in italics in table 1.3. The element which is the junction of the EV column and LV row is known as the pivotal element. Now the pivotal element (PE) in this case is 6. You can refer to the table 1.3 to identify the pivotal element. In the next iteration **s_1** would not be in the basis and **x_1** would have entered the basis. This is shown in the table below.

Table 1.6 – I Iteration empty table

Basis	x_1	x_2	s_1	s_2	s_3	s_4	Solution	Ratio
Z								
x_1								
s_2								
s_3								
s_4								

Now we can perform the iteration to improve upon the solution towards optimality using the change of basis by Guass-Jordan method.

The new pivotal equation or row (NPR) = old pivotal row / pivotal element.

The new variable (x_1) that has entered the basis is the new pivotal row.

The LV row is the old pivotal row and the pivotal element has already been identified as 6 (the element in the junction of EV column and LV row).

Now divide every element of the old pivotal row (that is the leaving variable row) by 6 (the pivotal element). The resulting new pivotal row is shown in table 1.7

NOTES

Table 1.7 – I Iteration table – with new pivotal row values.

Basis	x_1	x_2	s_1	s_2	s_3	s_4	Solution	Ratio
Z								
x_1	1	2/3	1/6	0	0	0	4	
s_2								
s_3								
s_4								

The other rows in the basis are computed as follows.

New Equation = Old Equation – (its EV column coefficient) * the NPR

I shall show how the Z equation is computed.

Please refer to the Table 1.5 to get the old Z equation. The values are

The EV column coefficient of the Z equation is -5. We will have to multiply this value (-5) to the NPR and then subtract it from the old Z equation. Alternatively what we will do

-5	-4	0	0	0	0	0

is to add a negative sign to the EV column coefficient and convert it into (+5), then multiply it with all elements of the NPR and then add to the old Z equation to get the new Z equation.

Let us see how this is done.

The NPR is (already computed in table 1.7)

The EV column coefficient is -5. With a negative sign added, it now becomes +5.

1	2/3	1/6	0	0	0	4

If the NPR values are multiplied by the +5 value, the resulting row is

Now this row is to be added to the old Z equation to get the new Z equation.

5	10/3	5/6	0	0	0	20

The values are shown in the table 1.8

NOTES

Table 1.8 – Computing new Z equation

Old Z equation	=	-5	-4	0	0	0	0	0
- (-5) * NPR	=	5	10/3	5/6	0	0	0	20
Sum (New Z equation)	**=**	**0**	**-2/3**	**5/6**	**0**	**0**	**0**	**20**

The computed new Z equation is filled in the I iteration table.

Table 1.9 – I Iteration table – with new Z equation

Basis	x_1	x_2	s_1	s_2	s_3	s_4	Solution	Ratio
Z	**0**	**-2/3**	**5/6**	**0**	**0**	**0**	**20**	
x_1	1	2/3	1/6	0	0	0	4	
s_2								
s_3								
s_4								

Similarly the new s_2 equation can be computed

Table 1.10 – Computing new s_2 equation

Old s_2 equation	=	1	2	0	1	0	0	6
- (1) * NPR	=	-1	-2/3	-1/6	0	0	0	-4
Sum (New s_2 equation)	**=**	**0**	**4/3**	**-1/6**	**1**	**0**	**0**	**2**

The new s_2 equation is filled in the I iteration table and is shown in Table 1.11.

NOTES

Table 1.11 – I Iteration table – with new s_2 equation

Basis	x_1	x_2	s_1	s_2	s_3	s_4	Solution	Ratio
Z	0	-2/3	5/6	0	0	0	20	
x_1	1	2/3	1/6	0	0	0	4	
s_2	**0**	**4/3**	**-1/6**	**1**	**0**	**0**	**2**	
s_3								
s_4								

We next compute the **s_3** equation using a similar procedure.

Table 1.12 – Computing new s_3 equation

The computed **s_3** equation is now filled in the I iteration table and is shown in table 1.13.

Old s_3 equation	=	0	1	0	0	1	0	2
- (0) * NPR	=	0	0	0	0	0	0	0
Sum (New s_3 equation)	=	**0**	**1**	**0**	**0**	**1**	**0**	**2**

Table 1.13 – I Iteration table – with new s_3 equation

We finally compute s_4 which is also a basic variable. The computation of s_4 is shown in the table 1.14 below.

Basis	x_1	x_2	s_1	s_2	s_3	s_4	Solution	Ratio
Z	0	-2/3	5/6	0	0	0	20	
x_1	1	2/3	1/6	0	0	0	4	
s_2	0	4/3	-1/6	1	0	0	2	
s_3	**0**	**1**	**0**	**0**	**1**	**0**	**2**	
s_4								

Table 1.14 – Computing new s_4 equation

The computed new equation is now filled in the I iteration table and the same is shown in table 1.15 below

Old s_4 equation	=	-1	1	0	0	0	1	1
- (-1) * NPR	=	1	2/3	1/6	0	0	0	4
Sum (New s_4 equation)	=	**0**	**5/3**	**1/6**	**0**	**0**	**1**	**5**

NOTES

Table 1.15 – I Iteration table – with new s_4 equation

The I-iteration table is now ready. We have to check whether optimality has been reached. The optimality condition states that if the coefficients of the non-basic variables in

Basis	x_1	x_2	s_1	s_2	s_3	s_4	Solution	Ratio
Z	0	-2/3	5/6	0	0	0	20	
x_1	1	2/3	1/6	0	0	0	4	
s_2	0	4/3	-1/6	1	0	0	2	
s_3	0	1	0	0	1	0	2	
s_4	**0**	**5/3**	**1/6**	**0**	**0**	**1**	**5**	

the Z equation are non-negative in a maximizing problem, then optimality is reached. The coefficients of the Z equation are

From the above table, it is seen that there is a negative coefficient and the corresponding

x_1	x_2	s_1	s_2	s_3	s_4
0	-2/3	5/6	0	0	0

variable is $\mathbf{x_2}$. Therefore we know that optimality has not been reached. We shall now proceed to the next iteration (that is II iteration). We have to identify the EV at this stage. The procedure is similar to what we had followed earlier. There is only one negative coefficient and the corresponding variable is $\mathbf{x_2}$. Hence $\mathbf{x_2}$ will be the EV. To find out the LV variable examine the Solution column and the $\mathbf{x_2}$ column. The values in the solution column will be the numerator and the corresponding values (only positive values) in the $\mathbf{x_2}$ column will be the denominator. Find the least positive ratio from the existing basic variables. The one with the least positive ratio will be the LV. The ratios are shown in table 1.16 below.

Table 1.16 – I Iteration table – with EV and LV variable highlighted

Basis	x_1	x_2	s_1	s_2	s_3	s_4	Solution	Ratio
Z	0	**-2/3**	5/6	0	0	0	20	-
x_1	1	**2/3**	1/6	0	0	0	4	6
s_2	**0**	**4/3**	**-1/6**	**1**	**0**	**0**	**2**	**3/2**
s_3	0	**1**	0	0	1	0	2	2
s_4	0	**5/3**	1/6	0	0	1	5	3

NOTES

From table 1.16 it is seen that the least positive ratio is 3/2 and this corresponds to variable **s_2**. Therefore the leaving variable (LV) is s2. Remember that the Z equation will not be considered for LV as it must never leave the basis. As we already know, the EV column is the pivotal column and the LV is the pivotal row. You very well know that the pivotal element is 4/3. We shall proceed to the II- iteration just the way we proceeded with the I-iteration. We first compute the NPR using the procedure as explained in the I-iteration. The NPR is as shown in table 1.17.

Table 1.17 – II Iteration table – with new pivotal row

Basis	x_1	x_2	s_1	s_2	s_3	s_4	Solution	Ratio
Z								
x_1								
x_2	0	1	-1/8	3/4	0	0	3/2	
s_3								
s_4								

I shall now give the blank formats for you compute the Z, x_1, s_3 and s_4, rows. You can try the calculations and try to <u>fill the values in the blank spaces provided</u>. You can cross check your answers with the II- iteration table that I have shown below. I would encourage you to do the calculations sincerely since this is the place where you would have plenty of scope to commit errors. So if the calculations done by you match with the answers that I have shown in the II –iteration table 1.21, then your levels of confidence would go up. To start with, try calculating the new Z equation. Then you proceed to calculate the other variables.

Table 1.18 – Computing new Z equation

Old Z equation	=	0	-2/3	5/6	0	0	0	20
- (-2/3) * NPR	=							
Sum (New Z equation)	=							

Now you may check with table 1.21 to see if your calculations of new Z equation match with the values given in the II-iteration table. Now you can proceed to calculate the new **s_3** equation and fill in the blank spaces.

NOTES

Table 1.19 – Computing new s_3 equation

Old s_3 equation	=	0	1	0	0	1	0	2
- (1) * NPR	=							
Sum (New s_3 equation)	=							

Now you may check with table1.21, to see if your calculations of the new $\mathbf{s_3}$ equation match with the values given in the II-iteration table. Now you can proceed to calculate the new $\mathbf{s_4}$ equation and fill in the blank spaces.

Table 1.20 – Computing new s_4 equation

Old s_4 equation	=	0	5/3	1/6	0	0	1	5
- (5/3) * NPR	=							
Sum (New s_4 equation)	=							

Since we have calculated all the required variables (rows) for the II iteration table, let us look at the II iteration table (that is table 1.21).

Table 1.21 – II Iteration table – with calculated values

Basis	X_1	x_2	s_1	s_2	s_3	s_4	**Solution**	**Ratio**
Z	0	0	3/4	1/2	0	0	**21**	
x_1	1	0	1/4	-1/2	0	0	<u>3</u>	
x_2	0	1	-1/8	3/4	0	0	<u>3/2</u>	
s_3	-1	1/3	-1/6	0	1	0	-2	
s_4	-5/3	5/9	-1/9	0	0	1	-5/3	

On having a close look at the Z equation in the table 1.21, you will understand that since all coefficients are non-negative (zero values are neither positive nor negative) we have reached an optimal and feasible solution. The calculated values of $\mathbf{x_1}$ = 3 and $\mathbf{x_2}$ = 3/2 are underlined in the table. For these values of $\mathbf{x_1}$ and $\mathbf{x_2}$ the value of the objective function Z = 21. You may remember that this is the same solution we have got through the graphical method for the same problem.

NOTES

How are you feeling after reading through this procedure? If you are excited to try out a problem similar to the one just solved, here's one for you.

Since we have fairly become familiar with the simplex procedure for two variables, let us now proceed to solve a three variable problem. Don't worry the procedure is exactly the same as in a two variable problem. So you can try the problem and cross check the answers for each iteration from the table given in the text.

Example 4:

Max Z = 22 x + 30y + 25z

Subject to: $2x + 2y + z <= 100$

$2x + y + z <= 100$

$x + y + 2z <= 100$

$x, y, z >= 0$.

Solution

We had earlier solved a similar two variable problem by first converting the inequalities into equalities by including slack variables in the constraint equations with a < = sign.

We will adopt a similar procedure in this problem also. Include a slack variable in each of the constraint equations and convert them into equalities.

$$2x + 2y + z + s_1 = 100$$

$$2x + y + z + s_2 = 100$$

$$x + y + 2z + s_3 = 100$$

Now there are six variables (n) in all and there are three constraint equations(m). Therefore the number of non-basic variables (n-m) is three. Obviously the three non-basic variables would be x, y and z. By the procedure for starting with a feasible solution we assume the values of the non-basic variables to be zero. That is x=y=z = 0. If the non-basic variables are = zero, then the basic variables $s_1 = 100$, $s_2 = 100$ and $s_3 = 100$. We can now proceed to frame the initial simplex table. At this point I would encourage you to prepare the table on your own and then check with the table in the text. It would help you to keep solving as you read the text. As we had done in the previous problem let us bring all the elements in the Z equation to the LHS and keep the RHS = 0.

$$Z - 22x - 30y - 25z - 0s_1 - 0s_2 - 0s_3 = 0$$

NOTES

Table 1.22 - Initial simplex table

Basis	x	y	z	s_1	s_2	s_3	Solution	Ratio
Z	-22	**-30**	-25	0	0	0	0	-
s_1	**2**	**2**	**1**	**1**	**0**	**0**	100	50
s_2	2	**1**	1	0	1	0	100	100
s_3	1	**1**	2	0	0	1	100	100

Since this is maximizing problem, the EV will be **y**. And in the direction of the EV column, the ratios are calculated. Since the minimum ratio is 50, the LV variable is s_1. Now proceed to calculating the NPR and all the other variables in the next iteration using the procedure that I had explained in the previous problem and check whether the values that you have calculated match with the values shown in the table below.

Table 1.23– I Iteration table

Basis	x	y	z	s_1	s_2	s_3	Solution	Ratio
Z	8	0	**-10**	15	0	0	1500	-
y	1	1	**1/2**	1/2	0	0	50	100
s_2	1	0	**1/2**	-1/2	1	0	50	100
s_3	**0**	**0**	**3/2**	**-1/2**	**0**	**1**	**50**	**100/3**

I trust that you would have calculated and checked with the I-iteration table. On observing the table we find that optimality is not reached (since all non-basic variables are not no-negative). There is a negative coefficient for z. Therefore z will be the EV now. And in the direction of the EV the ratios are calculated. Since 100/3 is the least positive ratio, the LV is $\mathbf{s_3}$. Now proceed to prepare the next iteration table using the usual procedure.

Table 1.24 – II Iteration table

Basis	x	y	z	s_1	s_2	s_3	Solution	Ratio
Z	8	0	0	35/3	0	20/3	**5500/3**	
Y	1	1	0	2/3	0	-1/3	100/3	
s_2	1	0	0	-2/3	1	-1/3	100/3	
z	0	0	1	-1/3	0	2/3	100/3	

NOTES

On observing the table we find that all the coefficients in the objective Z equation are non-negative. Therefore we can conclude that optimality is reached along-with feasibility. The calculated values are x = 0 (since optimality is reached with x in the basis), y = 100/3 and z = 100/3. The objective Z value is 5500/ 3. You can cross check the value of Z by substituting the value of x, y and z in the objective equation and see whether the value works out to be 5500/3. It is shown as 22 * 0 + 30* 100/3 + 25 * 100/3 = 5500 /3.

Well now that we have solved both two and three variable problems through the primal simple method it is time for us to move to problems with mixed constraints. As mentioned earlier in the text, problems with mixed constraints can be either solved by Big-M (also known as Penalty method) or by 2-Phase method. We will solve the same problem by both methods so that you can appreciate the relative merits and de-methods of the two methods.

1.3.2 - Big-M or Penalty Method.

We will try to understand the algorithm using an example.

Example 5:

Min Z = $4x_1 + x_2$

Subject to: $3x_1 + x_2 = 3$

$4x_1 + 3x_2 >= 6$

$x_1 + 2x_2 <= 4$

$x_1, x_2 >= 0$

Solution:

We see here that the problem has mixed constraints. That is all of them are not of < = type like the problems that we solved through primal simplex method.

In case of problems with mixed constraints, the Big – M or Penalty method can be used to solve. We have to introduce an artificial variable ‘A’ wherever there is a constraint with ‘=’ sign. For constraints with ‘>=’ sign, an artificial variable ‘A’ and a surplus variable ‘s’ would have to be introduced. And for constraints with ‘< =’ sign, slack variables would be introduced. The coefficients of the artificial variable ‘A’ in the objective Z equation is a large positive integer denoted by “M”. The coefficients of the slack and surplus variables in the objective Z equation are zero. The coefficient of the artificial, surplus and slack variables in the constraint equations wherever they appear are equal to1. Therefore the inequalities in the constraint equations are now transformed as

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$3x_1 + x_2 + A_1 = 3$

$4x_1 + 3x_2 + A_2 – s_1 = 6$

$x_1 + 2x_2 + s_2 = 4$

From the above equations it is seen that A_1 and A_2 are artificial variables, s_1 is surplus variable and so is subtracted (because the LHS is > = RHS) and s_2 is a slack variable.

In the constraint equations we find that there are 6 variables (n) and 3 equations (m). Therefore the number of non-basic variables is n-m=3. We can conveniently state that x_1 and x_2 are two non-basic variables. But how do we decide the other non-basic variable?

We will have to first choose the three (because if there are three non-basic variables then obviously the remaining are basic variables) basic variables. It is also simple logic that the number of basic variables is equal to the number of constraint equations (since each basic variable makes the presence of its constraint equation felt in the initial feasible simple table). Therefore our choice of basic variables would be to ensure that all the constraint equations are present in the initial feasible simplex table. For the first constraint equation 'A_1' is basic variable. For the second constraint equation we can either choose 'A_2' or 's_1' to be the basic variable. But it is advisable to choose 'A_2' as a basic variable (due to a positive coefficient). And in the third constraint we have no choice but to choose 's_2' as the basic variable. We have now decided on the basic variables. They are A_1, A_2 and s_2. Therefore the non-basic variables will be x_1, x_2 and s_1. As is always the practice let us assume that the non-basic variables are equal to zero (to start with a basic feasible solution). Then the basic variables will be $A_1 = 3$, $A_2 = 6$ and $s_2 = 4$.

The objective Z equation is transformed as

Min $Z = 4x_1 + x_2 + M A_1 + M A_2 + 0 s_1 + 0 s_2$

This is re-written as

$Z - 4x_1 - x_2 - M A_1 - M A_2 - 0 s_1 - 0 s_2 = 0$

We now prepare the initial basic simplex table.

Table 1. 25 - Initial simplex table (un-modified)

Basis	x_1	x_2	s_1	s_2	A_1	A_2	Solution	Ratio
Z	-4	-1	0	0	-M	-M	0	
A_1	3	1	0	0	1	0	3	
A_2	4	3	-1	0	0	1	6	
s_2	1	2	0	1	0	0	4	

NOTES

Now which is the EV? If you have said x_1 then you are absolutely WRONG? Because this is a minimizing problem and so the EV is the variable with the most positive coefficient in the Z equation. There is no variable with a positive coefficient in the Z equation. Also we find that there is some inconsistency in the solution of the Z equation. The coefficients of the artificial variables in the objective Z equation are 'M', and the artificial variables are not equal to zero. Therefore the solution in the Z equation would be in terms of 'M' and not zero. We have to transform the objective Z equation as per the under-mentioned procedure. Multiply the A and A rows of the initial simplex table by 'M' (that is the coefficient of the artificial variables) and then add it to the existing Z equation.

Existing Z Equation	-4	-1	0	0	-M	-M	0
M * A_1	3M	1M	0	0	1M	0	3M
M * A_2	4M	3M	-1M	0	0	1M	6M
New Z equation	7M – 4	4M - 1	-M	0	0	0	9M

We now have the modified initial simplex table with the new Z equation. All other rows are the same as the un-modified table.

Table 1.26- Initial simplex table – with new Z equation (modified)

Basis	x_1	x_2	s_1	s_2	A_1	A_2	Solution	Ratio
Z	**7M – 4**	4M - 1	-M	0	0	0	9M	
A_1	**3**	**1**	**0**	**0**	**1**	**0**	**3**	1
A_2	**4**	3	-1	0	0	1	6	3/2
s_2	**1**	2	0	1	0	0	4	4

Now we have to identify the EV, which is the variable with the most positive coefficient in the objective Z equation. Choose a large positive value for M, something like 100. Then for x_1 it is 700 – 4 and for x_2 it is 400 – 1. So the coefficient for x_1 is the most positive. Therefore x_1 is the EV. We find the LV as per the known procedure (that is the least positive ratio). The LV is A_1. We now proceed to the next iteration table using the well known procedure. May I once again encourage you to calculate the NPR and the other rows and then check with the table values given below.

NOTES

Table 1.27 – I -Iteration table

Basis	x_1	x_2	s_1	s_2	A_1	A_2	Solution	Ratio
Z	0	**(5M+1)/3**	-M	0	(4-7M)/3	0	2M+4	
x_1	1	**1/3**	0	0	1/3	0	1	
A_2	**0**	**5/3**	**-1**	**0**	**-4/3**	**1**	**2**	
s_2	0	**5/3**	0	1	-1/3	0	3	

On observing the table we find that the non-basic variables are still positive. Therefore optimality is not reached. The new EV will be $\mathbf{x_2}$ and the new leaving variable will be $\mathbf{A_2}$.

Going ahead to the next iteration with our known procedure, the II iteration table is shown below.

Table 1.28 – II -Iteration table

Basis	x_1	x_2	s_1	s_2	A_1	A_2	Solution	Ratio
Z	0	0	**1/5**	0	(-5M+8)/5	-(5M+1)/5	18/5	-
x_1	1	0	**1/5**	0	3/5	-1/5	3/5	3
x_2	0	1	**-3/5**	0	-4/5	3/5	6/5	-
s_2	**0**	**0**	**1**	**1**	**1**	**-1**	**1**	1

Have we reached the optimal solution? If you say yes, well I must say that you are once again mistaken. Although it appears as though optimality has been reached, it is not so. Looking carefully you will find that $\mathbf{x_1}$, $\mathbf{x_2}$, and $\mathbf{s_2}$ are in the basis. Therefore they are the basic variables now. This means the non-basic variables in this iteration are $\mathbf{s_1}$, $\mathbf{A_1}$ and $\mathbf{A_2}$. Out these non-basic variables we find that $\mathbf{s_1}$ has a positive coefficient. Therefore $\mathbf{s_1}$ is the EV and $\mathbf{s_2}$ is the LV (based least positive ratio).

We now proceed with the next iteration (I know you are hoping we reach the final iteration). The calculated table is shown below.

NOTES

Table 1.29 – III -Iteration table

Basis	x_1	x_2	s_1	s_2	A_1	A_2	Solution	Ratio
Z	0	0	0	-3/5	(-5M+7)/5	-M/15	**17/5**	
x_1	1	0	0	-1/5	2/5	0	2/5	
x_2	0	1	0	3/5	-1/5	0	9/5	
s_1	0	0	1	1	1	-1	1	

We now find that all the non-basic variables in the objective Z equation are non-positive (since this is a minimizing problem). Therefore we have reached optimality.

The values are $x_1 = 2/5$, $x_2 = 9/5$ and Z = 17/5.

We will now try to solve the same problem through the 2-Phase method and this will help you decide which is simpler for you to solve.

1.3.3 - Two-Phase Method.

In the Two-Phase method the problem is solved in two stages or phases.

In **Phase – I**, we find a basic solution of the resulting equation that minimizes the sum of the artificial variables. If the minimum value of the sum is positive, then we conclude that the LPP has no feasible solution, else we proceed to Phase –II.

In **Phase - II**, we use the feasible solution obtained in Phase I as a starting basic feasible solution for the original problem.

Again don't worry yourself too much if you don't understand the above statements at this point. I shall explain clearly with an example that we will now solve.

Example:

Min $Z = 4x_1 + x_2$

Subject to: $3x_1 + x_2 = 3$

$4x_1 + 3x_2 >= 6$

$x_1 + 2x_2 <= 4$

$x_1, x_2 >= 0$

Solution:

In this problem we have mixed constraints and so let us use artificial, slack and surplus variables (as we did in Big-M method) to convert the inequalities into equalities. The three constraint equations

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$3x_1 + x_2 = 3$

$4x_1 + 3x_2 > = 6$

$x_1 + 2x_2 < = 4$ are re-written as

$3x_1 + x_2 + A_1 = 3$

$4x_1 + 3x_2 + A_2 - s_1 = 6$

$x_1 + 2x_2 + s_2 = 4$ where A_1 and A_1 are artificial variables, s_1 is a surplus variable and s_2 is a slack variable.

This is similar to what we had done in Big-M method. Now instead of using the original objective equation we frame a new objective equation which minimizes the sum of the artificial variables present in the constraint equations. There are two artificial variables A_1 and A_2. Therefore the new objective equation to be used in Phase-I is written as

Min (R) = $A_1 + A_2$

We now solve for the new objective equation and the existing constraints. The non-basic and basic variables are identified similar to the Big-M method. For the first constraint equation 'A_1' is basic variable. For the second constraint equation we can either choose 'A_2' or 's_1' to be the basic variable. But it is advisable to choose 'A_2' as a basic variable (due to a positive coefficient). And in the third constraint we have no choice but to choose 's_2' as the basic variable. We have now decided on the basic variables. They are A_1, A_2 and s_2. Therefore the non-basic variables will be x_1, x_2 and s_1. As is always the practice let us assume that the non-basic variables are equal to zero (to start with a basic feasible solution). Then the basic variables will be $A_1 = 3$, $A_2 = 6$ and $s_2 = 4$.

The objective equation is re-written as

$R - A_1 - A_2 = 0$

We now frame the initial simplex table for Phase-I.

Table 1. 30 - Initial simplex table (un-modified)

Basis	x_1	x_2	s_1	s_2	A_1	A_2	Solution	Ratio
R	0	0	0	0	-1	-1	0	
A_1	3	1	0	0	1	0	3	
A_2	4	3	-1	0	0	1	6	
s_2	1	2	0	1	0	0	4	

Again here we find that there is no variable with the most positive coefficient in the objective equation. Also we find that there is inconsistency in the solution value (situations similar to the Big-M method). Therefore we multiply the A_1 row and A_2 row by the

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coefficients of the artificial variables and add them to the existing objective equation. Unlike in the Big-M method, where the coefficients were a large positive integer M, here the coefficients of the artificial variables in the objective equation are 1. Therefore the new objective equation (R) is calculated as shown below.

Existing R Equation	0	0	0	0	-1	-1	0
1 * A_1	3	1	0	0	1	0	3
1 * A_2	4	3	-1	0	0	1	6
New R equation	7	4	-1	0	1	1	9

We now write the new R equation in the modified initial simplex table and proceed to solve the problem.

Table 1. 31– Modified Initial simplex table with new R equation

Basis	x_1	x_2	s_1	s_2	A_1	A_2	Solution	Ratio
R	7	4	-1	0	1	1	9	-
A_1	**3**	**1**	**0**	**0**	**1**	**0**	**3**	**1**
A_2	**4**	3	-1	0	0	1	6	3/2
s_2	**1**	2	0	1	0	0	4	4

Now we have the variable $\mathbf{x_1}$ with the most positive coefficient 7, and so this becomes the EV. The LV is again the variable with the least positive ratio with a strictly positive denominator in the direction of the EV (as we have done in the previous problems). Hence the LV is $\mathbf{A_1}$. We now proceed to the next iteration by calculating the NPR and other rows as we had done in the previous problems. The I-Iteration table is shown below for your verification (needless to say that you would have done the calculations and are referring to the table below for just cross-checking)

Table 1.32 – I Iteration Table

Basis	x_1	x_2	s_1	s_2	A_1	A_2	Solution	Ratio
R	0	**5/3**	-1	0	-7/3	0	2	-
x_1	1	**1/3**	0	0	1/3	0	1	3
A_2	**0**	**5/3**	**-1**	**0**	**-4/3**	**1**	**2**	**6/5**
s_2	0	**5/3**	0	1	-1/3	0	3	9/5

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On observing the I iteration table we find that there is a positive coefficient in . Therefore optimality is not reached. Since there is a positive coefficient in $\mathbf{x_2}$, this becomes the EV for the next iteration. The LV (by our known principles) is $\mathbf{A_2}$. We now proceed to the next iteration and the calculated values are shown in the table below.

Table 1. 33– II Iteration Table

Basis	x_1	x_2	s_1	s_2	A_1	A_2	Solution	Ratio
R	0	0	0	0	-1	-1	0	
x_1	1	0	1/5	0	3/5	-1/5	3/5	
x_2	0	1	-3/5	0	-4/5	3/5	6/5	
s_2	0	0	1	1	1	-1	1	

On careful observation of the II iteration table we find that there are no positive values in the objective equation. Therefore we can conclude optimality is reached in Phase-I. Also we find that the solution value in the Objective R equation is non-positive, therefore we can proceed to Phase –II.

Phase II

We now use the feasible solution (omitting the artificial variables) obtained in Phase I and continue with the original objective equation. The feasible solution table from phase I without the artificial variable columns is shown below.

Table 1. 34 – Feasible Solution Table from Phase I (without artificial variables)

Basis	x_1	x_2	s_1	s_2	Solution	Ratio
R	0	0	0	0	0	
x_1	1	0	1/5	0	3/5	
x_2	0	1	-3/5	0	6/5	
s_2	0	0	1	1	1	

We now replace the objective R equation with the original Z equation. The original Z equation was

Min (Z) = $4\ x_1 + x_2 + 0s_1 + 0\ s_2$

This is re-written as

$Z - 4\ x_1 - x_2 – 0s_1 – 0\ s_2 = 0$

Therefore the new initial table for Phase II is prepared as given below.

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Table 1. 35 – Initial Table (unmodified) for Phase II (with original Z equaton)

Basis	x_1	x_2	s_1	s_2	Solution	Ratio
Z	-4	-1	0	0	0	
x_1	1	0	1/5	0	3/5	
x_2	0	1	-3/5	0	6/5	
s_2	0	0	1	1	1	

We once again find that there is no positive coefficient in the objective equation to be selected as the EV. Also there is no consistency in the solution value (since x_1 = 3/5 and x_2 = 6/5). So we multiply the $\mathbf{x_1}$ and $\mathbf{x_2}$ row with their coefficients (4 and 1) from the objective Z equation and add the same to the existing Z equation.

We shall now write the new Z equation in the modified table.

Existing Z Equation	-4	-1	0	0	0
4* x_1	4	0	4/5	0	12/5
1 * x_2	0	1	-3/5	0	6/5
New Z equation	0	0	1/5	0	18/5

Table 1.36 – Initial Table (modified) for Phase II (with original Z equaton)

ıg Z on	-4	-1	0	0	0
	4	0	4/5	0	12/5
	0	1	-3/5	0	6/5
equation	0	0	1/5	0	18/5

We find that there is a positive coefficient for $\mathbf{s_1}$. So this is the EV. The LV is $\mathbf{s_2}$.

We proceed to the next iteration with the usual procedure. The next iteration table is shown for your verification.

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Table 1.37 – I Iteration Table for Phase II

Basis	x_1	x_2	s_1	s_2	Solution	Ratio
Z	0	0	0	-1/5	**17/5**	
x_1	1	0	0	-1/5	2/5	
x_2	0	1	0	3/5	9/5	
s_2	0	0	1	1	1	

We find that there are all the coefficients in the objective Z equation are non-positive. Therefore we have reached optimality. The solution values are x = 2/5, x = 9/5 and Z = 17/5.

Now that we have done both Big-M and 2-Phase method for a mixed constraint problem, you can decide to adopt any one of the methods (unless otherwise specified) to solve a mixed constraint problem.

The algorithms that we have so far learnt have started with a feasible solution and then proceeded towards optimality. Suppose we have a situation where the initial basic solution resembles optimality, that is all the non-basic variables in a maximizing problem are non-negative and all non-basic variables in the minimizing problem are non-positive in the objective Z equation, then we have to adopt a procedure known as the DUAL SIMPLEX METHOD.

1.3.4 Dual Simplex Method

To adopt the dual simplex method, the starting table must have an optimum objective row with at-least one infeasible (<0) basic variable. To maintain optimality and simultaneously move towards feasibility at each new iteration, the following tow conditions are employed.

Dual Fasibility Condition: The leaving variable (LV) is the basic variable having the most negative value. Ties are broken arbitrarily. If all the basic variables are non-negative, then the algorithm ends.

Dual Optimality Condition: The entering variable (EV) is determined from among the non-basic variables as the one corresponding to the minimum ratio of the non-basic variable x_j, determined by the modulus of $Z_j/á_{rj}$ such that $á_{rj} < 0$. $á_{rj}$ is the constraint coefficient of the table associated with the row of the LV x_r and the column of the EV x_j. Ties are broken arbitrarily.

Do not get too disturbed if you are not able to understand the above two conditions at the moment. You will understand them as we progress to solve a problem.

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Example 6:

Solve Min (Z) = $3x_1 + 2x_2$

Subject to : $3 x_1 + x_2 > = 3$

$4 x_1 + 3 x_2 > = 6$

$x_1 + x_2 < = 3$

$x_1, x_2 > = 0$

Solution:

Let us convert all the constraint equations to < = type . The third constraint equation is already < = type but we will have to multiply the first and the second constraint equations throughout by (-1) in-order to convert them into a <= constraint. By doing this, the constraint equations are written as given under.

$3 x_1 - x_2 < = -3$ (Constraint 1)

$4 x_1 - 3 x_2 < = -6$ (Constraint 2)

$x_1 + x_2 < = 3$ (Constraint 3)

We now convert the inequalities into equalities by introducing slack variables (since all equations are <=0 type) in the constraint equations.

The equations are now written as

$3 x_1 - x_2 + s_1 = -3$ (Constraint 1)

$4 x_1 - 3 x_2 + s_2 = -6$ (Constraint 2)

$x_1 + x_2 + s_3 = 3$ (Constraint 3)

We have 5 variables and 3 constraint equations. By now I guess you must very familiar with way the basic and non-basic variables are determined. The basic variables are s_1, s_2, and s_3. While the non-basic variables are x_1 and x_2. Assuming the values of the non-basic variables = 0, the values of the basic variables are $s_1 = -3$, $s_2 = -6$, and $s_3 = 3$.

Let us re-write the objective equation by taking all the variables to the LHS and having the RHS = 0. By doing this we have the equation written like this.

$Z - 3x_1 - 2x_2 - 0 s_1 - 0 s_2 - 0 s_3 = 0$ We find that there are no positive coefficients for the non-basic variables for us to treat as the EV (as we normally do in the primal simplex method). This indicates that the objective Z equation resembles optimality. Also we find that two of the basic variables $s_1 = -3$, $s_2 = -6$ have < 0 values in the initial solution. This is a typical situation where we can adopt the dual simplex method.

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As mentioned earlier in the dual simplex method we start with an optimal infeasible solution and proceed towards feasibility while retaining optimality. In this method we will first select the LV (based on dual feasibility condition) and then select the EV (based on dual optimality condition). Let us first prepare the initial dual simplex table.

Table 1.38 – Initial Dual Simplex Table

Basis	x_1	x_2	s_1	s_2	s_3	Solution
Z	-3	**-2**	0	0	0	0
s_1	-3	**-1**	1	0	0	-3
s_2	**-4**	**-3**	**0**	**1**	**0**	**-6**
s_3	1	**1**	0	0	1	3

As we had mentioned earlier, in this minimizing problem we find there are no positive coefficients that we can select as the EV. So by the dual feasibility condition let us first select the LV. The LV is the basic variable with the most negative value in the solution column. By this condition we find that $\mathbf{s_2}$ is the basic variable with the most negative value. Therefore $\mathbf{s_2}$ is the LV. In-order to determine the EV we must have a look at the dual optimality condition. It shows an equation like this: modulus of $Z_j/á_{rj}$ such that $á_{rj} < 0$. $á_{rj}$. elements whose value is less than 0. This is the $á_{rj} < 0$. These elements are $\mathbf{x_1}$ and $\mathbf{x_2}$. Fromese two columns select the values from the objective Z equation. These values are Z_j es.

For $\mathbf{x_1}$ it is -3 / -4 and for $\mathbf{x_2}$ it is -2/-3.

Ignoring the signs we find that $\mathbf{x_2}$ has the least ratio among the two variables. Therefore $\mathbf{x_2}$ is the EV. Hence we proceed to the next iteration based on the earlier known procedure. The I-iteration table is shown below.

Table 1.39 – I – Iteration Table

Basis	x_1	x_2	s_1	s_2	s_3	Solution
Z	**-1/3**	0	0	-2/3	0	4
s_1	**-5/3**	**0**	**1**	**-1/3**	**0**	**-1**
x_2	**4/3**	1	0	-1/3	0	2
s_3	**-1/3**	0	0	1/3	1	1

NOTES

On observing the above table we find that there is a negative value in the solution column. Therefore we have not reached the optimal solution. Since the negative value falls in the $\mathbf{s_1}$ row this will be our leaving variable (LV). If $\mathbf{s_1}$ is our LV, can you identify the EV based on our earlier explanation? Yes: you got it right it is $\mathbf{x_1}$. Now let us proceed to the next iteration table with our usual procedure.

Table 1.40 – II – Iteration Table

Basis	x_1	x_2	s_1	s_2	s_3	Solution
Z	0	0	-1/5	-3/5	0	**21/5**
x_1	1	0	-3/5	1/5	0	3/5
x_1	0	1	4/5	-3/5	0	6/5
s_3	0	0	-1/5	2/5	1	6/5

Now on seeing the aboe table we find that there are no negative values in the solution column. Therefore we can conclude that optimality is reached. The value of $\mathbf{x_1}$ = 3/5 and that of $\mathbf{x_2}$ = 6/5. The value of Z = 21 /5.

DUAL SIMPLEX METHOD

Precondition:

1. Solution must be infeasible and optimal

Infeasibility condition: To adopt the dual simplex method the starting table must have an optimum objective row with at least one infeasible (less than zero) basic variable. To maintain optimality and simultaneously move towards feasibility at each new iteration the following two conditions are employed.

2. Dual feasibility condition:

The leaving variable is the basic variable having the most negative value. Ties are broken arbitrarily. If all the basic variables are non negative the algorithm ends.

3. Dual Optimality:

The entering variable is determined from among the non basic variables as the one corresponding to Min Non Basic Variable (x_j) mod $(Zj/á_{rj})$ where $á_{rj} > 0$ Where $á_{rj}$ is the constraint coefficient of the table associated with the leaving variable x_r and the column of the entering variable x_j.

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Solve the following by Dual simplex method.

Min (Z) = 3x1 + 2x2

S.t: 3x1 + x2 >= 3

4x1 + 3x2 >= 6

x1 + x2 <=3

x1 ,x2 >= 0

New Constraints:S

-3x1 – x2 +s1 = -3

-4x1 – 3x2 +s2 =-6

x1 +x2 +s3 = 3

Z – 3x1 – 2x2 -0s1 -0s2 -0s3 =0

Var (n) = 5 CE (m) = 3 NBV =2 (x1, x2)

S1 = -3; s2 = -6 ; s3 =3;

Basis	x1	x2	s1	s2	s3	sol
Z	-3	**-2**	0	0	0	0
s1	-3	**-1**	1	0	0	-3
s2	**-4**	**-3**	**0**	**1**	**0**	**-6**
s3	1	**1**	0	0	1	3
Z	**-1/3**	0	0	-2/3	0	4
s1	**-5/3**	**0**	**1**	**-1/3**	**0**	**-1**
x2	**4/3**	1	0	-1/3	0	2
s3	**-1/3**	0	0	1/3	1	1
Z	0	0	-1/5	-3/5	0	**21/5**
x1	1	0	-3/5	1/5	0	**3/5**
x2	0	1	4/5	-1/3	0	**6/5**
s3	0	0	-1/5	2/5	1	6/5

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1.4 PRINCIPLE OF DUALITY

Write the dual of

2x1 + 3x2 + x3 <= 300

x1 + x2 + 3x3 <= 300

x1 + 3x2 + x3 <= 240

x1 ,x2 ,x3 >= 0

Coefficient of primal problem   Coefficient of Dual (transpose of primal)

$$\begin{vmatrix} 2 & 3 & 1 & 300 \\ 1 & 1 & 3 & 300 \\ 1 & 3 & 1 & 240 \\ 2 & 2 & 4 & - \end{vmatrix} \qquad \begin{vmatrix} 2 & 1 & 1 & 2 \\ 3 & 1 & 3 & 2 \\ 1 & 3 & 1 & 4 \\ 300 & 300 & 240 & - \end{vmatrix}$$

Dual: Min (Z) = 300y1 + 300y2 +240y3

S.t:

2y1 + y2 + y3 >= 2

3y1 + y2 + 3y3 >= 2

y1 + 3y2 + y3 >= 4

y1 , y2 ,y3 >=0

2) **Write the dual of the following primal**

Max (Z) = 2x1 +x2

x1 +2x2 <= 10

x1 +x2 <=6

x1 – x2 <=2

x1 – 2x2 <=1

Coefficient of primal   Coefficient of dual

$$\begin{vmatrix} 1 & 2 & 10 \\ 1 & 1 & 6 \\ 1 & -1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & - \end{vmatrix} \qquad \begin{vmatrix} 1 & 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & -2 & 1 \\ 10 & 6 & 2 & 1 & - \end{vmatrix}$$

NOTES

Min (Z) = 10y1 + 6y2 +2y3 +y4

2y1 + y2 –y3 -2y4 >= 1

y1+y2+y3+y4>=2

1. Write the dual of the following primal

Min (Z) = 4x1 + x2

3x1 + x2 =3

4x1 +3x2 >= 6

x1 +2x2 <=4

x1 ,x2 >=0

3x1+x2 >=3 -3x1-x2<=-3

3x1+x2<=3 3x1 +x2 <=3

-4x1-3x2<=-6 -4x1-3x2<=-6

x1+2x2<=4 x1+2x2<=4

Coefficient of primal

$$\begin{vmatrix} -3 & -1 & -3 \\ 3 & 1 & 3 \\ -4 & -3 & -6 \\ 1 & 2 & 4 \\ 4 & 1 & - \end{vmatrix}$$

Coefficient of dual

$$\begin{vmatrix} -3 & 3 & -4 & 1 & 4 \\ -1 & 1 & -3 & 2 & 1 \\ -3 & 3 & -6 & 4 & - \end{vmatrix}$$

Max (Z) = -3y1+3y1’-6y3+4y4

-3y1+3y1’-4y3+y4>= 4

-y1+y1’-3y3+2y4>=1

y1’ – y1 can be taken as y1 so rewriting the above equations we get

Max (Z) = 3y1-6y3+4y4

3y1-4y3+y4>= 4

y1-3y3+2y4>=1

With this we have learnt the algorithms associated with simplex methods.

NOTES

1.5 SENSITIVITY ANALYSIS

Sensitivity Analysis: Sensitivity Analysis is the process of measuring the effects of changing the parameters and characteristics of the model on optimality.

This can be done by

1. Making changes in the RHS constants of the constraints.

If you were to look at the first problem that we formulated (concerning exterior and interior paints) you will remember that the limitations on the raw material M1 was a maximum availability of 24 tons a week. Suppose the company is in a position to buy 30 tons of M1, will the quantities of exterior and interior paints produced change? Yes it will. This is the focus of sensitivity analysis.

2 Making changes in the objective function coefficients.

Similarly if the selling price of the exterior and interior paints change (the objective function coefficients) then again the quantities of exterior and interior paints produced

3. Adding a new constraint.

Now suppose a new constraint is imposed such as not more than 2 tons of exterior paint can be sold. Then again this will have a bearing on the quantities of exterior and interior paints produced

4. Adding a new variable.

If the company opts to produce water proofing paints in addition to exterior and interior paints with the same available resources, then again the quantities of exterior and interior paints produced.

Measuring all these changes is the focus of sensitivity analysis.

1) Making changes in RHS of constraints

Analyze the change in the optimal solution if the RHS of the constraints have a change in their values from [9 5 1 1] and [12 6 4 1]

Max (Z) = 2x1 + x2

Subject to:

x1 + 2x2 <= 10

x1 + x2 <=6

x1 – x2 <= 2

NOTES

x1 – 2x2 <=1

S.t: x1 +2x2 +s1 =10

x1 + x2 +s2 =6

x1 –x2 +s3 =2

x2 – 2x2 +s4 =1

Z- 2x1 – x2 – 0s1 -0s2 -0s3 -0s4=0

Basics	**x1**	x2	s1	s2	s3	s4	sol	Ratio
Z	**-2**	-1	0	0	0	0	0	
s1	**1**	2	1	0	0	0	10	10
s2	**1**	1	0	1	0	0	6	6
s3	**1**	-1	0	0	1	0	2	2
s4	**1**	**-2**	**0**	**0**	**0**	**1**	**1**	1
Z	0	**-5**	0	0	0	2	2	
s1	0	**4**	1	0	0	-1	9	9/4
s2	0	**3**	0	1	0	-1	5	5/3
s3	**0**	**1**	**0**	**0**	**1**	**-1**	**1**	1
x1	1	**-2**	0	0	0	1	1	
Z	0	0	0	0	5	**-3**	7	
s1	0	0	1	0	-4	**3**	5	5/3
s2	**0**	**0**	**0**	**1**	**-3**	**2**	**2**	1
x2	0	1	0	0	1	**-1**	1	
x1	1	0	0	0	2	**-1**	3	
Z	0	0	0	3/2	1/2	0	10	
s1	0	0	1	-3/2	1/2	0	2	
s4	0	0	0	1/2	-3/2	1	1	
x2	0	1	0	1/2	-1/2	0	2	
x1	1	0	0	1/2	1/2	0	4	

NOTES

Initial solution: x1 =4 x2 =2 Z= 10

[Basic variable in optimal table] = [Technical coefficient of optimal

Tables w.r.t the basic variables

in the initial table] * [New RHS values]

CASE I:

s1 =	1	-3/2	½	0	9
s4	0½	-3/2	1	*	5
x2	0½	-1/2	0		1
x1	0	½	½	0	1
s1	=	2			
s4		2			
x2		2			
x1		3			

Optimal solution is not affected

CASE II:

s1 =	1	-3/2	½	0	12
s4	0½	-3/2	1	*	6
x2	0½	-1/2	0		4
x1	0	½	½	0	1
s1	=	5			
s4		-2			
x2		1			
x1		5			

Iterating further towards optimality:

Z	0	0	0	3/2	**1/2**	0	11
s1	0	0	1	-3/2	**1/2**	0	5
s4	**0**	**0**	**0**	**1/2**	**-3/2**	**1**	**-2**
x2	0	1	0	1/2	**-1/2**	0	1
x1	1	0	0	1/2	**1/2**	0	5
Z	0	0	0	5/3	0	1/3	31/3
s1	0	0	1	-4/3	0	1/3	13/3
s3	0	0	0	-1/3	1	-2/3	4/3
x2	0	1	0	1/3	0	-1/3	5/3
x1	1	0	0	2/3	0	1/3	13/3

NOTES

Optimal solution: x1 = 13/3 x2 = 5/3 Z = 31/3

2) Making changes in Objective function coefficient:

Max (Z) = 10x1 + 15x2 + 20x3

S.t: 2x1 + 4x2 +6x3 <=24

3x1 + 9x2 + 6x3 <= 30

x1 ,x2 ,x3 >=0

Find the range of objective function coefficient c1 of variable x1 such that optimality is unaffected.

S.t 2x1 + 4x2 +6x3 + s1 =24

3x1 + 9x2 + 6x3 + s2 = 30

Z – 10x1 - 15x2 - 20x3 -0s1 -0s2 = 0

Basis	x1	x2	x3	s1	s2	sol	Ratio
Z	-10	-15	**-20**	0	0	0	
s1	**2**	**4**	**6**	**1**	**0**	**24**	**4**
s2	3	9	**6**	0	1	30	5
Z	**-10/3**	-5/3	0	10/3	0	80	
x3	**1/3**	2/3	1	1/6	0	4	12
s2	**1**	**5**	**0**	**-1**	**1**	**6**	**6**
Z	0	15	0	0	10/3	100	
x3	0	-1	1	1/2	-1/3	2	
x1	1	5	0	-1	1	6	

Initial solution x1 =6 x3 =2 Z= 100

NOTES

Range of C1

Zx2 = 15 – {[20 C1] -1 } =35 -5C1 <=0 C1>= 7
5

Zs1= 0 – {[20 C1] -1/2 } =C1 - 10 <=0
-1

Zs2 = 0 – {[20 C1] -1/3 } =C1 >= 20/3 = 6.666
1

7<= C <= 10

Determine the range of C2 w.r.t Variable x2

Zx2 = C2 - {[20 C1] -1 } =C2 -30 <= 0 **C2 <= 30**

5

3) Introduce a new constraint

Max (Z) = 6x1 + 8x2

S.t: 5x1 +10x2 <= 60

4x1 + 4x2 <= 40

x1 ,x2 >=0

S.t: 5x1 +10x2 + s1 =60

4x1 + 4x2 +s2 = 40

Z – 6x1 -8x2 -0s1 -0s2 = 0

Basis	x1	x2	s1	s2	sol	Ratio
Z	-6	-8	0	0	0	
s1	**5**	**10**	**1**	**0**	**60**	**6**
s2	4	4	0	1	40	10
Z	**-2**	0	4/5	0	48	
x3	**1/2**	1	1/10	0	6	12
s2	**2**	**0**	**-2/5**	**1**	**16**	**8**
Z	0	0	2/5	1	64	
x2	0	1	1/5	-1/4	2	
x1	1	0	-1/5	1/2	8	

Initial Solution: x1 = 8 x2 = 2 Z =64

NOTES

CASE I

If new constraint 7x1 + 2x2 <= 65 is added check whether the new constraint makes the optimal solution infeasible.

7x1 + 2x2 <= 65 ➔ Substitute x1 = 8 and x2 =2 we get 60 <= 65

The new constraint when added doesn't make the optimal solution infeasible.

CASE II:

6x1 + 3x2 <= 48 Makes the optimal solution infeasible.

So substituting for x1 and x2 from the initial optimal table.

6x1 + 3x2 +s3 =48

x2 +1/5s1 -1/4s2 = 2 ➔ x2 = 2 -1/5s1 +1/4s2

x1 -1/5s1 +1/2s2 = 8 ➔ x1 = 8+1/5s1 -1/2s2

6(8 +1/5s1 -1/2s2) + 3(2 -1/5s1 +1/4s2) +s3 =48

Simplifying we get 3/5s1 – 9/4s2 + s3 = -6

Introducing the new constraint as above in the optimal table and further iterating

Basis	x1	x2	s1	**s2**	s3	sol
Z	0	0	2/5	**1**	0	64
x2	0	1	1/5	**-1/4**	0	2
x1	1	0	-1/5	**1/2**	0	8
s3	**0**	**0**	**3/5**	**-9/4**	**1**	**-6**
Z	0	0	2/3	0	4/9	184/3
x2	0	1	2/15	0	-1/9	8/3
x1	1	0	-1/15	0	2/9	20/3
s2	0	0	-4/15	1	-4/9	8/3

New optimal solution x1 = 20/3 x2 = 8/3 Z = 184/3

NOTES

4) Adding a new variable:

A new product P3 is introduced in the existing product mix. The profit per unit of the new product in the first constraint is 6 hours and 5 hours in II constraint per unit respectively. Check whether the new variable affects optimality after the new variable has been incorporated.

Max (Z) = 6x1 + 8x2 +20x3

S.t

5x1 + 10x2 +6x3 <= 60

4x1 + 4x2 +5x3 <= 40

$$Cx3 = [Cx3] - [8\ 6] \begin{matrix} 1/5 & -1/4 \\ -1/5 & ½ \end{matrix} * \begin{matrix} 6 \\ 5 \end{matrix}$$

$$= 20 – [8\ \ 6] \begin{matrix} -1/20 \\ 13/10 \end{matrix} = 63/5$$

Introducing the coefficient of X3 in the initial optimal table and then further iterating

Basis	x1	x2	x3	s1	s2	sol
Z	0	0	**-63/5**	2/5	1	64
x2	0	1	**-1/20**	1/5	-1/4	2
x1	**1**	**0**	**13/10**	**-1/5**	**1/2**	**8**
Z	126/13	0	0	**-20/13**	76/13	1840/13
x2	**1/26**	**1**	**0**	**5/26**	**-3/13**	**30/13**
x3	10/3	0	1	**-2/13**	5/13	80/13
Z	10	8	0	0	4	160
S1	1/5	26/5	0	1	-6/5	12
X3	4/5	4/5	1	0	1/5	8

Optimal solution: x3 =8 Z = 160

NOTES

Exercises

Have you understood?

1. Is it possible to have multiple solutions for an LP problem by graphical solution? If yes, when?
2. What is the purpose of an iso-profit line?
3. When will you adopt primal simplex, Big-M or Two-Phase and Dual simplex methods?
4. When will you adopt primal simplex, Big-M or Two-Phase and Dual simplex methods?
5. Can sensitivity analysis have real time applications? Think of one for each case.
6. Try to formulate an LPP on your own using simple statements.

Solve

1. The standard weight of a special brick is 5 kg and it contains two ingredients B1 and B2. B1 costs Rs 5 per kg and B2 costs Rs 8 per kg. Strength considerations dictate that the brick contains not more than 4 kg of B1 and a minimum of 2 kg of B2. Formulate the problem as an Linear Programming Model.
2. A company produces two types of leather belts A and B. A is of superior quality and B is of inferior quality. The respective profits are Rs 10 and Rs 5 per belt. The supply of raw material is sufficient for making 850 belts per day. For belt A, a special type of buckle is required and 500 are available per day. There are 700 buckles available for belt B per day. Belt A needs twice as much time as that required for belt B and the company can produce 500 belts if all of them were of type A. Formulate a LP model for the above problem.
3. Max $Z = 2x_1 + x_2$

   Subject to: $x_1 + 2x_2 <= 10$

   $x_1 + x_2 <= 6$

   $x_1 - x_2 <= 2$

   $x_1 - 2x_2 <= 1$

   $x_1, x_2 >= 0$
4. Max $Z = 3x_1 + 2x_2$

   Subject to: $2x_1 + x_2 <= 2$

   $3x_1 + 4x_2 >= 6$

   $x_1, x_2 >= 0$

5. Min $Z = x_1 - 2x_2 - 3x_3$

   Subject to: $-2x_1 + x_2 + 3x_3 = 2$

   $2x_1 + 3x_2 + 4x_3 = 1$

   $x_1, x_2, x_3 >= 0$

Summary

This unit would have given you an insight on the application of LP models to real time situations and the methodology to solve such problems.

NOTES

NOTES

UNIT II

LINEAR PROGRAMMING EXTENSIONS

INTRODUCTION

This unit will provide insight into the transportation and assignment models. Both transportation and assignment models are extended LPP's. Transportation models are concerned with matching the aggregate supply and aggregate demand while still minimizing the costs or maximizing revenues. Assignment models are concerned with allocations on a one to one basis with an objective of minimizing costs or maximizing revenues. There are very many applications in real time based on the two models are how you apply them in real time situations is only limited to your comprehension.

Learning Objectives

The learning objectives in this unit are

1. To arrive at Initial Basic Feasible solutions in Transportation Models.
2. To check for optimality and to arrive at optimality in Transportation Models.
3. To solve for and learn the different assignment models.

2.1 TRANSPORTATION MODELS

A transportation model is a subclass of an LP problem with focus on transporting goods from points of origin (POO) such as factories to points of destination (POD) such as retail outlets at minimum possible cost, time or any other resource. The requirements at the POD indicate the demand and the capacities of the POO indicate the supply position.

With this brief introduction, let us look into a typical transportation problem.

2.1.1 Minimising Problems

Example 1: There are three factories F_1, F_2, and F_3 capable of producing 56, 82 and 77 units of a product per week respectively. This product has to be shipped to three retail outlets namely R_1, R_2, and R_3 which have requirements of 72, 102 and 41 units of the same product per week. The cost of transporting one unit from the three factories to the retail outlets are given in the matrix below. Find the optimal shipping cost.

NOTES

Table 2.1 - Cost Matrix

Factory / Retail Outlet	R_1	R_2	R_3
F_1	4	8	8
F_2	16	24	16
F_3	8	16	24

Solution:

Step 1:

In any transportation problem first identify whether it is a minimizing problem or a maximizing problem. This problem is a minimizing case because you have been asked to find the optimal cost (read the last sentence of the problem). All minimizing problems would be related to time, cost or any other similar resource. Maximising problems would be related to revenue or profits.

Step 2:

Let us now accommodate the supply and demand data also in the cost matrix. Your new matrix will appear like this

Table 2.2 - Cost Matrix with supply and demand values

Factory / Retail Outlet	R_1	R_2	R_3	Supply
F_1	4	8	8	56
F_2	16	24	16	82
F_3	8	16	24	77
Demand	72	102	41	

We know that factory $\mathbf{F_1}$ can supply 56 units and $\mathbf{F_2}$ can supply 82 units and $\mathbf{F_3}$ can supply 77 units. Similarly retail outlet $\mathbf{R_1}$ needs 72 units, $\mathbf{R_2}$ needs 102 units and $\mathbf{R_3}$needs 41 units.

NOTES

Step 3:

An important check before proceeding is to see whether the cumulative supply = cumulative demand. That is **Ó S = Ó D,** which means 56+82 +77 must be equal to 72 +102+41. It happens to be so in this case. The total of supply = total of demand = 215. This is known as the balanced condition in a transportation model. The new matrix will appear as below.

Table 2.3 - Cost Matrix with supply and demand values - Aggregated

Factory / Retail Outlet	R_1	R_2	R_3	Supply
F_1	4	8	8	56
F_2	16	24	16	82
F_3	8	16	24	77
Demand	72	102	41	215

Step 4:

We can now proceed to find the initial basic feasible solution (IBFS). The initial basic feasible solution is a solution which is a possible (feasible) solution but may or may not be the optimal.

There are three ways of computing the initial basic feasible solution. They are

1. The north – west corner (NWC) rule
2. The least cost method (LC) and
3. The Vogel's Approximation method (Popularly known as VAM)

Note: I would suggest that you to adopt VAM (if the other two methods are not specifically asked to be adopted) because it gives you a better IBFS than the other two methods.

However we will discuss all the three methods in an effort to enlighten you better.

IBFS through NWC Rule

Table 2.4 – Choice of cell – By NWC Rule

Factory / Retail Outlet	R_1	R_2	R_3	Supply
F_1	$\boxed{4}$	8	8	56
F_2	16	24	16	82
F_3	8	16	24	77
Demand	72	102	41	215

NOTES

The ighlighted cell is the north-western corner of the cost matrix. So we make the first allocation in this cell. This cell indicates that if 56 units are transported from $\mathbf{F_1}$ to $\mathbf{R_1}$ then the cost per unit of transportation is Rs 4/-. You must be careful to choose the minimum of the two values (that is between 56 and 72, choose the lower value). The logic is that, although $\mathbf{R_1}$ needs 72 units $\mathbf{F_1}$ cannot supply more than 56 units. Therefore the balance need for $\mathbf{R_1}$ must be met from some other factory. After all the units from a factory is allocated strike out that row or write the word "Done". Similarly when all the need of a retail store is fulfilled cross out that row or write the word "Done".

Table 2.5 – Allocation of cell – By NWC Rule – Step 1

Factory / Retail Outlet	R_1	R_2	R_3	Supply	Status
F_1	4 x 56	8	8	56 – 56 =0	Done
F_2	16	24	16	82	
F_3	8	16	24	77	
Demand	72 - 56 = 16	102	41	215	
Status					

We have to make further allocations by moving progressively (diagonally) to the south-eastern corner of the matrix. The next cell where an allocation can be made is from $\mathbf{F_2}$ to $\mathbf{R_1}$. This is because retail store $\mathbf{R_1}$ is yet to receive 16 units and that is to be provided by $\mathbf{F_2}$.

Table 2.6 – Allocation of cell – By NWC Rule – Step 2

Factory / Retail Outlet	R_1	R_2	R_3	Supply	Status
F_1	4 x 56	8	8	56	Done
F_2	16 x 16	24	16	82 – 16 = 66	
F_3	8	16	24	77	
Demand	72 – 56 -16 = 0	102	41	215	
Status	Done				

Table 2.6 shows that 16 units have been allocated from $\mathbf{F_2}$ to $\mathbf{R_1}$. Also notice that $\mathbf{F_2}$ can now supply only 66 units to other retail outlets since it has already supplied 16 units to $\mathbf{R_1}$. To progress further, the next obvious allocation will be from F_2 to R_2, in-order to exhaust the capacity of $\mathbf{F_2}$. This is now shown in Table 2.7.

NOTES

Table 2.7 – Allocation of cell – By NWC Rule – Step 3

Factory / Retail Outlet	R_1	R_2	R_3	Supply	Status
F_1	4 x 56	8	8	56	Done
F_2	16 x 16	24 x 66	16	82 – 16 – 66 = 0	Done
F_3	8	16	24	77	
Demand	72 – 56 -16 = 0	102 – 66 = 36	41	215	
Status	Done				

You will now notice that $\mathbf{F_2}$ had a balance of 66 units which has been allocated to $\mathbf{R_2}$. But $\mathbf{R_2}$ requires a total of 102 units. Therefore $\mathbf{F_2}$ has only partially the demand of $\mathbf{R_2}$. The balance of 36 units for $\mathbf{R_2}$ has to be supplied from $\mathbf{F_3}$. This is shown in Table 2.8.

Table 2.8 – Allocation of cell – By NWC Rule – Step 4

Factory / Retail Outlet	R_1	R_2	R_3	Supply	Status
F_1	4 x 56	8	8	56	Done
F_2	16 x 16	24 x 66	16	82 – 16 – 66 = 0	Done
F_3	8	16 x 36	24	77 – 36 = 41	
Demand	72 – 56 -16 = 0	102 – 66 - 36 = 0	41	215	
Status	Done	Done			

From Table 2.8 it is seen that 36 units have been allocated from $\mathbf{F_3}$ to $\mathbf{R_2}$. Therefore all the demand for $\mathbf{R_2}$ has been met and so the status as been updated to “Done”. $\mathbf{F_3}$ can now supply only 41 units and amazingly that is the same amount required at $\mathbf{R_3}$. The final allocation is shown in Table 2.9.

NOTES

Table 2.9 – Allocation of cell – By NWC Rule – Step 4

Factory / Retail Outlet	R_1	R_2	R_3	Supply	Status
F_1	4 x 56	8	8	56	Done
F_2	16 x 16	24 x 66	16	82 – 16 – 66 = 0	Done
F_3	8	16 x 36	24 x 41	77 – 36 - 41= 0	Done
Demand	72 – 56 -16 = 0	102 – 66 - 36 = 0	41 – 41 = 0	215	
Status	Done	Done	Done		

We now find that all allocations have been completed and it is time for us to check whether we have done all allocations correctly. The first aspect is to check whether the allocated totals in each row and each column are equal to the boundary values. For example if you look at row $\mathbf{F_2}$, you will see that 16 units have been allocated to $\mathbf{R_1}$ and 66 units have been allocated to $\mathbf{R_2}$ from $\mathbf{F_2}$. At the same time the maximum capacity of $\mathbf{F_2}$ has neither been under-utilized or over-utilized. Similarly check for all rows and columns to see that the units allocated add up to the boundary values.

We now calculate the total transportation cost done through N-W corner rule method. Look at the highlighted cells. They indicate the cells where allocations have been made. In every cell the first value represents the cost of transporting one unit and the second value shows the number of units transported. For example, in the first cell, 4 x 56, indicates that 56 units are transported from factory $\mathbf{F_2}$ to retail store $\mathbf{R_1}$ and the transportation cost is Rs 4/- per unit. Therefore the cost of transporting 56 units is Rs 224/-. Similarly the other costs are calculated as given below.

4 x 56 + 16 x 16 + 24 x 66 + 16 x 36 + 24 x 41 = Rs 3624/-. Therefore the total cost of transportation through the N-W corner rule method is Rs 3624/-.

Now remember that we have an IBFS (if you are wondering what IBFS means, refer to earlier portions of this text) but not and the optimal (best) solution.

Least coast method

We now look for another IBFS through the LC method. In the LC method the logic is that we start allocating from the least cost cell. Let us start with the initial cost matrix.

NOTES

Table 2.10 – Cost Matrix with supply and demand values

Factory / Retail Outlet	R_1	R_2	R_3	Supply	Status
F_1	4	8	8	56	
F_2	16	24	16	82	
F_3	8	16	24	77	
Demand	72	102	41		
Status					

It is seen that the least cost, a value of 4, is in the **F_1 R_1** cell. So we allot 56 units to this cell.

Table 2.11 – Cost Matrix with supply and demand values

Factory / Retail Outlet	R_1	R_2	R_3	Supply	Status
F_1	4 x 56	8	8	56	Done
F_2	16	24	16	82	
F_3	8	16	24	77	
Demand	72 – 56 = 16	102	41		
Status					

We find that the there are three least cost cells with a value of 8. They are at **F_1 R_2**, **F_1 R_3**, and **F_3 R_1**. In case of ties, we have to select a cell where maximum allocation is possible. At **F_1 R_2**, and **F_1 R_3** , we cannot allocate any quantity since the entire F row is already completed. So our only choice is at **F_3 R_1**. We can allocate 16 units here. This is reflected in our next table.

Table 2.12 – Cost Matrix with supply and demand values

Factory / Retail Outlet	R_1	R_2	R_3	Supply	Status
F_1	4 x 56	8	8	56	Done
F_2	16	24	16	82	
F_3	8 x 16	16 x 61	24	77 – 16 – 61 = 77	Done
Demand	72 – 56 – 16 = 0	102 – 61 = 41	41		
Status	Done				

NOTES

If we now look at the other least cost values, we have one 16 at $F_2 R_3$ and another at $F_3 R_2$. Out of these two we can make a maximum allocation of 61 at $F_3 R_2$. We next allocate 41 units to $F_2 R_3$. This is shown in the following Table 2.13.

Table 2.13 – Cost Matrix with supply and demand values

Factory / Retail Outlet	R_1	R_2	R_3	Supply	Status
F_1	4 x 56	8	8	56	Done
F_2	16	24	16 x 41	82 – 41 = 41	
F_3	8 x 16	16 x 61	24	77 – 16 – 61 = 77	Done
Demand	72 – 56 – 16 = 0	102 – 61 = 41	41 – 41 = 0		
Status	Done		Done		

We finally have only one cell where we can make an allocation. And that cell is $F_2 R_3$. We can allocate 41 units here and that completes our allocation. This is shown in the next Table 2.14.

Table 2.14 – Cost Matrix with supply and demand values

Factory / Retail Outlet	R_1	R_2	R_3	Supply	Status
F_1	4 x 56	8	8	56	Done
F_2	16	24 x 41	16 x 41	82 – 41 - 41 = 0	Done
F_3	8 x 16	16 x 61	24	77 – 16 – 61 = 0	Done
Demand	72 – 56 – 16 = 0	102 – 61 - 41= 0	41 – 41 = 0		
Status	Done	Done	Done		

We have now completed the allocations through the least cost method. The cost works out to be 4 x 56 + 24 x 41 + 16 x 41 + 8 x 16 + 16 x 61 = Rs 2968. We now find that the least cost method has given us a better result than the N-W corner rule method.

However as I had mentioned earlier, VAM gives us the best IBFS.

We shall now work out the IBFS thro' the VAM. For this purpose we need the original matrix which is shown in table 2.15. You need to read the following lines carefully. In every row and every column subtract the least cost from the next highest cost. These differences are known as penalties.

Table 2.15 – Cost Matrix with supply and demand values

Factory / Retail Outlet	R_1	R_2	R_3	Supply	Column Penalty - I	Status
F_1	4	8	8	56	4	
F_2	16	24	16	82	**8**	
F_3	8 x 72	16	24	77 – 72 = 5	**8**	
Demand	72 – 72 = 0	102	41	215		
Row Penalty – 1	4	**8**	**8**			
Status	Done					

We must choose the row or column with the highest penalty to identify the cell to make an allocation. However we find here that a highest penalty of 8 occurs in 4 places that is at F2, F3, R2 and R3. In such cases select the cell where least cost occurs. We find that out of F2, F3, R2 and R3 the least cost occurs at R2, R3 and F3. These are the highlighted cells. The tie is still not broken. So we now select a cell where maximum allocation can be made. This is possible at the junction of R1and F3. So we allocate 72 units to this cell. R1 is completed and we have to repeat the same process.

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Table 2.16 – Cost Matrix with supply and demand values

Factory / Retail Outlet	**R_1**	**R_2**	**R_3**	**Supply**	**Column Penalty 1**	**Column Penalty 2**	**Status**
F_1	4	8 x 56	8	56 – 56 = 0	4	4	Done
F_2	16	24	16	82	**8**	8	
F_3	8 x 72	16	24	77 – 72 = 5	**8**	8	
Demand	72 – 72 = 0	102 – 56 = 46	41	215			
Row Penalty 1	4	**8**	**8**				
Row Penalty 2	Done	**8**	**8**				
Status	Done						

We now find the same situation as before. I guess you can now follow the old rules of selecting the cell with least cost first and if ties still exist then choose the where a maximum allocation can be made. Let us hope you identify the correct cell to make an allocation. Yes! you are right its is the junction of F1 and R2. We allocate 56 units to this cell and the row R1 is done. We now proceed with calculation of penalties and before.

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Table 2.16 – Cost Matrix with supply and demand values

Factory/ Retail Outlet	R_1	R_2	R_3	Supply	Column Penalty 1	Column Penalty 2	Column Penalty 3	Status
F_1	4	8 x 56	8	56 – 56 = 0	4	4	Done	Done
F_2	16	24	16 x 41	82 – 41 = 41	**8**	8	8	
F_3	8 x 72	16	24	77 – 72 = 5	**8**	8	8	
Demand	72 – 72 = 0	102 – 56 = 46	41 – 41 = 0	215				
Row Penalty 1	4	**8**	**8**					
Row Penalty 2	Done	**8**	**8**					
Row Penalty 3	Done	**8**	**8**					
Status	Done		Done					

The tie does seem to leave us in this stage also. So let's follow the old rules. Let's see if you identify the cell. Yes it is the junction of F2 and R3. We allocate 41 units here and column R3 is done with.

NOTES

Table 2.17 – Cost Matrix with supply and demand values

Factory / Retail Outlet	R_1	R_2	R_3	Supply	Column Penalty 1	Column Penalty 2	Column Penalty 3	Column Penalty 4	Status
F_1	4	8 x 56	8	56 – 56 = 0	4	4	Done	Done	Done
F_2	16	24	16 x 41	82 – 41 = 41	**8**	8	8	**8**	
F_3	8 x 72	16 x 5	24	77 – 72 - 5 = 0	**8**	8	8	**8**	Done
Demand	72 – 72 = 0	102 – 56 – 5 = 41	41 – 41 = 0	215					
Row Penalty 1	4	**8**	**8**						
Row Penalty 2	Done	**8**	**8**						
Row Penalty 3	Done	**8**	**8**						
Row Penalty 4	Done	**8**	**Done**						
Status	Done		Done						

If you observe closely there are only two cells where allocations can be made they are the junctions of F2 and R2 and F2 and R3. Out of these two cells the junction of F3 and R2 has the least cost. That is 16. Therefore we allocate 5 units here from F3 to R2. Hence F3 is done with. Finally we have 41 units to be allocated from F2 to R2. This is the last cell to be allocated and it not essential that we find the penalties and only then proceed to allocation. The final allocation is shown in Table 2.18.

NOTES

Table 2.17 – Cost Matrix with supply and demand values

Factory / Retail Outlet	R_1	R_2	R_3	Supply	Column Penalty 1	Column Penalty 2	Column Penalty 3	Column Penalty 4	Status
F_1	4	8 x 56	8	56 – 56 = 0	4	4	Done	Done	Done
F_2	16	24 x 41	16 x 41	82 – 41 - 41 = 0	**8**	8	8	**8**	Done
F_3	8 x 72	16 x 5	24	77 – 72 - 5 = 0	**8**	8	8	**8**	Done
Demand	72 – 72 = 0	102 – 56 – 5 - 41 = 0	41 – 41 = 0	215					
Row Penalty 1	4	**8**	**8**						
Row Penalty 2	Done	**8**	**8**						
Row Penalty 3	Done	**8**	**8**						
Row Penalty 4	Done	**8**	**Done**						
Status	Done	Done	Done						

We now find that all allocations have been completed. The cost works out to be 8 x 56 + 24 x 41 + 16 x 41 + 8 x 72 + 16 x 5 = Rs. 2744.

We now find that out of the three methods the VAM has given us the least possible cost in the IBFS.

We now have to check whether the IBFS is the optimal (best) possible solution.

NOTES

Check for Optimality.

Checking 'IBFS through VAM' for optimality.

We'll first check whether the IBFS thro' VAM is optimal.

Before proceeding to the check for optimality, we must first check whether m + n -1 = Number of allocations (where m is the number of rows and n is the number of columns). We have made 5 allocations, and this is equal to 3 + 3 – 1. Therefore we can proceed with our check for optimality. If this condition is not satisfied, it is known as a condition of "degeneracy". If degeneracy occurs we will have to resolve it before we proceed to the check for optimality. Don't worry about it now, as this problem satisfies the condition

Let us name all the rows as u_i and the rows as v_j. So the first row is u1, the second row u2 and the third row is u3. Similarly the columns are v1, v2 and v3. The allocated cells are shown with names assigned to the rows and columns.

Table 2.18 – Cost Matrix with supply and demand values

		v1	**v2**	**v3**	
	Factory / Retail Outlet	**R_1**	**R_2**	**R_3**	**Supply**
u1	**F_1**	4	8 x 56	8	56
u2	**F_2**	16	24 x 41	16 x 41	82
u3	**F_3**	8 x 72	16 x 5	24	77
	Demand	72	102	41	215

We need to take the cost values of the allocated cells and write them as a summation of ui + vj.

That is u1 + v2 = 8,

u2 + v2 = 24

u2 + v3 = 16

u3 + v1 = 8

u3 + v2 = 16

One word of caution here, you need to take the cost values and <u>not</u> the number of units assigned to the cells.

We next assume u1 = 0. Read the next few lines carefully and if required repeatedly.

NOTES

If u1 =0 then v2 =8.

If v2 = 8 then u2 = 16

If u2 = 16 then v3 = 0

If v2 =8 then u3 = 8 and

If u3 = 8 then v1 = 0.

We next form the ui + vj matrix. The already allocated cells are filled with 'X'. The other cells are calculated as a summation of ui + vj. For example u1+v1 =0, u1 + v3 = 0, u2 + v1 = 16 and u3 + v3 = 8.

Table 2.19 – (ui + vj) matrix

	v1	**v2**	**v3**
u1	0	X	0
u2	16	X	X
u3	X	X	8

We next calculate the Cij – (ui +vj) matrix. Here Cij represents the original cost values of the required cells. The computed values are shown in Table 2.20. The calculations are done like this The cost value in u1 v1 cell is 4. The u1 + v1 value is 0 (refer Table 2.19), therefore c11 – (u1 + v1) = 4 – 0 = 4. Likewise calculate all the other cell values (except the allocated cells).

Table 2.20 –Cij - (ui + vj) matrix - VAM

	v1	**v2**	**v3**
u1	4	X	8
u2	0	X	X
u3	X	X	16

Since the Cij – (ui+vj) matrix contains all non-negative values, we can conclude that the solution obtained by VAM is optimal (the best possible solution).

Now I would encourage you to check if the IBFS obtained through the Least Cost (LC) method is optimal or not, by our known u-v method. Please compute the Cij – (ui+vj) for the IBFS obtained through LC method. The result of your computation is likely to be as under.

Table 2.21 –Cij - (ui + vj) matrix – LC method

	v1	v2	v3
u1	X	-4	4
u2	0	X	X
u3	X	X	16

There is a negative value in the cell u1v2 and this means that the IBF solution we have thro' LC method is not optimal.

So we'll have to iterate towards optimality using the MODI (Modified Distribution) or Stepping Stone method.

MODI method

Refer to Table 2.21. The presence of a -4 in the cell u1 v2 indicates that if an allocation is made in this cell, you are likely to get a better solution. So we need to follow a simple procedure for re-distribution.

Once again read the following lines carefully and if required repeatedly.

Starting from the cell with a negative value, we need to complete a closed loop using only horizontal and vertical lines such that the corners are already allocated cells. The closed loop will appear like this.

Table 2.22 –Cij - (ui + vj) matrix – LC method

	v1	v2	v3
u1	X	-4	4
u2	0	X	X
u3	X	X	16

Starting from the cell with a negative value, that is u1 v2, assign + and – signs alternatively for all corners. So u1 v2 will have a + sign, u3v2 will have a – sign, u3 v1 will have a + sign and finally u1 v1 will have a – sign.

Now these signs will have real meaning in the allocations done using the LC method.

NOTES

Table 2.23 – Cost Matrix with supply and demand values

Factory / Retail Outlet	(v1) R_1	(v2) R_2	(v3) R_3	Supply
(u1) F_1	4 x 56	8	8	56
(u2) F_2	16	24 x 41	16 x 41	82
(u3) F_3	8 x 16	16 x 61	24	77
Demand	72	102	41	

Out of the corners with the negative values which is u1v1 and u3v2 we'll have to select the one with the least allocation. That is cell u1v1 with an allocation of 56 units. Now this value is the one that has to be added or subtracted to the corners of the closed loop depending on the signs assigned to the corners.

The table 2.24. shows the modified distribution.

Table 2.24 – Cost Matrix with supply and demand values

Factory / Retail Outlet	(v1) R_1	(v2) R_2	(v3) R_3	Supply
(u1) F_1	4 x (56-56=0)	8 x 56	8	56
(u2) F_2	16	24 x 41	16 x 41	82
(u3) F_3	8 x (16+56=72)	16 x (61-56=5)	24	77
Demand	72	102	41	

So the new distribution will be like this.

Table 2.25 – Cost Matrix with supply and demand values

Factory / Retail Outlet	(v1) R_1	(v2) R_2	(v3) R_3	Supply
(u1) F_1	4	8 x 56	8	56
(u2) F_2	16	24 x 41	16 x 41	82
(u3) F_3	8 x 72)	16 x 5	24	77
Demand	72	102	41	

NOTES

Does this distribution look familiar to you? Yes this is the same allocations we had thro'VAM.

Normally after every modified distribution we must perform the check for optimality. But in this case since the distribution is the same as the one thro' VAM and we had already checked for optimality, we stop and conclude that this distribution is optimal. That is the allocations by LC method were not optimal which was reflected in our test for optimality and so we got a modified distribution.

I now suggest that you perform the test for optimality for the IBFS got thro' N-W corner rule and iterate towards optimality.

OK that brings us to the close of the transportation model. Now for a quick recap, first compute the IBFS, then check for optimality. If optimal stop else proceed to MODI and continue till optimality is reached.

We'll now see how a maximising problem in transportation model is solved.

2.1.2. Maximising Problem

A maximizing problem in a transportation model is identified if you are asked to find maximum profit or maximum revenue or maximum sales.

We'll look at an example.

Example 2.1.2.1.

A company manufacturing air-coolers has plants located at Mumbai and Kolkota with capacities of 200 units and 100 units respectively. The company supplies its air-coolers to four showrooms at Ranchi, Delhi, Lucknow and Kanpur which have a maximum demand of 75, 100, 100 and 30 units respectively. Due to the differences in raw material cost and transportation cost, the profit/unit in rupees differs and this is shown in the table below.

Table 2.26 – Profit Matrix

	Ranchi	**Delhi**	**Lucknow**	**Kanpur**
Mumbai	90	90	100	110
Kolkota	50	70	130	85

Plan the production program so as to maximize the profit.

Solution

We know it is a problem of maximizing case. Before that let us remember our old rules. The aggregate supply must be equal to aggregate demand. But we find the aggregate

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supply = 300 units but the aggregate demand = 305 units. We need to include a dummy factory which theoretically will supply the required extra 5 units. The following table shows the necessary inclusion of the dummy row.

Note: In case the aggregate supply is > aggregate demand, then include a dummy column (i.e., a dummy outlet) which will absorb the excess supply.

Table 2.27 – Profit Matrix – With supply, demand and dummy row.

	Ranchi	Delhi	Lucknow	Kanpur	Supply
Mumbai	90	90	100	110	200
Kolkota	50	70	130	85	100
Dummy	0	0	0	0	5
Demand	75	100	100	30	305

We now find that the aggregate supply and aggregate demand are equal. Our next concern is to convert the maximizing problem into a minimizing case. To do this we'll have to select the highest profit from the profit matrix and subtract all other profit values from this value.

Table 2.28 – Profit Matrix – Converted to minimizing case

	Ranchi	Delhi	Lucknow	Kanpur	Supply
Mumbai	40	40	30	20	200
Kolkota	80	60	0	45	100
Dummy	130	130	130	130	5
Demand	75	100	100	30	305

Now we can follow the usual procedure for getting our IBFS. We will follow VAM (as we know that it gives us the best IBFS).

NOTES

Table 2.29 – IBFS – Stage 1

	Ranchi	Delhi	Lucknow	Kanpur	Supply	Row Penalty 1	Status
Mumbai	40	40	30	20	200	10	
Kolkota	80	60	**0 x 100**	45	100	**45**	**Done**
Dummy	130	130	130	130	5	0	
Demand	75	100	100	30	305		
Column Penalty 1	40	20	30	25			
Status			**Done**				

I have not explained how we have allocated 100 units from Kolkota to Lucknow on purpose. If you have not understood, you can refer to how we have solved the previous problem thro' VAM. We continue to the next stage.

Table 2.29 – IBFS – Stage 2

	Ranchi	Delhi	Lucknow	Kanpur	Supply	Row Penalty 1	Row Penalty 2	Status
Mumbai	40	40	*30*	*20 x 30*	200 – 30 = 170	10	20	
Kolkota	*80*	*60*	*0 x 100*	*45*	*100*	***45***	*Done*	*Done*
Dummy	130	130	*130*	*130*	5	0	0	
Demand	75	100	*100*	*30*	305			
Column Penalty 1	40	20	*30*	*25*				
Column Penalty 2	90	90	*Done*	***110***				
Status			*Done*	*Done*				

NOTES

Table 2.29 – IBFS – Stage 3

	Ranchi	Delhi	Lucknow	Kanpur	Supply	Row Penalty 1	Row Penalty 2	Row Penalty 3	Status
Mumbai	40	*40 x 100*	*30*	*20 x 30*	200 – 30 - 100 = 70	10	20	0	
Kolkota	*80*	*60*	*0 x 100*	*45*	*100*	***45***	*Done*	*Done*	*Done*
Dummy	130	*130*	*130*	*130*	5	0	0	0	
Demand	75	*100*	*100*	*30*	305				
Column Penalty 1	40	*20*	*30*	*25*					
Column Penalty 2	90	*90*	*Done*	***110***					
Column Penalty 3	90	***90***	*Done*	*Done*					
Status		*Done*	*Done*	*Done*					

We have allocated all except two cells. That is Ranchi has not got its quota of 75 units. This can be supplied from Mumbai plant (70 units) and the remaining 5 units from the Dummy plant.

Table 2.29 – IBFS – Stage 3

	Ranchi	Delhi	Lucknow	Kanpur	Supply	Row Penalty 1	Row Penalty 2	Row Penalty 3	Status
Iumbai	40x70	*40 x 100*	*30*	*20 x 30*	200 – 30 – 100 - 70 = 0	10	20	0	Done
'olkota	*80*	*60*	*0 x 100*	*45*	*100*	***45***	*Done*	*Done*	*Done*
)ummy	130x5	*130*	*130*	*130*	5 – 5 = 0	0	0	0	Done
)emand	75 – 70 – 5 = 0	*100*	*100*	*30*	305				
:olumn enalty 1	40	*20*	*30*	*25*					
:olumn enalty 2	90	*90*	*Done*	***110***					
:olumn enalty 3	90	***90***	*Done*	*Done*					
tatus	Done	*Done*	*Done*	*Done*					

NOTES

The allocations are now completed. The profit values are 90 x 70 + 90 x 100 + 110 x 30 + 130 x 100 + 0 x 5 = 31,600. If you look carefully we have taken the original profit values for computing the total profit.

If you are interested you may try the IBFS thro' N-W corner rule and L-C method.

We now have to proceed towards check for optimality.

Check for optimality.

As usual we need to check whether the number of allocations = m + n -1. We have 5 allocations. But m + n -1 = 6 (Since we have three rows (m) and four columns (n)). This is known as 'Degeneracy' in a transportation model. That is, if the number of allocations is less than m + n – 1, then it is known as a case of degeneracy. To resolve this we will have to allocate an infinitely small positive quantity known as 'e', (pronounced as 'epsilon') to an unallocated least cost cell. Looking into our allocation table we find that Mumbai to Lucknow is the cell with the least cost of 30. Therefore we allocate 'e' to this cell.

	Ranchi	Delhi	Lucknow	Kanpur	Supply
Mumbai	40x70	*40 x 100*	*30 x?*	*20 x 30*	200 – 30 – 100 - 70 = 0
Kolkota	*80*	*60*	*0 x 100*	*45*	*100*
Dummy	130x5	*130*	*130*	*130*	5 – 5 = 0
Demand	75 – 70 – 5 = 0	*100*	*100*	*30*	305

Now we have m + n -1 = 6 = Number of allocations. We can now proceed with the check for optimality using the U-V method. Trusting that you have understood the process in the previous problem, I urge you to try computing the values

u1 + v1 = 40

u1 + v2 = 40

u1 + v3 = 30

u1 + v4 = 20

u2 + v3 = 0

u3 + v1 = 130

If u1 = 0 then v1 = 40, v2 = 40, v3 = 30 and v4 = 20.

If v1 = 40 then u3 = 90

If v3 = 30 then u2 = - 30

Now we have the ui + vj matrix given below.

ui + vj matrix

X	X	X	X
10	10	X	-10
X	130	120	110

We now proceed to Cij – (ui + vj) matrix.

Cij – (ui + vj) matrix

X	X	X	X
70	50	X	55
X	0	10	20

We find that all values in the Cij – (ui + vj) matrix are non-negative. This shows that the solution is optimal.

To recollect a bit: This problem was a maximizing case, an unbalanced case and one with degeneracy. But you were lucky that the IBFS was found to be optimal so we need have to progress to MODI method.

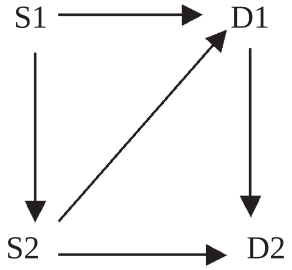
By now you have quite a grip on both minimizing and maximizing cases in transportation models. So let us now proceed on to transshipment models.

2.1.3 Transhipment Models

Trans-shipment models are cases where you have intermediate or transient nodes before the goods reach the final destination.

There are two possible cases.

1. *The sources and destinations act as transient nodes.* If you refer to drawing the given shown below, you will understand that S2 and D1 are the transient nodes.



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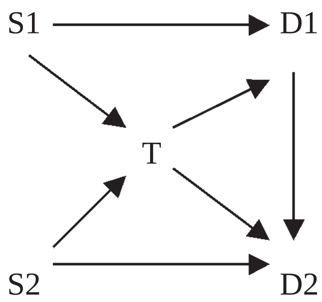
That is, instead of shipping from S1 and S2 directly to D1 and D2, shipping takes place from S1 to S2 and then from S2 to D1 and D2. Again some units are shipped from D1 to D2. So S2 and D1 are the transient nodes.

2. There are exclusive transient nodes present for trans-shipment besides the sources and destination. This is shown in the diagram below.



The diagram above shows that T,

When we have a case of trans-shipment then a value known as 'buffer stock' which is > = the aggregate of supply and demand is added to the values of supply or demand at the transient nodes.

The rules are as follows.

Supply at pure supply nodes = original supply.

Supply at transient nodes = original supply + buffer stock

Demand at pure demand nodes = original demand

Demand at transient nodes = original demand + buffer stock.

We shall now look at an example for a trans-shipment model.

Example 3: A firm with sources S1 and S2 wishes to ship its products to two destinations D1 and D2. The number of units available at S1and S2 are 5 units and 25 units respectively and the product demanded at D1 and D2 are 20 and 10 units respectively. The firm instead of shipping directly from the supply points to the destinations is considering the possibility of transshipment. The transportation cost in rupees is given in the table.

	S1	**S2**	**D1**	**D2**
S1	0	2	3	4
S2	2	0	2	4
D1	3	2	0	1
D2	4	4	1	0

Find the optimal shipping schedule.

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Solution: Looking at the cost values in the matrix it is evident that all the four nodes (two sources and two destinations are to be treated as transient nodes). Therefore let us include a buffer value >= the aggregate of supply and demand. The aggregate of supply and demand are 30 units. Therefore our buffer value can be 30 units or more. We'll choose 30 units as our buffer value. Let us prepare our combined table with supply and demand values. In this problem there are no pure supply or demand nodes.

	S1	S2	D1	D2	Supply
S1	0	2	3	4	5 + 30
S2	2	0	2	4	25 + 30
D1	3	2	0	1	0 + 30
D2	4	4	1	0	0 + 30
Demand	0 + 30	0 + 30	20 + 30	10 + 30	150

I guess you must have understood the way we have assigned the supply and demand values. 30 indicates the buffer value. There can be no supply from the destinations and similarly there can be no demand from the sources hence the value '0' is seen in such cells.

The problem can now be solved as an usual minimizing transportation case. As always I would suggest you obtain the IBFS through VAM.

I hope you solve thro' VAM and check whether the allocations shown below are the ones you have made.

	S1	S2	D1	D2	Supply
S1	0 x 30	2	3	4 x 5	35
S2	2	0 x 30	2 x 25	4	55
D1	3	2	0 x 25	1 x 5	30
D2	4	4	1	0 x 30	30
Demand	30	30	50	40	150

The cost works out to be, 0 x 30 + 4 x 5 + 0 x 30 + 2 x 25 + 0 x 25 + 1 x 5 + 0 x 30 = 75. We next have to progress to check for optimality. Once again I suggest you compute the ui-vj matrix and check with the result given below. m + n – 1 = 7 = number of allocations. So we don't have a case of degeneracy. The computed ui + vj matrix is given below.

ui + vj matrix

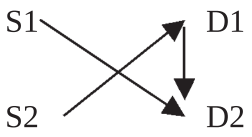
X	1	3	X
-1	X	X	3
-3	-2	X	X
-4	-3	-1	X

The next activity is to compute the Cij – (ui+vj) matrix. You can cross check with the values below.

Cij – (ui + vj) matrix

X	1	0	X
3	X	X	1
6	4	X	X
8	7	2	X

Since the Cij – (ui + vj) matrix contains all non-negative values, the IBFS thro' VAM is the optimal one. Therefore the allocations can be interpreted as below. The cells with '0' cost values are dummy in nature. So the only relevant allocations are S1 to D2 (5 units), S2 to D1 (25 units) and D1 to D2 (5 units). The allocations are shown below.



S1 sends 5 units to D2. S2 sends 25 units to D1. But D1 needs only 20 units so D1 sends 5 units to D2. So D2 gets 5 units from S1 and the remaining 5 units from D1. Therefore it is seen that D1 acts as the transient node.

This brings us to the end of the transportation models. I urge you to work out the exercise questions.

Exercises

Have you understood?

1. What is the need to use VAM?
2. What is degeneracy in a transporation model?
3. What is a transshipment model?
4. How is a maximizing problem in a transportation model solved?
5. How is optimality checked in transportation model.

NOTES

TRANSPORTATION MODELS

1. Solve the following Transportation Model.

	D_1	D_2	D_3	CARACITY
S_1	2	2	3	10
S_2	4	1	2	15
S_3	1	3	1	40
DEMAND	200	15	30	65

Ans:

S_1-D_1 =10,

S_2-D_1 =10,

$S_{2-}D_2$ =15,

S_3-D_2 =10,

S_3-D_3 = 30.

Total Cost=125

2. Solve the following Transportation problem

	D_1	D_2	D_3	D_4	SUPPLY
O_1	11	13	17	14	250
O_2	16	18	14	10	300
O_3	21	24	13	10	400
DEMAND	200	225	275	250	950

Ans:

O_1-D_1=200,

O_1-D_2=50,

O_2-D_2=175,

O_2-D_4=125,

O_3-D_3=275,

O_3-D_4=125,

Total Cost=12075.

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3. Solve the following Transportation Problem

	D_1	D_2	D_3	D_4	SUPPLY
O_1	6	1	9	3	70
O_2	11	5	2	8	55
O_3	10	12	4	7	70
DEMAND	85	35	50	45	

Ans:

O_1-D_1=40,

O_1-D_2=30,

O_2-D_2=5,

O_2-D_3=50,

O_3-D_1=25,

O_3-D_4=45, Total Cost=960.

4. Solve the following Transportation Problem

	D_1	D_2	D_3	SUPPLY
F_1	10	8	8	8
F_2	10	7	10	7
F_3	11	9	7	9
F_4	12	14	10	4
DEMAND	10	10	8	

Ans:

F_1-D_1=6,

F_1-D_2=2,

F_2-D_2=7,

F_3-D_2=1,

O_3-D_3=8,

F_4-D_1=4,

Total Cost=238

NOTES

5. Solve the following to Maximize Profit

	D_1	D_2	D_3	D_4	SUPPLY
O_1	40	25	22	33	100
O_2	44	35	30	30	30
O_3	38	38	28	30	70
DEMAND	40	20	60	30	

Ans:

O_1-D_1=20,

O_1-D_2=30,

O_1-D_3=50,

O_2-D_1=20,

O_2-D_3=10,

O_3-D_2=20,

O_3-D_3=50,

Total Cost=5130.

Summary

An overview and the methodology to solve transportation models have been explained for your perusal.

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UNIT III

INTEGER LINEAR PROGRAMMING AND GAME THEORY

INTRODUCTION

This unit will provide insight into integer programming and game theory models. Integer programming carries a lot of meaning when the computed optimal values are constrained to be integers, such as the number of men or women to be assigned to a job. Game theory is applicable in business situations where the competitors react to each other's strategies using their counter.

Learning Objectives

The learning objectives in this unit are

1. To arrive at optimal integer values using Branch & Bound Algorithm
2. To arrive at optimal integer values using Gomory's cutting plane Algorithm
3. To compute the strategies on the different game theory models using a variety of algorithms.

3.1 INTEGER PROGRAMMING

In a general linear programming problem all basic variables including the slack and surplus variables are permitted to take non- negative real values (non integer values).

For example, the values of the variables can be x1=4.2 or x2 = 5.1. But these values have little meaning if x1 and x2 represent the number of men and women to be assigned to a particular job. So it is funny to say that we must assign 4.2 men and 5.1 women on the job. In this context integer programming is meaningful. So we say x1=4 and x2= 5 thus assigning integer values to the variables. But we can't be sure that by just converting non-integers into integers we have retained optimality. So we have two algorithms, to facilitate integer programming, namely.

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1. Branch and Bound (B&B)
2. Gomory's cutting plane method (GCP).

We will use B&B in the graphical method and the GCP method in the simplex method in our text. By now, you are aware that the graphical method is used whenever we have two variables and the simplex method can be used whenever we have more than three variables.

The only distinguishing feature between a usual LPP and an integer programming problem (IPP) is the constraint as mentioned below.

Constraint: x1, x2 >= 0 **and are integers**

If in an integer programming problem all (or) some of the variables are permitted to take non negative integer values then the problem is called a pure integer programming problem. If only some of the variables are constrained to be integers then it is called a mixed integer programming problem.

Common approach to solve:

It is possible to ignore the integer restrictions and solve the problem as a regular LPP and non integer solution can be rounded off to the nearest integer value. However this can lead to deviation from optimality and may also violate the feasibility constraints. Therefore an integer programming problem can be solved either by

a) Branch and Bound algorithm (Graphical – 2 Variables)
b) Gomory's Cutting Plane algorithm (Computational 2 Variables and more)

Branch and Bound Algorithm:

In branch and bound algorithm the problem is first solved as an ordinary LPP and then the solution space is systematically partitioned into sub problems by deleting portions that contain no feasible solution.

Gomory's Cutting plane algorithm:

First solve the problem as an ordinary LPP and then introduce additional constraints (as fractional cuts) one after the other until an integer solution is obtained.

We shall first solve a problem by the B&B (Graphical) method.

3.1.1 Graphical Method

Example 3.1.1:

Max (Z) = 2x1 + 3x2

S.t: 6x1 + 5x2 <= 25

x1+ 3x2 <= 10

x1 , x2 > = 0 and are **non negative integers**

Solution

NOTES

Objective Equation: Max (Z) = 2x1 + 3x2

Just as we plot the graphical solution in the usual LPP we represent the constraint equations as straight lines in the graph using at-least two coordinates.

Constraints: 6x1 + 5x2 <= 25: Co-ordintes of the first constraint are (0,5) , (4.16,0)

x1+ 3x2 <= 10: Co-ordintes of the second constraint are (10,0) , (0,3.33)

The plotted non-integer solution is shown in figure 3.1 below.

As seen in figure 3.1, we have the non-integer Initial Basic Solution as : x1 = 1.9, x2 = 2.7 Z= 11.9. But this Initial Basic optimal solution is (1.9, 2.7) infeasible because x1 and x2 are positive non-integers. We need integer values for x1 and x2. So we need to adopt B&B method. The values of x1 =1.92. Therefore x1 will take only non integer values between 1 and 2. So an integer solution of x1 will lie before 1 or greater than 2. Hence we need to add two more constraints such as x1 <=1 and x1>=2. Figure 3.2 shows the graph with the two constraints namely x1<=1 and x1> = 2.

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LINEAR PROGRAMMING -- GRAPHICAL SOLUTION

Title: int-prg-1

Summary of Optimal Solution:
Objective Value = 11.92
x1 = 1.92
x2 = 2.69

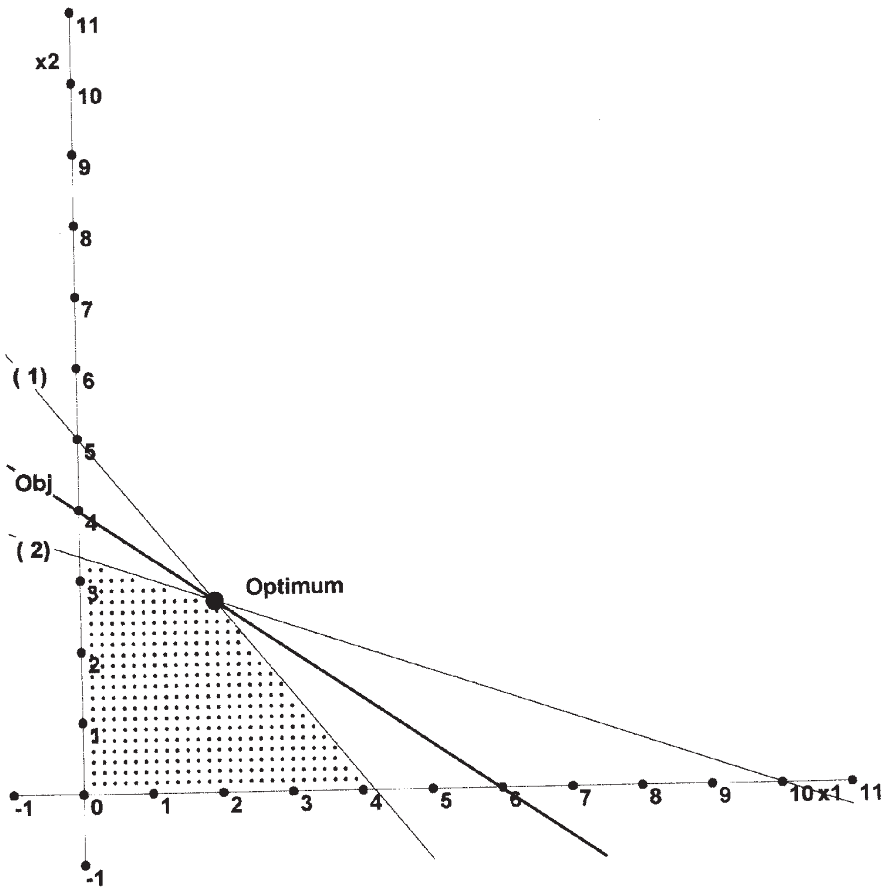

Figure 3.1

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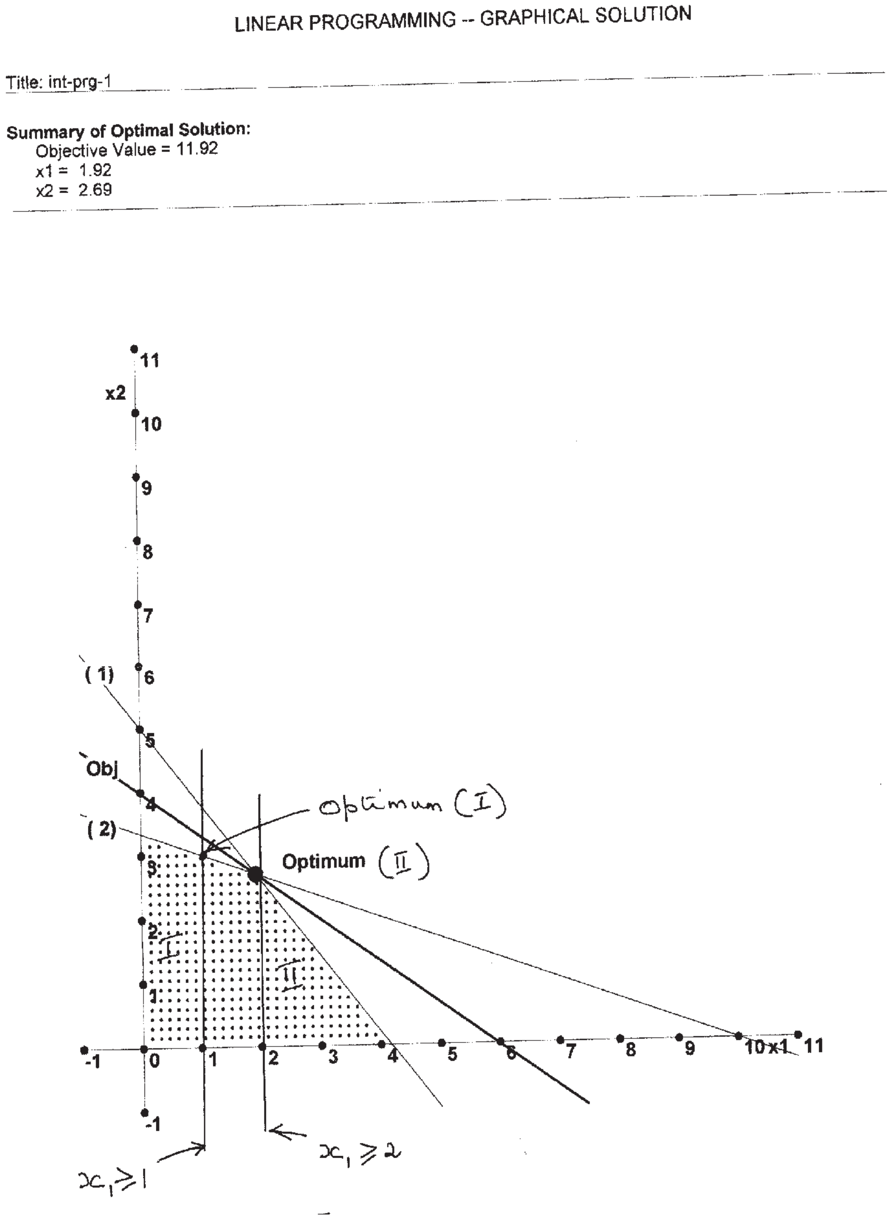

Figure 3.2

Figure 3.2 shows that the solution space is partitioned into I and II. The optimum point of solution space I is (1, 3) and that of solution space II is (2, 2.69). From solution space-I, we now have an integer solution as x1=1 and x2 =3 which gives us a Z value of 11. But we need to check other possibilities also to see if we can get a better solution. From solution space II we know that x1 = 2 but x2 = 2.69 which is still a non-integer value. So we add two more constraints x2 <=2 and x2 >=3 (Because we don’t have integer solutions between 2 and 3 for x2). This is shown in figure 3.3.

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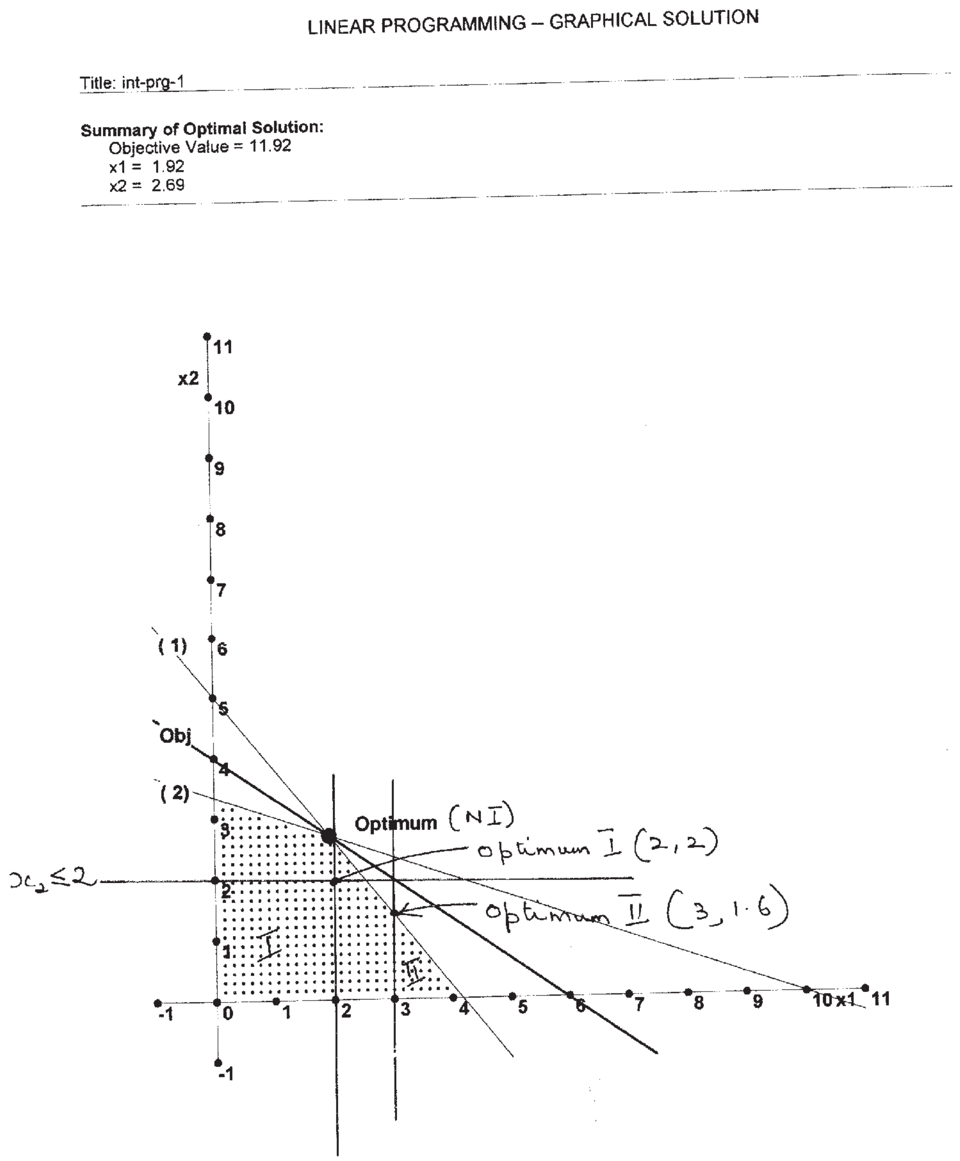

Figure 3.4

From figure 3.3 we see that there is no feasible solution space above x1 >=2 (Based on the previous stage) and x2 >=3. That is, there is no solution space for x2>=3 at x1 >=2. But for solution space II at x1 >=2 there is a solution space shown by II at x2 <=2. This yields us an optimum coordinate at (2.5, 2). So we again have a non integer solution at x1 = 2.5. So we add two more constraints at x1 <=2 and x1 >= 3 (Because we do not have integer solutions for x1 between 2 and 3). This is further shown in figure 3.4.

Here we find that there is a feasible solution at (2,2). But the value of Z=10. This is a

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feasible integer solution but not better than the one we had at x1=1 and x2 = 3 giving us a Z value of 11. The solution space II gives us a value of (3,1.6) .This means we once again have a non-integer value at x2 = 1.6. So we include two constraints namely x2<=1 and x2>=2. This is shown in figure 3.5.

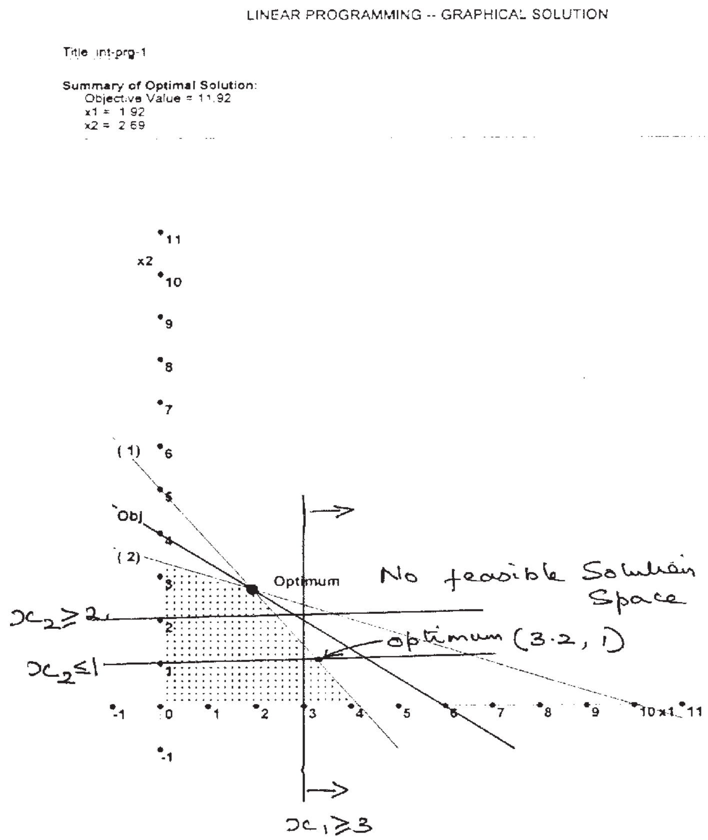

Figure 3.5

On examining figure 3.5 we see that there is no feasible solution space at x1>=3 and x2 <=2. But at x1 > = 3 and x1 < = 1 there is again a non-integer solution available at (3.2,1). So if we probe further we need to go at x1>=4 (where there is no feasible solution space) or x1<=3 (where we don't get a better solution than before). So we can conclude that x1=1, x2=3 yields the optimal integer solution Z = 11. A summary of our stages is shown below.

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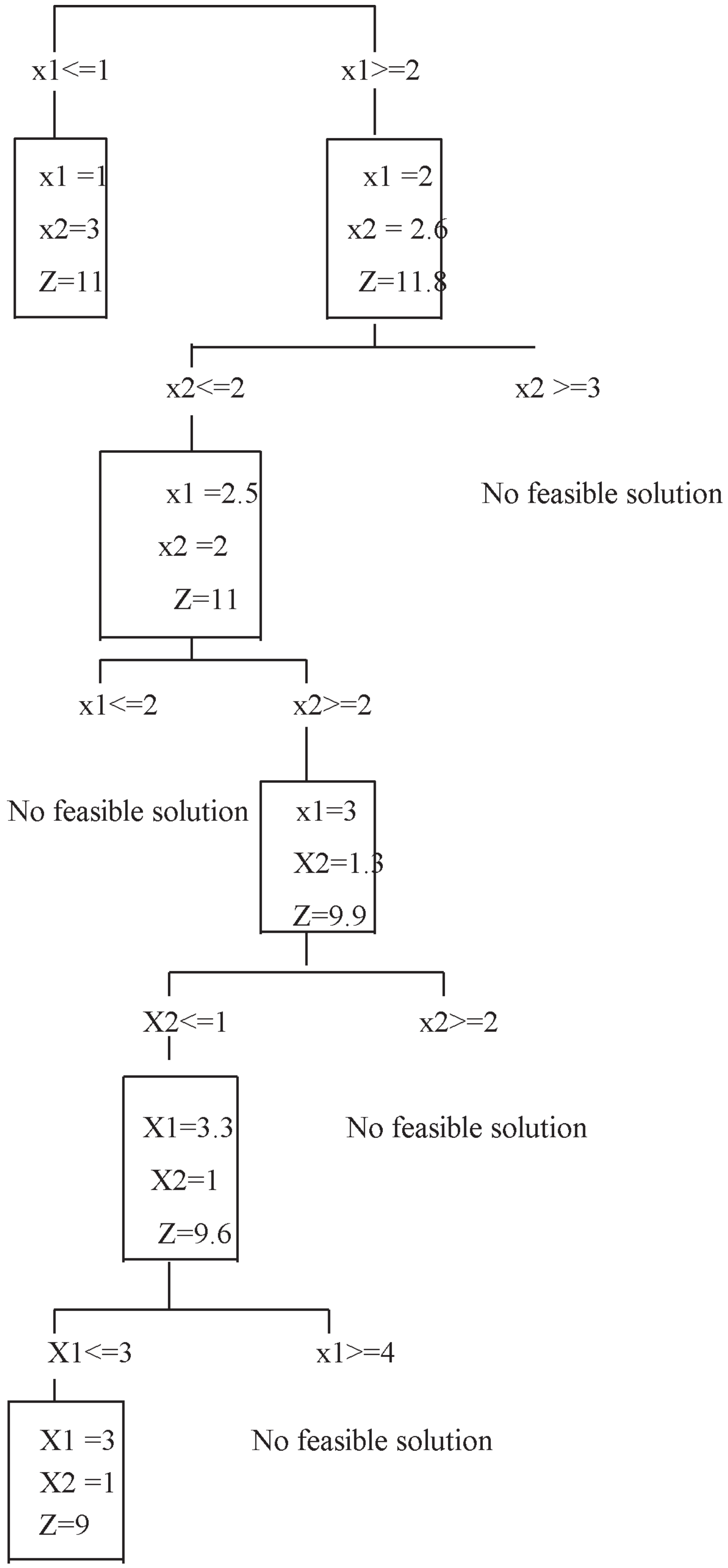

Optimal and Feasible Solution: x1 =1, x2 =3, Z= 11

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3.1.2 Gomory's Cutting Plane Method

We shall now solve a problem through the gomory's cutting plane method.

Example 3.1.2

Max (Z) = 7x1 + 10x2

S.t: -x1 + 3x2 <= 6

7x1 + 3x2 <= 6

x1 , x2 >0 and are integers

We shall solve this problem using the gomory's cutting plane method.

Since the constraint equations are all <= type we add slack variables to the constraints to convert them into equalities.

Adding slack variables:

-x1 + 3x2 +s1 = 6

7x1 + x2 + s2 = 35

The number of variables (n) = 4 and the number of equations (m) = 2. Therefore we have n – m = 2 (Non-basic variables) these are likely to be x1 and x2. This is shown below.

Var (n) = 4 CE (m) = 2 ; NBV = n-m= 2 (x1 ,x2)

If we substitute NBV =0 in constraint equations we have s1 = 6; and s2 =35. Remember these are all our old principles which we used while solving the simplex method. The objective equation is written as Z – 7x1 -10x2 – 0s1 – 0s2 =0. We now prepare the initial simplex table as given below. We go ahead and solve through the primal simplex method, using our known principles. If you have doubts in this table, I recommend that you do a quick review of the primal simplex method explained in the first unit.

NOTES

Table 3.1

Basics	x1	x2	s1	s2	Sol	Ratio	
Z	-7	**-10**	0	0	0		
s1	**-1**	**3**	**1**	**0**	**6**	**2**	
s2	7	**1**	0	1	35	35	
Z	**-31/3**	0	10/3	0	20		
x2	**-1/3**	1	1/3	0	2		
s2	**22/3**	**0**	**-1/3**	**1**	**33**		
Z	0	0	63/22	31/22	133/2		
x2	**0**	**1**	**7/22**	**1/22**	**7/2**	**Source**	**Row**
x1	1	0	-1/22	3/22	9/2		

Based on Table 3.1 we see that we have a non-integer optimal solution at x1=9/2, x2 = 7/2 which yields a value of Z = 133/2. But what we need is an integer solution for x1 and x2.

This is done by the gomory's cutting plane method that we shall discuss now. Since both x1 and x2 have non-integer values, one of them can be chosen as the source row. Generally the variable with a higher fractional value would be chosen first. But in this case, we find that both x1 and x2 have equal fractional values. That is, x1 = 3 + ½ and x2 = 4 + ½ . Since both variables have equal fractional values we choose one of them. In this case we choose x2 as our source row. Now this source row is taken from the non-integer optimal simplex table and written in the equation form. This is shown below.

x2 + 7/22 s1 + 1/22 s2 = 7/2 ——————————— 1

We have to split the coefficients of every variable and the constant into an integer and a <u>strictly positive fraction</u>. The variable x2 has only an integer but no fractional part. S1 has no integer part but only a fractional part which is positive with a value of 7/22. s2 also does not have an integer part but only a positive fractional part with a value of 1/22. The constant in the RHS is 7/2 which has both an integer and a fractional part. The integer part being 3 and the fractional part being ½. To make things simple if 7 is divided by 2 the answer is 3 ½. Therefore the integer part is 3 and the fractional part is ½.

If we split the coefficients then equation 1 is written as

(1+0)x2 + (0+7/22) s1 + (0+1/22 s2) = (3+1/2) ——————————— 2

Equation 2 has all positive fractions and hence we can proceed further.

NOTES

We now expand and bring all the fractional parts to the RHS and move all the integer parts to the LHS. The equation 2 would now change as given below.

x2 – 3 = - 7/22 s1 - 1/22 s2 +1/2 ———————— 3

We now just ignore the RHS of the equation which has the integer part and make the equation =0. The equation now becomes as shown below.

-7/22 s1 - 1/22 s2 +1/2 = 0———————— 4

We now move the constant fraction ½ to the RHS. The new equation is as shown below.

-7/22 s1 - 1/22 s2 = -1/2———————— 5

Since the source row was from a constraint equation with a <= sign, we treat this equation 5 also as one with <= sign. This is shown in the equation below.

-7/22 s1 - 1/22 s2 <= -1/2———————— 6

So we must add a slack variable s3 to the LHS to convert the inequality into an equality. The new equation 7 is known as the gomory's fractional cut.

-7/22 s1 - 1/22 s2 + s3 = -1/2———————— 7

This equation is now introduced to the existing non-integer optimal table. From this point forth we must interate towards an optimal integer solution.

The non-integer optimal solution table is shown below.

	x1	x2	**s1**	s2	s3	Sol	
Z	0	0	**63/22**	31/22	0	133/2	
x2	0	1	**7/22**	1/22	0	7/2	
x1	1	0	**-1/22**	3/22	0	9/2	

The table with the gomory's fractional cut is shown below.

Introducing Gomory's Fractional Cut							
	x1	x2	s1	s2	s3	Sol	
Z	0	0	63/22	31/22	0	133/2	
x2	0	1	7/22	1/22	0	7/2	
x1	1	0	-1/22	3/22	0	9/2	
s3	**0**	**0**	**-7/22**	**-1/22**	**1**	**-1/2**	

NOTES

We now find from the table above that it has one infeasible constraint introduced through the gomory's fractional cut.

We have already seen in Unit I that if the solution table is optimal but infeasible then we can solve using the dual simplex method. So you may have to give a quick review of the dual simplex method found in unit I and then proceed.

As per the dual simplex procedure, the row s3 is the leaving variable and s1 is the entering variable.

This is highlighted in the table below.

Introducing Gomory's Fractional Cut							
	x1	x2	**s1**	s2	s3	Sol	
Z	0	0	**63/22**	31/22	0	133/2	
x2	0	1	**7/22**	1/22	0	7/2	
x1	1	0	**-1/22**	3/22	0	9/2	
s3	**0**	**0**	**-7/22**	**-1/22**	**1**	**-1/2**	

After iterating through our usual procedure we the new table as shown below.

	x1	x2	**s1**	s2	s3	Sol	
Z	0	0	0	1	9	62	
x2	0	1	0	0	1	3	
x1	1	0	0	1/7	-1/7	32/7	
s1	0	0	1	1/7	-22/7	11/7	

We now find that x2 is now an integer value but x1 is still a fractional value. So we choose x1 to be the new source row. Just as we had done earlier, we need to write the source row equation as a combination of an integer and a strictly positive fraction. So we have the equation as

x1 + 1/7s2 – 1/7s3 =32/7

Splitting as an integer part and a fractional part, we have

(1+0)x1 + (0+1/7) s2 +(0 – 1/7) s3 = 4 + 4/7

Except s3 all other variables have a positive fraction. We have to now convert the negative fraction of s3 into a positive fraction. This is shown in the equation below.

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(1+0)x1 + (0+1/7) s2 +(-1 + 6/7) s3 = 4 + 4/7

We have introduced a negative integer and converted the negative fraction into a positive fraction. Just as we had done before we shall expand the equation.

x1 + 1/7s2 -1s3 + 6/7s3 = 4 + 4/7

We take all the fractions to the RHS and bring all the integers to the LHS.

x1 – 4 –s3 = -1/7s2 – 6/7s3 +4/7

We remove the integer parts and equate to 0

-1/7s2 – 6/7s3 +4/7 = 0

We now move the constant fraction to the RHS.

-1/7s2 – 6/7s3 = -4/7

Since the original constraint equation is <=, we introduce the same in the equation.

-1/7s2 – 6/7s3 <= -4/7

We now introduce a slack variable s4 to make it into an equality.

-1/7s2 – 6/7s3 + s4 = -4/7

This is the second fractional cut which we will introduce in the existing optimal table to bring in infeasibility. This is shown in the table below.

Basis	x1	x2	s1	**s2**	s3	s4	Sol
Z	0	0	0	**1**	9	0	62
x2	0	1	0	**0**	1	0	3
x1	1	0	0	**1/7**	-1/7	0	32/7
s1	0	0	1	**1/7**	-22/7	0	11/7
s4	**0**	**0**	**0**	**-1/7**	**-6/7**	**1**	**-4/7**

Using our usual dual simplex method we solve and we have the following table.

Basis	x1	x2	s1	**s2**	s3	s4	Sol
Z	0	0	0	0	3	7	**58**
x2	0	1	0	0	1	0	**3**
x1	1	0	0	0	-1	1	**4**
s1	0	0	1	0	-4	1	1
s2	0	0	0	1	6	-7	4

NOTES

We now find that both x1 and x2 are now positive integer values with the value of Z = 58.

So we now have an optimal and feasible solution.

A summary table of our computations is shown below.

Introducing Gomory's Fractional Cut							
	x1	x2	**s1**	s2	s3	Sol	
Z	0	0	**63/22**	31/22	0	133/2	
x2	0	1	**7/22**	1/22	0	7/2	
x1	1	0	**-1/22**	3/22	0	9/2	
s3	**0**	**0**	**-7/22**	**-1/22**	**1**	**-1/2**	
Z	0	0	0	1	9	62	
x2	0	1	0	0	1	3	
x1	**1**	**0**	**0**	**1/7**	**-1/7**	**32/7**	**Source**
s1	0	0	1	1/7	-22/7	11/7	
Basics	x1	x2	s1	**s2**	s3	s4	Sol
Z	0	0	0	**1**	9	0	62
x2	0	1	0	**0**	1	0	3
x1	1	0	0	**1/7**	-1/7	0	32/7
s1	0	0	1	**1/7**	-22/7	0	11/7
s4	**0**	**0**	**0**	**-1/7**	**-6/7**	**1**	**-4/7**
Z	0	0	0	0	3	7	**58**
x2	0	1	0	0	1	0	**3**
x1	1	0	0	0	-1	1	**4**
s1	0	0	1	0	-4	1	1
s2	0	0	0	1	6	-7	4

With this we have come to the end of integer programming. Please try out the exercises given in the end of the unit to improve your problem solving skills.

NOTES

3.2 GAME THEORY

A competitive situation in business can be treated similar to a **game**. There are two or more players and each player uses a strategy to out play the opponent.

A strategy is an action plan adopted by a player in-order to counter the other player. In our of game theory we have two players namely Player A and Player B. The basic assumption would be that

Player A – plays to **Maximize profit** (offensive) - Maxi (min) criteria

Player B – plays to **Minimize losses** (defensive) - Mini (max) criteria

The Maxi (Min) criteria is that – Maximum profit out of minimum possibilities

The Mini (max) criteria is that – Minimze losses out of maximum possibilities.

Game theory helps in finding out the best course of action for a firm in view of the anticipated counter-moves from the competing organizations. A competitive situation is a competitive game if the following properties hold good:

1. The number of competitors is finite, say N.
2. A finite set of possible courses of action is available to each of the N competitors.
3. A play of the game results when each competitor selects a course of action from the set of courses available to him. In game theory we make an important assumption that all the players select their courses of action simultaneously. As a result, no competitor will be in a position to know the choices of his competitors.
4. The outcome of a play consists of the particular courses of action chosen by the individual players. Each outcome leads to a set of payments, one to each player, which may be either positive, or negative, or zero.

Managerial Applications of the Theory of Games

The techniques of game theory can be effectively applied to various managerial problems as detailed below:

1. Analysis of the market strategies of a business organization in the long run.
2. Evaluation of the responses of the consumers to a new product.
3. Resolving the conflict between two groups in a business organization.
4. Decision making on the techniques to increase market share.
5. Material procurement process.
6. Decision making for transportation problem.
7. Evaluation of the distribution system.
8. Evaluation of the location of the facilities.
9. Examination of new business ventures and
10. Competitive economic environment.

NOTES

Some terminologies that might be helpful to you.

If the Maxi (min) of A = Mini (max) of B then it is known as the **Saddle Point**→ This is also the **Value of the game**

We call it a **Zero Sum game** because the Gain of A – Loss of B = 0. In other words, the gain of Player A is the Loss of Player B.

Game of Pure strategies – If both players play only one strategy through out the game then it is known as a game of Pure Strategies.

Game of Mixed Strategies – One or both players use combination of strategies

In all problems relating to game theory, first look for saddle point, then check out for rule of dominance and see if you can reduce the matrix.

We will first solve a simple game with a saddle point.

3.2.1 Solution through Saddle Point

Example 3.2.1.

1) **Solve the following game.**

		Player B				
		I	II	III	IV	V
	1	16	4	0	4	-2
Player	2	10	8	6	10	12
A	3	2	6	4	8	14
	4	8	10	2	2	0

Solution

The given matrix shows the pay-off values of the players. That is, if Player A plays strategy 1 and Player B plays strategy 1 then, Player A gains 16. If Player A plays strategy 1 and Player B plays strategy 5 than A looses 2. The other values represent the payoffs for the respective strategies.

Player A plays maxi(min) criteria. That is he wants to maximize his profits even if B plays its best strategies. For example, if you look at Strategy 1 of A; A will make a profit of 16 against strategy 1 of B, 4 against Strategy 2 of B, 0 against Strategy 3 of B, 4 against Strategy 4 of B and -2 against Strategy 5 of B. A value of -2 means that Player A will loose 2 and Player B will gain 2. In all other cases Player A gains. So among all possibilities A feels he will loose 2 against the best Strategy of B, if Player A plays strategy 1. So we write

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the value of -2 in the last column. Similarly we write the minimum payoff for the other rows. This is shown in the matrix below.

	Player B						
		I	II	III	IV	V	**Maxi(min)**
Player A	1	16	4	0	4	-2	-2
	2	10	8	6	10	12	**6**
	3	2	6	4	8	14	2
	4	8	10	2	2	0	0

Similarly Player B plays the Mini(max) criterion that is to minimize losses. That is, for every strategy Player B would like to know the losses for the best strategy of Player A. So we see that for Strategy 1 of Player B the best strategy of A is 1 which yields a payoff of 16 (This is the maximum loss for Player B for its Strategy 1). Similarly for Strategy 2 of Player B the maximum value from A is 10, for Strategy 3 of Player B the maximum value from A is 6, for Strategy 4 of Player B the maximum value from A is 10, and for Strategy 5 of Player B the maximum value from A is 14. These values are shown in the table below.

	Player B						
		I	II	III	IV	V	**Maxi(min)**
Player A	1	16	4	0	4	-2	-2
	2	10	8	6	10	12	**6**
	3	2	6	4	8	14	2
	4	8	10	2	2	0	0
	Mini(max)	16	10	**6**	10	14	

What we have done so far is find the minimum possible gains of Player A (Maximin column) and the maximum possible losses of Player B (Minimax row). So now out of the minimum possibilities, Player A would like to maximize the profits. That is out of -2, 6, 2, and 0 the highest value is 6. This means that if Player A plays strategy 2 it will gain 6 even if Player B plays its best strategy. So strategy 2 is the best for Player A to maximize profits under most difficult condition. So the Maximin value of Player A is 6. Similarly for Player B Strategy 3 is the best option since it will make a minimum loss of 6 against all the possible strategies of Player A under adverse conditions. So the Minimax value of Player B is 6. Since the Maximin value of Player A = The Minimax value of Player B we have a saddle point. This is also the value of the game. And this is a zero sum game, since the gain of A = 6 = the loss of B. Therefore 6 – 6 = 0 (zero sum game). This is highlighted in the table below.

NOTES

	Player B						
		I	II	III	IV	V	**Maxi(min)**
	1	16	4	0	4	-2	-2
Player	2	10	8	$\boxed{\mathbf{6}}$	10	12	**6**
A	3	2	6	4	8	14	2
	4	8	10	2	2	0	0
	Mini(max)	16	10	**6**	10	14	

Saddle Point: Maxi (min) of A = Mini (max) of B = Value of Game =6

Game of Pure strategies

A's Strategy (0,100%, 0, 0)

B's Strategy (0, 0,100%, 0, 0)

3.2.2 Solution Using Rule of Dominance

We shall now try to solve a problem using dominance rule. The rule of dominance is as stated below.

Rule of Dominance:

If every value of one strategy of A is lesser than that of the other strategy of A,

Then A will play the strategy with greater values and remove the strategy with the lesser payoff values.

If every value of one strategy of B is greater than that of other strategy of B , B will play the lesser value strategy and remove the strategy with higher payoff values..

Example 3.2

Solve the following game

	Player B			
		1	2	3
Player A	I	0	-2	7
	II	2	5	6
	III	3	-3	8

NOTES

Solution

As always we must look for saddle point. The Maximin and Minimax values are shown in the matrix below.

	Player B				
		1	2	3	Maxi(min)
Player A	I	0	-2	7	-2
	II	2	5	6	**2**
	III	3	-3	8	-3
	Mini(max)	**3**	5	8	

The maximin value of Player A is 2 and the minimax value of Player B is 3 . So there is no saddle point. But we now have a clue that the value of the game lies between 2 & 3. The value of the game will always lie between the maximin and the minimax values of the players.

Since there is no saddle point we now proceed towards reducing the matrix by rule of dominance. Let us examine the strategies of Player A. At present no strategy of A has all its values less than another strategy of A.

Let us examine the strategies of Player B. Clearly, the third strategy of B has all values higher than that of strategies 1 and 2. Even if the values of strategy 3 of B are greater than that of one of the strategies of B, then it can be removed. The logic behind this is that player B will avoid playing strategy 3 but will play strategy 1 or 2 instead. So we remove strategy 3 from Player B. The resulting matrix is shown below.

		Player B	
		1	2
Player A	II	2	5
	III	3	-3

If we now observe, we see that the first strategy of Player A has all its values lesser than the second strategy of Player A. Therefore Player A will not play strategy one but will play strategy 2. So we remove the first strategy for player A. The resulting matrix is shown below.

NOTES

		Player B	
		1	2
Player A	II	2	5
	III	3	-3

So Player B plays only strategies 1 and 2 and Player A plays only strategy 2 and 3. We must now solve the reduced 2 x 2 matrix using the method of Oddment.

3.2.3 Solution through Method of oddment.

In the method of oddment, find the difference between the payoffs of the second strategy of Player A but write it against the third strategy. For example, the difference between the payoff values of the second strategy is 5 – 2 = 3. This value is written for the third strategy of Player A. Similarly the difference between the payoffs of the third strategy is 3 – (-3) = 6. This value is written against that of the second strategy. A similar activity is done for the strategies of Player B. The values of the oddments are shown in the matrix below.

Method of oddment:

	1	2	
II	2	5	6
III	3	-3	3
	8	1	9

We have to carefully note that the row total = 6 + 3 = 9 = 8 + 1. From this we have the strategies of Player A and Player B given below.

A's Strategy (0, 2/3, 1/3)

The interpretation is that Player A does not play strategy 1 but plays strategy 2 for 2/3 of the time and plays strategy 3 for 1/3 of the time.

B's strategy (8/9, 1/9, 0)

Similarly Player B plays strategy 1 for 8/9 of the time, strategy 2 for 1/9 of the time but does not play strategy 3.

The value of the game can be calculated by four possibilities. The payoff values are multiplied by the corresponding proportion of time the strategies are played.

Value = 2*2/3 + 1/3*3 = 7/3

= 5*2/3 – 3*1/3= 7/3

= 2*8/9 + 5*1/9 = 7/3

= 3*8/9 – 3*1/9 = 7/3

Please note that by any possibility the value of the game must be the same.

3.2.4 Solution by Graphical Method

If one of the players, play only two strategies or if the game can be reduced such that one of the players play only two strategies. Then the game can be solved by the graphical method.

Example 3.3: Solve the following game

	Player B				
		y1	y2	y3	y4
Player A	x1	19	6	7	5
	x2	7	3	4	6
	x3	12	8	18	4
	x4	8	7	13	-1

Solution

We look for saddle point as usual in the given data. This is shown in the table below.

	Player B		
		y2	y4
	x1	6	5
Player	x2	3	6
A	x3	8	4
	x4	7	-1

There is no saddle point however we know that the value of the game lies between 5 and 6.

NOTES

We now try to reduce the matrix by using rule of dominance. You can see that all values of Strategy y1 of Player B is greater than the corresponding values of Strategy y2. Similarly all values of Strategy y3 are greater than the corresponding values of y2. Therefore strategies y1 and y3 of B are removed. This is shown in the matrix below.

	Player B		
		y2	y4
	x1	6	5
Player	x2	3	6
A	x3	8	4
	x4	7	-1

Any one player not playing more than two strategies graphical method can be used to solve the game.

Now observing the strategies of A we see that the values of strategy x4 has all its values lesser than the strategy x3. Therefore the strategy x4 is removed from Player A. This is shown in the matrix below.

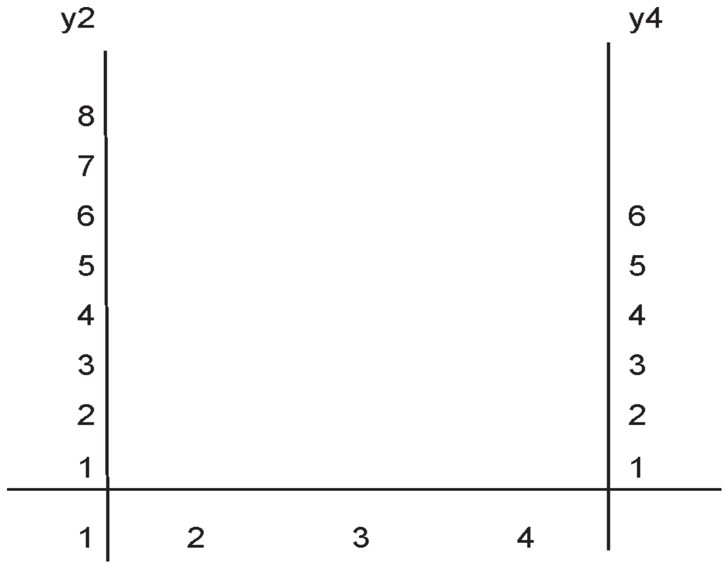
	Player B		
		y2	y4
	x1	6	5
Player	x2	3	6
A	x3	8	4

Now that we have only two strategies from B's perspective and three strategies from A' perspective, we can solve the game from this stage either through sub-games or through graph.

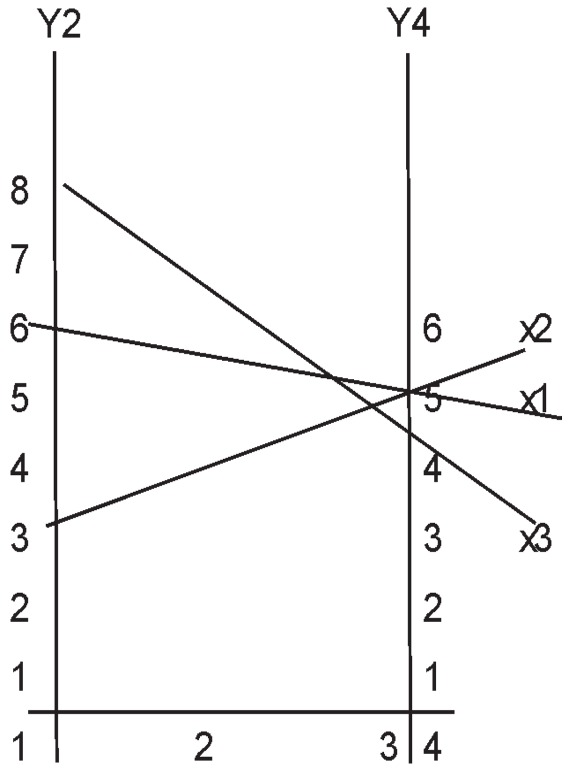
We will first solve through the graphical method.

To plot the graph, space the two strategies y2 and y4 in the graph sheet 4 units apart as shown below. Then in each on the vertical lines mark the range of y2 (max 8) and y4 (max 5).

NOTES

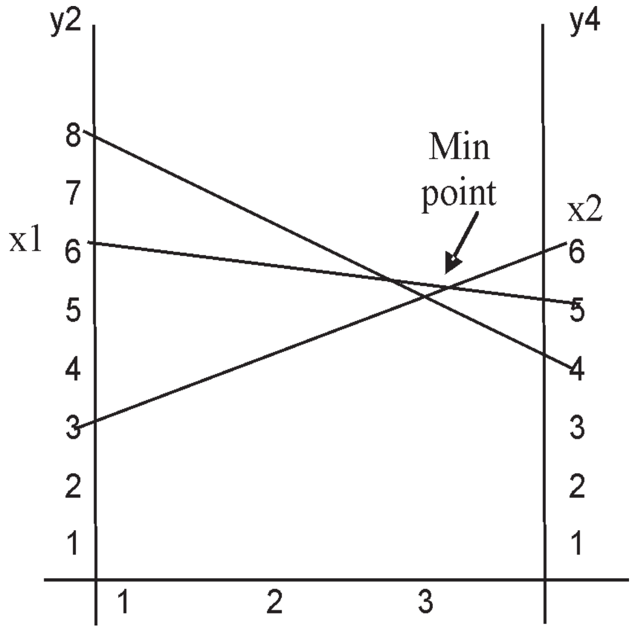


Now we can plot the three strategies of Player A. x1 is with values of 6 and 5 for y2 and y4 respectively. x2 is with 3 and 6 and x3 with 8 and 4. This is shown in the graph below.



NOTES

Since the game is from B's perspective we have to choose the space above the lines as the maxmising space. In this maximizing space we must locate the minimum point (since B plays Maximin criteria). If Player A was playing two strategies, then we would have chosen the space below the lines and would have chosen the maximum point (since Player A plays Minimax criteria). The chosen space and the minimum point (shown by the arrow) is shown in the graph below.



By this we know that strategy x3 of player A is eliminated. Thus we have strategy x1 and x2 of Player A and y2 and y4 of Player B.

So using this we can compute the proportion of strategies and the value of the game as shown below.

SA =	6	5	3
	3	6	1
1	3	4	

Value = 6*3/4 + 3*1/4 = 21/4

A's strategy (¾, ¼, 0, 0)

B's strategy (0, 1/4, 0, 3/4)

We shall now solve this by sub-games. Oddments of all combinations are done.

NOTES

$$SA = \begin{array}{cc|c} 6 & 5 & 3 \\ 3 & 6 & 1 \\ \hline 1 & 3 & 4 \end{array}$$

$$SB = \begin{array}{cc|c} 6 & 5 & 4 \\ 8 & 4 & 1 \\ \hline 1 & 2 & \end{array}$$

Sub-game SB is not feasible since the row totals and column totals are not equal.

The values of the sub-games are as given below.

$$SC = \begin{array}{cc|c} 3 & 6 & 4 \\ 8 & 4 & 3 \\ \hline 2 & 5 & 7 \end{array}$$

SA= 21/4

SB= Infeasible

SC = 36/7

Choose the game with the minimum pay off, and the value of the game = 21/4

Value = 6*3/4 + 3*1/4 = 21/4

A's strategy (¾, ¼, 0, 0)

B's strategy (0, 1/4, 0, 3/4)

Choose the game with the minimum pay off, and the value of the game = 21/4

Example 3.4: Solve the following game by graphical method.

	Player B					
		y1	y2	y3	y4	Maxi(min)
Player	x1	3	3	4	0	0
A	x2	5	4	3	7	3
	Mini(max)	5	4	4	7	

Try to draw the graph on your own and see which strategy gets eliminated. You will see that in graph y1 strategy gets eliminated

NOTES

Using subgames method to find the value of the game

	y2	y3	
x1	3	4	1
x2	4	3	1
	1	1	2

	y3	y4	
x1	4	0	4
x2	3	7	4
	7	1	8

	y2	y4	
x1	3	0	3
x2	4	7	3
	7	1	Infeasible

Value = 3/2 + 4/2 = 7/2

A's strategy (1/2, 1/2)

B's strategy (0, ½, ½, 0) or (0, 0, 7/8, 1/8)

3.2.5 Solution by method of Matrices

Example 3.5: Solve the game by method of matrices.

Player B

		1	2	3
Player A	1	7	1	7
	2	9	-1	1
	3	5	7	6

Solution:

NOTES

We shall solve this problem using the method of matrices. Subtract every column from the column to its left. That is c1 – c2 and c2 – c3. Similarly subtract every row from the row above. That is r1 – r2 and r2 – r3. The resulting matrix is shown below.

	Player B					
		1	2	3	c1 - c2	c2 - c3
Player A	1	7	1	7	6	-6
	2	9	-1	1	10	-2
	3	5	7	6	-2	1
	r1 - r2	-2	2	6		
	r2 - r3	4	-8	-5		

The oddments are prepared and the values are calculated based on the principle of determinants.

$$\text{Oddment: } A1 = \begin{vmatrix} 10 & -2 \\ -2 & 1 \end{vmatrix}$$

$$= 10 - 4 = 6$$

$$A2 = \begin{vmatrix} 6 & -6 \\ -2 & 1 \end{vmatrix} = -(6 - 12)$$

$$A3 = \begin{vmatrix} 6 & -6 \\ -2 & 1 \end{vmatrix} = -12 + 60 = 48$$

$$B1 = \begin{vmatrix} 2 & 6 \\ -8 & -5 \end{vmatrix} = -10 + 48 = 38$$

$$B2 = \begin{vmatrix} -2 & 6 \\ 4 & -5 \end{vmatrix} = -(10 - 24) = 14$$

$$B3 = \begin{vmatrix} -2 & 2 \\ 4 & -8 \end{vmatrix} = 16 - 8 = 8$$

NOTES

Based on the values calculated by the principle of determinants, the following matrix is prepared.

		Player B			
		1	2	3	
Player A	1	7	1	7	6
	2	9	-1	1	6
	3	5	7	6	48
		38	14	8	60

The Strategies of Player A are (1/10, 1/10, 8/10)

The Strategies of Player B are (19/30, 7/30, 4/30) and the

Value of the game = 7*6/60 + 9*6/60 + 5*48/60 = 56/10 = 5.6

3.2.6 Solution by using convex linear combinations

Example 3.6: Solve the game by using convex linear combinations

		Player B					
		1	2	3	4	5	6
Player A	1	4	2	0	2	1	1
	2	4	3	1	3	2	2
	3	4	3	7	-5	1	2
	4	4	3	4	-1	2	2
	5	4	3	3	2	2	2

Solution:

Although the name convex linear combinations sounds a little complex it nothing but a simple modification of the method of matrices.

As usual we look for saddle point and find none. But we know that the value of the game will lie between 1 and 2. This is shown in the matrix below.

NOTES

Player B

		1	2	3	4	5	6	Maxi(min)
Player A	1	4	2	0	2	1	1	0
	2	4	3	1	3	2	2	1
	3	4	3	7	-5	1	2	-5
	4	4	3	4	-1	2	2	-1
	5	4	3	3	-2	2	2	-2
	Mini(max)	4	3	7	3	2	2	

Now we try to apply the rule of dominance to reduce the matrix. We can remove strategy 1 of Player B since all its values are greater than that of strategy 2. This is shown in the matrix below.

Player B

		2	3	4	5	6
Player A	1	2	0	2	1	1
	2	3	1	3	2	2
	3	3	7	-5	1	2
	4	3	4	-1	2	2
	5	3	3	-2	2	2

Similarly we remove strategy 2 of Player B in comparison with strategy 4. This is shown in the table below.

Player B

		3	4	5	6
Player A	1	0	2	1	1
	2	1	3	2	2
	3	7	-5	1	2
	4	4	-1	2	2
	5	3	-2	2	2

NOTES

We can now remove strategy 6 of Player B in-comparison with strategy 4. This is shown in the table below.

		Player B		
		3	4	5
Player A	1	0	2	1
	2	1	3	2
	3	7	-5	1
	4	4	-1	2
	5	3	-2	2

We cannot remove any more strategies of Player B by using the rule of dominance.

Now on observing the strategies of Player A we can see that strategy 1 can be removed in-comparison with strategy 2. This is shown in the table below.

		Player B		
		3	4	5
Player A	2	1	3	2
	3	7	-5	1
	4	4	-1	2
	5	3	-2	2

We now find we cannot apply the rule of dominance any further unless we apply the rule of convex linear combinations. Now have a look at strategy 3 and 4 of Player B. If we take the column averages they work out to be 2, 1, 1.5 and 0.5. This is shown in the Table below.

NOTES

Player B

		3	4	5	Average of 3 & 4
Player A	2	1	3	2	2
	3	7	-5	1	1
	4	4	-1	2	1.5
	5	3	-2	2	0.5

We find that the average of the strategies 3 & 4 of B are lesser than or equal to the values of strategy 5. Therefore Player B will avoid playing strategy 5 in comparison with strategy 3 & 4. So we remove the strategy 5 from Player B. This is shown in the Table below.

Player B

		3	4	5	Average of 3 & 4
Player A	2	1	3	2	2
	3	7	-5	1	1
	4	4	-1	2	1.5
	5	3	-2	2	0.5

Now you can remove strategy 5 of Player A in-comparison to strategy 4 using our regular rule of dominance. This is shown in the Table below.

NOTES

	Player B			
		3	4	Average of 3 & 4
Player A	2	1	3	2
	3	7	-5	1
	4	4	-1	1.5
				0.5

Now for player A the averages of strategies 2 and 3 dominate strategy 4. These averages are shown in the Table below.

	Player B			
		3	4	Average of 3 & 4
Player A	2	1	3	2
	3	7	-5	1
	4	4	-1	1.5
				0.5
	Averages of 2 & 3	4	1	

Therefore we shall remove strategy 4 from Player A and this is shown in the Table below.

NOTES

Player B

		3	4	Average of 3 & 4
Player A	2	1	3	2
	3	7	-5	1
				1.5
				0.5
	Averages of 2 & 3	4	1	

Thus we have only strategies 2 and 3 of Player A and strategies 3 and 4 of Player B. We can now easily come to a solution using the oddment method as shown below.

Method of oddment:

	y3	y4	
x2	1	3	12
x3	7	-5	2
	8	6	14

Value = 6/7 + 7/7 = 13/7

A's strategy (0, 6/7, 1/7, 0, 0)

B's strategy (0, 0, 4/7, 3/7, 0)

3.2.7 Solution by LPP

Example 3.7: Solve the game by LPP method.

Last but not the least method, we shall solve the game using linear programming. As usual we look for saddle point and find none. But we know that the value of the game lies between -2 and 3.

NOTES

		1	2	3	Maxi(min)
Player A	1	3	-4	2	-4
	2	1	-3	-7	-7
	3	-2	4	7	**-2**
	Mini(max)	**3**	4	7	

Player B (columns)

Let x1, x2, and x3 be the probabilities with which player A chooses strategies. Let y1, y2, y3 be the probabilities with which player B chooses strategies.

Add a positive quantity greater than the maximum possible value of game (i.e. 3).

Assume k=3 add to every element.

	1	2	3
1	6	-1	5
2	4	0	-4
3	1	7	10

$6y1 - y2 + 5y3 <= V$

$4y1 - 4y3 \quad <= V$

$y1 + 7y2 + 10y3 <= V$

$y1 + y2 + y3 = 1$

Dividing all equations by V and substituting the following:

yj/V = Yj we get the following

$6Y1 - Y2 + 5Y3 <= 1$

$4Y1 - 4Y3 \quad <= 1$

$Y1 + 7Y2 + 10Y3 <= 1$

$Y1, Y2, Y3 >= 0$

$Max (1/V) = Y1+Y2+Y3 = Max (Z)$

Adding slack variables:

NOTES

Z-Y1-Y2-Y3-0s1-0s2-0s3 = 0

6Y1 – Y2 +5Y3 +s1 = 1

4Y1 – 4Y3 + s2 = 1

Y1 + 7Y2 + 10Y3 +s3 = 1

Now we solve by our known primal simplex method.

The consolidated table is shown below.

Basics	Y1	Y2	Y3	s1	s2	s3	sol	Ratio
Z	**-1**	-1	-1	0	0	0	0	
s1	**6**	**-1**	**5**	**1**	**0**	**0**	**1**	**1/6**
s2	**4**	0	-4	0	1	0	1	¼
s3	**1**	7	10	0	0	1	1	1
Z	0	**-7/6**	-1/6	1/6	0	0	1/6	
Y1	1	**-1/6**	5/6	1/6	0	0	1/6	
s2	0	**2/3**	-22/3	-2/3	1	0	1/3	½
s3	**0**	**43/6**	**55/6**	**-1/6**	**0**	**1**	**5/6**	**5/43**
Z	0	0	57/43	**6/43**	**0**	**7/43**	13/43	
Y1	1	0	45/43	7/43	0	1/43	8/43	
s2	0	0	-352/43	- 28/43	1	-4/43	11/43	
Y2	0	1	55/43	-1/43	0	6/43	5/43	

1/V = 13/43 ➔ V= 43/13

Y1 = y1/V ➔ y1 = Y1.V = 43*8 / 13*43 = 8/13

y2 = y2.V = 43*5 / 13*43 = 5/13

x1 = X1. V = 43*6 / 13*43 = 6/13

x2 = X2.V = 43*7 / 13*43 = 7/13

We subtract the value of k=3 from the computed value of the game.

Value = 43/13 – 3 = 4/13

NOTES

Value of the game = 4/13

A's strategy (8/13 , 5/13 , 0)

B's strategy (6/13 , 0 , 7/13)

Have you understood?

1. What is the significance of integer programming?
2. When can branch and bound algorithm be applied?
3. What is a zero sum game?
4. When can the dominance rule be applied?
5. Is a solution through LP a feasible one in game theory?

Exercises in B& B Algorithm.

1. Min (Z) = 20x1 + 10x2

S.t: x1 + 2x2 <= 40 (40, 0), (0, 20)

3x1 + x2 >= 30 (10, 0), (0, 30)

4x1 + x2 >= 60 (15, 0), (0, 60)

x1 , x2 >=0 and non negative integers

Iso Profit line: 20x1 + 10x2 = 200 (10, 0), (0, 20)

Optimal and Feasible Solution: x1 =15, x2 =0, Z= 300

2. Min (Z) = 9x1 + 10x2

S.t: x1 <= 9

X2 <= 8

4x1 + 3x2 >= 4 (1,0) , (0,1.33)

x1 , x2 >=0 and are integers

Optimal and feasible solution: x1 =1, x2 =0, Z=9

Acknowledgements

The author wishes to acknowledge that the solutions to the graphical methods in integer programming were drawn using TORA – a package which is given along-with the first book in reference (Hamdy A. Taha)

Exercises for Gomory’s Cutting Plane method.

NOTES

1. MAX (Z) =$4X_1+3X_2$

Sub To

X_1+2X_2d”4

$2X_1+X_2$d”6

X_1, X_2e”0 and Integers

Ans: X_1=3, X_2=0, Z=12

2. MAX (Z) =$3X_1+4X_2$

Sub To

$3X_1+2X_2$d”8

X_1+4X_2d”10

X_1, X_2e”0 and Integers

Ans: X_1=0, X_2=4, Z=16

3. MIN (Z) =$7X_1+6X_2$

Sub To

$2X_1+3X_2$d”12

$6X_1+5X_2$d”30

X_1, X_2e”0 and Integers

Ans: X_1=5, X_2=0, Z=35

4. MAX (Z) =X_1+4X_2

Sub To

$2X_1+4X_2$d”7

$5X_1+3X_2$d”15

X_1, X_2e”0 and Integers

Ans: X_1=1, X_2=1, Z=5.

EXERCISES ON GAME THEORY

1. Solve the following Game

PLAYER B

1 3 11

PLAYER A 8 5 2

Ans:

Strategy of A: 3/11, 8/11

Strategy of B: 0, 9/11, 2/11

Value: 49/11

2. Solve the following Game

PLAYER B

1 -3

PLAYER A 3 5

-1 6

4 1

2 2

-5 0

Ans:

Strategy of A: 0, 3/5, 0, 2/5, 0, 0

Strategy of B: 1/5, 4/5

Value: 17/5

3. Solve the following Game

PLAYER B

3 -2 4

PLAYER A -1 4 2

2 2 6

Ans:

Strategy of A: 0, 0, 1

Strategy of B: 4/5, 1/5, 0

Value: 2/5

NOTES

4. Solve the following Game

PLAYER B

	1	7	2	
PLAYER A		6	2	7
	5	1	6	

Ans:

Strategy of A: 2/5, 3/5, 0

Strategy of B: 1/2, 1/2, 0

Value: 4

Summary

This unit helps you appreciate the significance of integer programming and the application of strategies in game theory based decision making.

NOTES

NOTES

UNIT IV

DYNAMIC PROGRAMMING, SIMULATION AND DECISION THEORY

INTRODUCTION

Dynamic Programming (DP) determines the optimum solution to a n-variable problem by decomposing it into n stages with each stage comprising a single-variable sub-problem. The predominant contribution of DP is the principle of optimality, which is a framework for decomposing the problem into stages. We will also be discussing simulation using random numbers which would be useful in modeling of real time systems. The last part of this unit discusses decision theory models.

Learning Objectives

1. To be able to solve deterministic and LP model problems using DP.
2. To simulate using monte-carlo simulation method using random numbers
3. To model decision making problems under situations of risk and uncertainty.

4.1 DYNAMIC PROGRAMMING

Dynamic programming is concerned with the theory of multi-stage decision making. Mathematically, a DPP is a decision making problem with ‘n’ variables. The problem is divided into ‘n’ sub problems or segments, each sub problem being a decision making problem with one variable only. The solution to a DPP is achieved sequentially starting from the initial stage to the next, till the final stage is reached.

Bellman’s Principle of Optimality

An optimal policy (a set of decisions) has the property that whatever the initial state and decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. We will essentially concentrate only on the deterministic models.

We shall now look at a problem which might be a little interesting to you.

NOTES

Example 4.1.1:

A student has opted for 3 courses namely X, Y, Z in a semester. He has totally 3 days available for study. He believes it is best to study an entire course in a day, i.e. 2 courses not studied in the same day. He may study a course for 1 day, 2 days, 3 days or not study at all. His estimates of marks depending on the number of days of study are given:

Study days/ course	X	Y	Z
0	20	40	20
1	40	40	40
2	40	80	80
3	80	95	80

How should the student study to maximize his grades?

SOLUTION:

The student knows the marks he would score depending on the time he spends on them.

Since dynamic programming is a multistage decision making process, we shall compute the marks that is likely to be scored by the student if he has to study only courses X and Y. The table below shows the days and the corresponding marks likely to be scored. This is just the original data put in a common format. The row represents the course Y and the column represents course X.

X/Y: day and marks		**0**	**1**	**2**	**3**
		40	**40**	**80**	**95**
0	**20**				
1	**40**				
2	**40**				
3	**80**				

If the student does not spend even a day for both courses X and Y still he would score 60 marks. That is, spending ‘0’ days for X and Y he would still score 60 marks. Similarly if he spends 0 days for course X and 1 day for course Y he would score 60 marks. If he spends 0 days on X and 2 days on Y, then he would score 100 marks. If he spends 0 days on X and 3 days on Y he will score 115 marks. You can fill the remaining values in the rows accordingly.

NOTES

X/Y: day and marks		0	1	2	3
		40	**40**	**80**	**95**
0	**20**	60	60	100	115
1	**40**	80	80	120	-
2	**40**	80	80	-	-
3	**80**	120	-	-	-

You may have noticed some cells not having the computed values and this is because they are infeasible. Since the maximum available days are 3, there is no need to compute marks for more than three days put together.

Now in each diagonal (that is, 0 and 0 days, 1 and 1 day, 2 and 2 days and 3 and 3 days for X and Y respectively) select the highest value of cumulative marks. This is shown as an * in the table.

So for 0 days it is 60

1 days it is 80

2 days it is 100 and

for 3 days it is 120

X/Y: day and marks		0	1	2	3
		40	**40**	**80**	**95**
0	**20**	*60	60	*100	115
1	**40**	*80	80	120	-
2	**40**	80	80	-	-
3	**80**	*120	-	-	-

Now we take the maximum marks of X & Y together along the column and the values of Z along the row. This is shown in the Table below.

X+Y/Z: day and marks		0	1	2	3
		20	**40**	**80**	**80**
0	**60**				
1	**80**				
2	**100**				
3	**120**				

NOTES

We compute the marks for all the three courses as before. This is shown in the Table below.

X+Y/Z: day and marks		0	1	2	3
		20	40	80	80
0	60	* 80	*100	*140	140
1	80	*100	120	*160	-
2	100	120	140	-	-
3	120	140	-	-	-

Just as in the previous stage we identify the maximum values along the diagonals. And we see that the maximum value is at 160.

X+Y/Z: day and marks		0	1	2	3
		20	40	80	80
0	60	* 80	*100	*140	140
1	80	*100	120	*160	-
2	100	120	140	-	-
3	120	140	-	-	-

Maximum marks is 160 when X is studied for 1 day (40 marks), Y for 0 days (40 marks) and Z for 2 days (80 marks).

Example 4.1.2:

A retail marketing company has 9 salesmen, which it wants to assign to 3 districts. Sales revenues generated:

Districts/salesman	0	1	2	3	4	5	6	7	8	9
A	38	41	48	58	66	72	83	85	89	93
B	40	42	50	60	66	75	82	84	87	90
C	60	64	68	78	90	102	09	110	113	115

Identify the number of salesmen to be allocated to each district to maximize sales revenue.

SOLUTION

You can try to solve this problem just like the previous problem. Every step is same except that it has a greater set of data.

NOTES

Stage 1:

A/B: district and revenue		0	1	2	3	4	5	6	7	8	9
		40	**42**	**50**	**60**	**66**	**75**	**82**	**84**	**87**	**90**
0	**38**	* 78	80	88	*98	104	113	120	122	125	128
1	**41**	* 81	83	91	101	107	116	123	125	128	-
2	**48**	* 88	90	98	108	114	123	130	132	-	-
3	**58**	* 98	100	108	118	124	*133	140	-	-	-
4	**66**	*106	108	116	126	132	141	-	-	-	-
5	**72**	112	114	122	132	138	-	-	-	-	-
6	**83**	*123	125	133	143	-	-	-	-	-	-
7	**85**	125	127	135	-	-	-	-	-	-	-
8	**89**	129	131	-	-	-	-	-	-	-	-
9	**93**	*133	-	-	-	-	-	-	-	-	-

Stage 2:

A+B/C: district and revenue		0	1	2	3	4	5	6	7	8	9
		60	**64**	**68**	**78**	**90**	**102**	**109**	**110**	**113**	**115**
0	**78**	*138	*142	146	156	*168	*180	*187	188	191	193
1	**81**	141	145	149	159	171	183	190	191	194	-
2	**88**	*148	152	156	166	178	*190	197	189	-	-
3	**98**	*158	162	166	176	188	*200	207	-	-	-
4	**106**	166	170	174	184	196	*208	-	-	-	-
5	**113**	173	177	181	191	203	-	-	-	-	-
6	**123**	183	187	191	203	-	-	-	-	-	-
7	**126**	186	190	194	-	-	-	-	-	-	-
8	133	193	197	-	-	-	-	-	-	-	-
9	143	203	-	-	-	-	-	-	-	-	-

Maximum sales is Rs.208 when 4 salesmen are sent to A, none to B and 5 to C.

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Example 4.1.3 Solve the following LPP through DP

Maximise $Z = 2x_1 + 5x_2$

Subject to: $2x_1 + x_2 <= 430$

$x_2 < = 230$

$x_1, x_2 >= 0$

Solution:

This problem has two variables therefore it will have two stages to be solved in.

Every constraint indicates a resource constraint, so there are two resources. This means the problem has two states.

Let the stages be indicated as f_1 and f_2 and the states be indicated as v (constraint 1) and w (constraint 2).

We start from stage 2, following backward recursive process.

Stage 2:

f_2 (v2, w2) = max [5 x_2]

Such that 0 <= x_2 <=v2 and

0 < = x_2 <= w2.

Thus the maximum of [5 x_2] occurs at x_2 = minimum of {v2, w2} and the solution for stage 2 is given in the Table below.

	Optimum Solution	
State	f_2 **(v2, w2)**	**x2**
v2,w2	5 min {v2,w2}	min {v2,w2}

Stage 1:

In stage 1 we write

f_1 (v1, w1) = maximum {2 x_1 + f2(v1 – 2x1, w1}

In order to arrive at a final solution we need a solution from stage 2 and this is obtained from setting v1 = 430 and w1 = 230, which give the result of

0 < = 2 x_1 < = 430. Because the min of (430 – 2 x_1, 230) is the lower envelope of the

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two intersecting lines, it results in

min $(430 – 2 x_1, 230) = 2 x_1 + 230,$ if $0 <= x_1 <= 100$

and

min $(430 – 2 x_1, 230) = 2 x_1 + 430 – 2 x_1,$ if $100 <= x_1 <= 215$

and

$f_1 (430, 230)$ = maximum of $2 x_1 + 1150$ if $0 <= x_1 <= 100$

or

$f_1 (430, 230)$ = maximum of $-8 x_1 + 2150$ if $100 <= x_1 <= 215$

A graphical solution would indicate to you that the optimum value of f_1 (430, 230)occurs at $x_1 = 100$,

So we have

	Optimum Solution	
State	**f_1 (v1, w1)**	**x_1**
430, 230	1350	100

We can get the optimum value of x_2 from

$v2 = v1 -2 x_1 = 430 – 200 = 230$ and

$w2 = w1 – 0 = 230$

Therefore $x_2 = 230$

The optimum solution is $x_1 =100$, $x_2 = 230$ and Z = 1350

4.2 SIMULATION

Simulation is a very good alternative to observing a real system. It creates an environment like a real situation where random forces act. Data from a real time situation is used to model an expected system using random numbers. Some simple problems based on simulation are illustrated below.

NOTES

Example 4.2.1:

During the past year a bank has recorded receipts and payment patterns:

RECEIPTS	PROBABILITY	PAYMENT	PROBABILITY
3000	0.2	4000	0.3
5000	0.3	6000	0.4
7000	0.4	8000	0.2
12000	0.1	10000	0.1

Using independent sets of random numbers for receipts and payments simulate the receipts and payments of the bank for a week if the opening balance is Rs.8000

SOLUTION:

We know that to simulate a process real time data is required. Looking at the data table we have a set of real time information.

RECEIPTS	PROBABILITY	PAYMENT	PROBABILITY
3000	0.2	4000	0.3
5000	0.3	6000	0.4
7000	0.4	8000	0.2
12000	0.1	10000	0.1

The way we interpret the probabilities associated with receipts and withdrawals is like this. Out of 100 days, receipts of Rs 3000 would have been for 20 days, 5000 for 30 days, 7000 for 40 days and 12000 for 10 days. This gives us the values of probability which adds up to 1. A similar interpretation is done for the payments. If this was the observed receipts and payment patterns we have to now simulate for a future period. For this we need to first calculate the cumulative probability. This is shown in the following Table. Each probability is added to the next probability to compute the cumulative probability.

NOTES

Receipts	Probability	Cumulative probability	Payments	Probability	Cumulative probability
3000	0.2	0.2	4000	0.3	0.3
5000	0.3	0.5	6000	0.4	0.7
7000	0.4	0.9	8000	0.2	0.9
12000	0.1	1	10000	0.1	1

Next for each cumulative probability we fix a range. If the cumulative probability is 0.2, if multiplied by a factor 100, we know the value is 20. So there are 20 values present. If we start at 0 we have the 20^{th} value at 19. This is shown in the Table below.

Receipts	Probability	Cumulative probability	Range	Payments	Probability	Cumulative probability	Range
3000	0.2	0.2	0-19	4000	0.3	0.3	0-29
5000	0.3	0.5	20-49	6000	0.4	0.7	30-69
7000	0.4	0.9	50-89	8000	0.2	0.9	70-89
12000	0.1	1	90-99	10000	0.1	1	90-99

Random no:

The problem requires us to simulate the receipts and payments for a period of 12 weeks. So we need 24 random numbers (12 for receipts and 12 for payments). These random numbers may either be given in the problem itself or may have to be drawn from the random number table (Appendix 1). The random numbers chosen from the table are given below.

Receipts: 20, 74, 94, 22, 93, 45, 44, 16, 04, 32, 03, 62

Payments: 25, 38, 57, 93, 40, 37, 57, 66, 37, 49, 42, 03

For the first week if the random number is 20, then this falls within the range of 20 – 49. Therefore the probable receipt is Rs. 5000. Similarly, the random number is 25 for payments which falls within the range of 0 -29 for payments. Therefore the expected payment is Rs 4000. According to the question, the opening balance is Rs 8000. So 8000 + 5000 – 4000 = 9000. This would be the balance at the end of the 1^{st} week. You can compute the remaining values in the table and compute the balance at the end of each week.

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Week	Receipts		Payments		Balance
	Random no	Amount	Random no	Amount	
0					8000
1	20	5000	25	4000	9000
2	74	7000	38	6000	10000
3	94	12000	57	6000	16000
4	22	5000	93	10000	11000
5	93	12000	40	6000	17000
6	45	5000	37	6000	16000
7	44	5000	57	6000	15000
8	16	3000	66	6000	12000
9	4	3000	37	6000	9000
10	32	5000	49	6000	8000
11	3	3000	42	6000	5000
12	62	7000	03	4000	8000

12 weeks make a quarter of a year. The average balance is 11,076.

Another similar problem is solved below for your benefit.

Example 4.2.2:

A company manufactures 200 mopeds. Depending upon raw materials, etc, the daily production has been varying from 196 to 204. Probability distribution is given:

NO. OF MOPEDS	196	197	198	199	200	201	202	203	204
PROBABILTY	.05	.09	.12	.14	.2	.15	.11	.08	.06

The mopeds are transported in a 3-storied lorry that can accommodate only 200 mopeds. Using the given set of random numbers generate space availability of the truck or days in which mopeds are kept waiting and find the average number of mopeds waiting and average space in the truck during the given period.

Random no: 82, 89, 78, 24, 53, 61, 18, 45, 04, 23, 50, 77, 27, 54, 10

NOTES

NO.OF MOPEDS	196	197	198	199	200	201	202	203	204
PROBABILITY	.05	.09	.12	.14	.2	.15	.11	.08	.06
CUM.PROBABILITY	.05	.14	.26	.40	.60	.75	.86	.94	1
RANGE	0-4	5-13	14-25	26-39	40-59	60-74	75-85	86-93	94-99

SOLUTION:

DAY	RANDOM NO	NO.OF MOPEDS	Space in Truck	MOPEDS WAITING
1	82	202	-	2
2	89	203	-	3
3	78	202	-	2
4	24	198	2	-
5	53	200	-	-
6	61	201	-	1
7	18	198	2	-
8	45	200	-	-
9	04	196	4	-
10	23	198	2	-
11	50	200	-	-
12	77	202	-	2
13	27	199	1	-
14	54	200	-	-
15	10	197	3	-

Average space-0.93, average mopeds waiting-0.66 (or) 1

We now have an example combining an application of a queing model to simulation. But we do not need random numbers to solve this problem. You can first read the problem to understand the core need to be solved.

Example 4.2.3:

Customers arrive at a milk booth for the required service. Assuming that inter arrivals and service times are constant and are given by 1.8 and 4 time units respectively. Simulate the system by hand computation for 14 time units. What is the average waiting time for the customer and percentage idle time? Assume simulation starts at 0 time units.

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SOLUTION:

Let us assume that the time units are in minutes. According to the problem, a customer arrives every 1.8 minutes and it takes 4 minutes to serve a customer. So assuming that the simulation starts at the 0^{th} minute, the first customer arrives in the 0^{th} minute. The next customer arrives in the 1.8^{th} minute. The second customer arrives in the 3.6^{th} minute. The next time event is the 4^{th} minute and it the time that the first customer has completed the service and leaves the system. You can try and completed the remaining time events and check with the table given below.

TIME	EVENT	CUSTOMER
0	Arrival	1
1.8	Arrival	2
3.6	Arrival	3
4	Departure	1
5.4	Arrival	4
7.2	Arrival	5
8	Departure	2
9	Arrival	6
10.8	Arrival	7
12	Departure	3
12.6	Arrival	8
14	arrival	9

The waiting time is calculated as follows:

Customer W.Time

1 —— 4-0=4
2 —— 8-1.8=6.2
3 —— 12-3.6=8.4
4 —— 14-5.4=8.6
5 —— 14-7.2=6.8
6 —— 14- 9= 5
7 —— 14-10.8=3.2
8 ——— 14-12.6=1.4
Average waiting time = 5.45 time units

NOTES

Exercises

SIMULATION

1. A manufacturing company keeps stock of a special product. Previous experience indicates the daily demand as given below.

Daily demand	5	10	15	20	25	30
Probability	0.01	0.2	0.15	0.5	0.12	0.02

Simulate the demand for next 10 days. Also find the average daily demand for the product based on the simulated data.

Ans: average demand = 17 units/day.

2.A Company manufactures 30 items per day and the demand for the items produced has the following distribution.

Sales (units)	Probability
27	0.1
28	0.15
29	0.20
30	0.35
31	0.15
32	0.05

The production cost and the sale price of the unit are Rs.40 and Rs.50. Any unsold product is disposed off at a loss of Rs.15. There is a penalty of Rs.5 per unit when the demand is not met. Simulate the profit or loss of the Company for the next 10 days. If the company decides to produce 29 items per day are there any disadvantages?

Ans: Total profit=Rs.2695.there is no disadvantage of producing 29 items because the profit remain the same.

4.3 DECISION THEORY

The types of decision making situations are summarized in the table below. We come across situations to take decisions all the time in our life. Some of them are so small that we do not categorise the situation into the problem, the alternatives and the outcomes associated with the alternatives. But some decisions are critical and we define the problem then list out the various alternative and evaluate the outcomes associated with the alternatives. Finally we choose the alternative that yields us the best possible outcome.

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	Problem	Alternatives	Outcome
Certainty	Known	Known	Known
Risk	Known	Known	(Probability known)
Uncertainty	Known	Known	Unknown

Decision making under situations of certainty require no further explanation. So we will try to solve problems associated with risk and uncertainty. We will solve three examples of decision making under risk using decision trees and the fourth based of Expected monetary value criterion.

Decision making under risk:

Example 4.3.1:

A pharmaceutical firm is planning to develop and market a new drug. The cost of extensive research to develop the drug has been estimated at Rs.1,00,000. The manager has found that there is a 60% chance that the drug would be developed successfully. The market potential is as given below:

Market condition (Potential)	Probability	Present Value (P.V.) profits
Large	0.1	50000
Moderate	0.6	25000
Low	0.3	10000

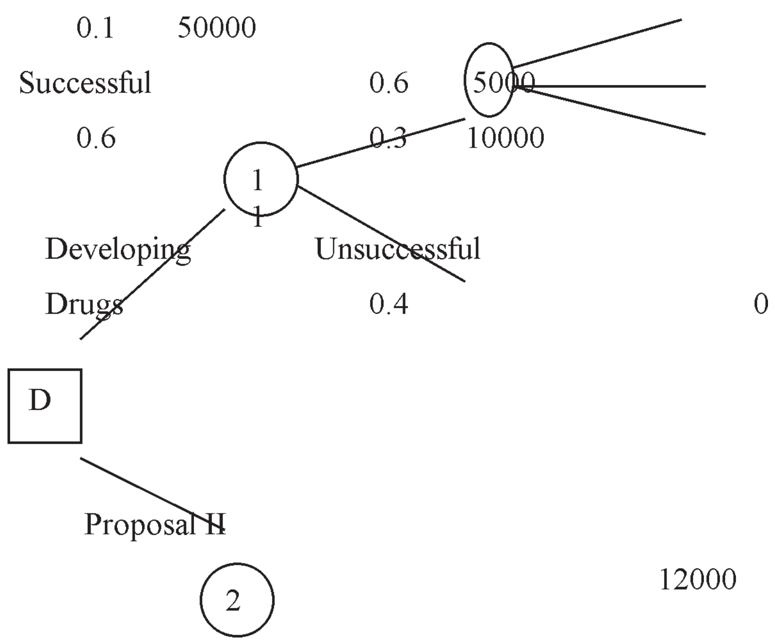
P.V. figures do not include the cost of research. While the firm is considering this proposal, a second proposal almost similar comes up for consideration. The second one also requires an investment of Rs.100000 but P.V. of all profits is 12,000. ROI in second proposal is certain.

(a) Draw a decision tree indicating all events and choices of the firm.
(b) What decision must the firm take regarding the investment of Rs. 100000?

SOLUTION

Decision trees are easy to construct just by using pure logic. Based on the given information we know that the firm has two proposals. These are shown in the first branching. In case the pharmaceutical firm decides to develop the drug, there is a 60% probability that the project would be successful and there is a 40% probability that it might fail. If the development of the drug is successful, then there is a 10% probability of a large market potential, 60% of a moderate market potential and 30% of a low market potential. The possible returns in rupees associated with the market potential is also shown in the diagram. The computations of the Returns on Investment (ROI) are shown below the diagram. We find that it would benefit the firm to go ahead and develop the drug instead of accepting the second proposal as the development of the new drug despite the possibility of being unsuccessful still yields are probable higher return.

ROI



ROI for 1st proposal = (50000*0.1*0.6) + (25000*0.6*0.6) + (10000*0.3*0.6)

= Rs. 13800

ROI for 2nd proposal = Rs. 12000

The 1st proposal to develop and market drugs gives better returns.

We have two similar situations illustrated in example 4.3.2 and 4.3.3. I would suggest that you read the question then try to draw the decision tree on your own and then verify the solution.

Example 4.3.2:

You have an option to invest Rs.10000 in the stock market by buying shares of company A or B. Shares of company A are risky but could yield 50% ROI during the next year. If the stock market conditions are not favorable, the stock will lose 20% of its value. Company B provides soft investments with 15% returns in bullish market and only 5% in

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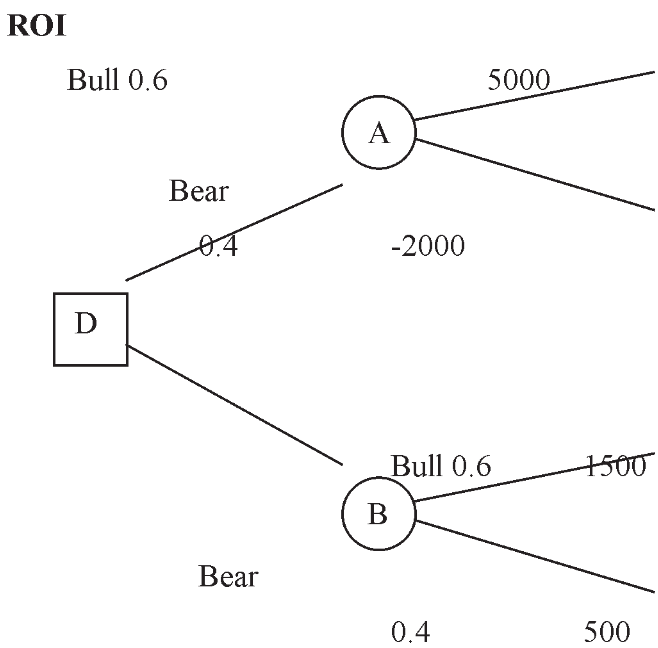
a Bearish market. All stock market journals predict 60% chance for a Bull market. In which company would you invest?

SOLUTION:



For A,

E.V.= 5000(0.6)+(-2000)(0.4)

=3000-800 = Rs.2200.

For B,

E.V.=1500(0.6)+500(0.4)

=900+200 = Rs.1100.

So, A is better. Since A gives higher ROI, we will invest on A.

Example 4.3.3:

A company is to decide about the size of the plant. Two alternatives are available. Plant A has annual capacity of 50000 units. Plant B has annual capacity of 25000 units. Demand is not known with certainty. But management has estimated probabilities for 3 different levels of demand.

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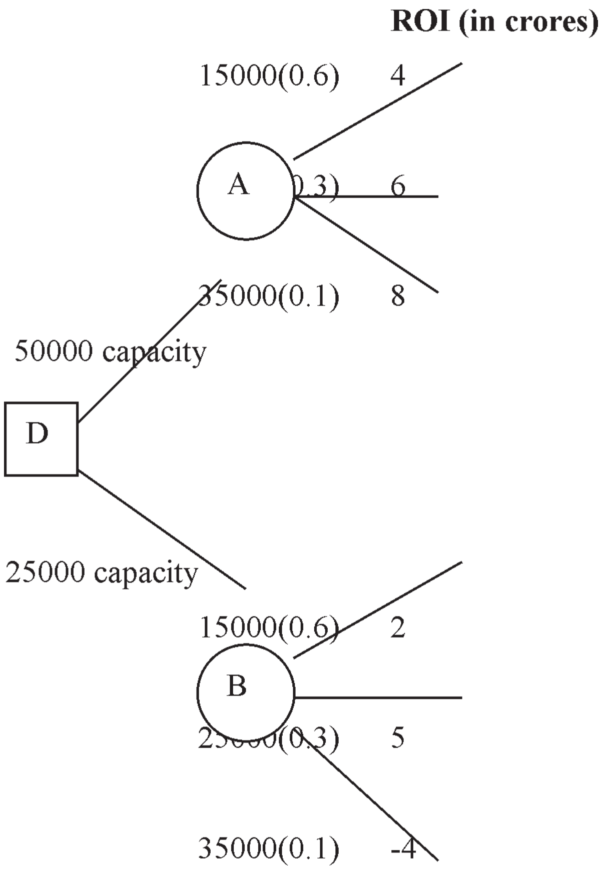
(Units) Demand	Probability	Profits in crores of Rs.	
		Plant A	Plant B
15000	0.6	4	2
25000	0.3	6	5
35000	0.1	8	4

Should the company opt for B?

SOLUTION:



Expected Value:

A=(4,00,00,000X0.6)+(6,00,00,000X0.3)+(8,00,00,000X0.1)

=2,40,00,000+1,80,00,0002+80,00,000

=5,00,00,000.

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B=(2,00,00,000X0.6)+(5,00,00,000X0.3)+(-4,00,00,000X0.1)

=1,20,00,000+1,50,00,000-40,00,000

=2,30,00,000.

So, A is a better option to invest.

Example 4.3.4:

A newspaper boy knows the following probabilities of selling a magazine:

No. of copies sold	Probability
10	0.1
11	0.15
12	0.2
13	0.25
14	0.3

Cost of one copy is 30 paise and selling price is 50 paise. If he cannot return unsold magazines, determine the number of units of magazine he must stock based upon EMV, EOL and EVPI criterions.

SOLUTION:

Based on past experience the newspaper boy knows that there is a 10% probability that 10 copies would be sold in a day, 15% probability that 11 copies would be sold in a day, 20% probability that 12 copies would be sold in a day, 25% probability that 13 copies would be sold in a day and 30% probability that 14 copies would be sold in a day (shown in the table below).

No. of copies sold	Probability	Percentage of Probability
10	0.1	10
11	0.15	15
12	0.2	20
13	0.25	25
14	0.3	30

His interest is in knowing the number of copies that he must stock in-order to maximize profits or to minimize losses. To maximize profits he will adopt the Expected Monetary

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Value (EMV) and to minimize losses he will adopt the Expected Opportunity Loss criterion (EOL) and if he knows the exact quantity to stock each day, he would maximize his returns through the Expected Value of Perfect Information (EVPI) criterion.

We'll first adopt the EMV criterion to help the newspaper boy know the number of copies to be stocked to maximize returns.

EMV CRITERION:

We'll first have to compute the conditional profit table which is shown below. The first column shows the expected demand and the second row shows the possible quantity stocked. If the expected demand is 10 newspapers and the boy stocks 10 newspapers he will make a profit of 200 paise (profit @ 20p per sold newspaper). If the demand for newspapers is 10 but he has a stock of 11 newspaper, he will have one unsold news paper. By selling 10 newspapers (demand quantity) he has made a profit of 200p but the unsold news papers has brought down his profit by 30p. So if he overstocks (that is 11 newspapers) and if the demand is only for 10 newspapers, then his profit will only be 170p shown in the table below. Similarly you can compute the other conditional profit values.

Conditional profit table:

Demand	Possible quantity of stock				
	10	**11**	**12**	**13**	**14**
10	200	170	140	110	80
11	200	220	190	160	130
12	200	220	240	210	180
13	200	220	240	260	230
14	200	220	240	260	280

Hope you have computed the conditional profit table and found it to be similar to the table shown above.

We now compute the expected profit values by multiplying the values in the conditional profit table by the probabilities associated with the expected demand values. These values are shown in the table below.

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Expected profit table:

Demand	Prob.	Possible quantity of stock				
		10	11	12	13	14
10	0.1	20	17	14	11	8
11	0.15	30	33	2.5	24	19.5
12	0.2	40	44	48	42	36
13	0.25	50	55	60	65	57.5
14	0.3	60	66	72	78	84
TOTAL		200	215	222.5*	220	205

We find that Expected profit is maximum when 12 units are stocked.

We next compute the conditional loss table. If the demand is 10 news papers and the boy stock exactly 10 news papers then his conditional loss is 0 paise. But if he stocks 11 newspapers when the demand is only for 10 news papers then his conditional loss is 30p. Similarly you can compute the other table values.

EOL CRITERION:

Conditional loss table:

If the demand is 10 news papers and the boy stock exactly 10 news papers then his conditional loss is 0 paise. But if he stocks 11 newspapers when the demand is only for 10 news papers then his conditional loss is 30p. Similarly you can compute the other table values.

Demand	Possible quantity of stock				
	10	11	12	13	14
10	0	30	60	90	120
11	20	0	30	60	90
12	40	20	0	30	60
13	60	40	20	0	30
14	80	60	40	20	0

NOTES

Expected loss table:

The expected loss table is computed by multiplying the values of the conditional loss table by the probabilities associated with the demand. This is shown in the table below.

Demand	Prob.	Possible quantity of stock				
		10	11	12	13	14
10	**0.1**	0	3	6	9	12
11	**0.15**	3	0	4.5	9	13.5
12	**0.2**	8	4	0	6	12
13	**0.25**	15	10	5	0	7.5
14	**0.3**	24	18	12	6	0
TOTAL		50	35	27.5*	30	45

We find that the Expected loss is least when 12 units are stocked.

EVPI CRITERION:

The EVPI value is computed by multiplying the probabilities associated with the demand values to the profit values of stocking the exact quantity. This is shown in the table below

Expected Demand	Probability of Demand	Quantity Stocked	Expected Profit	Probability x expected profit
10	**0.1**	10	200	20
11	**0.15**	11	220	33
12	**0.2**	12	240	48
13	**0.25**	13	260	65
14	**0.3**	14	280	84
Total profit under EVPI				250

The following equation is also true about EVPI.

EVPI = EMV + EOL

= 222.5 + 27.5 = 250

NOTES

Decision making under uncertainty

QUESTION.1:

A company would like to manufacture and sell three types of perfumes. The company can produce 20000 units, 10000 units, and 2000 units of all the three types. Assuming that the company can sell all its products, payoff matrix is as follows:

Expected Sales	20000	10000	2000
Type of Perfume			
I	250	15	10
II	40	20	5
III	60	25	3

Determine the type of perfume and the quantity to be produced based upon the following criteria:

a. Maxi-min
b. Maxi-max
c. Mini-max regret
d. Laplace
e. Hurwicz

SOLUTION:

a) Based on Maximin criteria we have the following choice

Expected Sales	20000	10000	2000	Maxi(min)
Type of Perfume				
I	250	15	10	**10**
II	40	20	5	5
III	60	25	3	3

From each row we select the minimum payoff value and then choose the maximum payoff value among the minimum.

Therefore based on maxi-min criteria we have Type I perfume as the choice to be manufactured.

NOTES

b) We now look for a choice based on Maxi-max criterion

Expected Sales	20000	10000	2000	Maxi(max)
Type of Perfume				
I	250	15	10	**250**
II	40	20	5	40
III	60	25	3	60

The maxi-max criterion is the most optimistic of criterion which assumes everything would be all-right. We choose the maximum payoff from each row and again choose the maximum payoff from the column. This gives us the result of 250.

Therefore based on maxi-max criterion we can produce and sell Type I perfume.

c) Mini-max Regret criterion

In-order make a choice based on mini-max regret criterion we need to first prepare the regret table. In every column subtract the value from the highest value in that column. In the first column the highest payoff is 250, so all values in that column are subtracted from 250. Similarly, in the second column the highest payoff is 25. So, all values are subtracted from this value. And in the third column, the highest value is 10 and all other values are subtracted from this value. The regret table is shown below.

Expected Sales	20000	10000	2000
Type of Perfume			
I	0	10	0
II	210	5	5
III	190	0	7

Now in the regret table we will adopt the Mini-max criterion. In every row choose the maximum value and then choose the minimum out of the row maximums. This is shown in the table below.

Expected Sales	20000	10000	2000	Mini-max
Type of Perfume				
I	0	10	0	**10**
II	210	5	5	210
III	190	0	7	190

Based on the mini-max criterion of the regret table, we choose Type I perfume.

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d) Laplace Criterion.

Laplace criterion is based on the principle of insufficient reason (the criterion of rationality) and equal probabilities are assigned to all payoffs or events.

There are three possibilities that is sales of 20000 units, 10000 units and 2000 units. So we assign 1/3 probabilities for each of the type of perfumes.

Laplace payoffs of the three types of perfumes are computed.

Type I: 250 x 1/3 + 15 x 1/3 + 10 x 1/3 = 91.63

Type II: 40 x 1/3 + 20 x 1/3 + 5 x 1/3 = 21.67

Type III: 60 x 1/3 + 25 x 1/3 + 3 x 1/3 = 29.33

Expected Sales	20000	10000	2000	Laplace Value
Type of Perfume				
I	250	15	10	**91.63**
II	40	20	5	21.67
III	60	25	3	29.33

Based on the Laplace criterion we choose the maximum of the three column values and the choice is naturally the Type I perfume.

e) Hurwicz Criterion

The Hurwicz criterion is also known as the criteria of realism or the weighted average criteria.

An á value is assigned to the highest payoff in each row and a (1- á) value is assigned to the lowest payoff value in each row.

Hurwicz: α(Max. Payoff) + (1-α)(Min. Payoff)

This is shown below

Type I: 250 x 0.7 + 10 x 0.3 = 178

Type II: 40 x 0.7 + 5 x 0.3 = 29.5

Type III: 60 x 0.7 + 3 x 0.3 = 42.9

NOTES

Expected Sales	20000	10000	2000	Hurwicz Value
Type of Perfume				
I	250	15	10	**178**
II	40	20	5	29.5
III	60	25	3	42.9

Based on the Hurwicz criterion the maximum value is 178 and the choice is to produce Type I perfume.

As per Maximin, Maximax, Minimax Regret, Laplace and Hurwicz criterions, Type I should be manufactured.

Exercises

Have you understood?

1. What are the types of decision making situations?
2. How does a decision tree help in arriving at decisions?
3. Is a solution through DP for LPP better than an LP algorithm?
4. What is a deterministic DP model?
5. What criteria are relevant for decision making under uncertainty?

DECISION MAKING UNDER RISK

1. A bakery collected the information on the sale of cakes for a 100-day period and the information is given below.

Sales per day	No of days	Probability
25	10	0.1
26	30	0.3
27	50	0.5
28	10	0.1

Construct the payoff table and the opportunity loss table. What is the optimal no of cakes that should be prepared each day? The cakes cost 0.80 paise and sells for Re.1.

Ans: 26 cakes

DECISION TREE

1.A company is contemplating whether to produce a new product .If it decides to produce the product it must either install a new division that needs 4 crores or work overtime with overtime expenses of 1.5 crores. If the company decides to install the new

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division it needs the approval of the government and there is a 70% chance of getting the approval. A market survey has revealed the following facts about the magnitude of sales for the product.

Magnitude of Sales	Probability	Resulting Profit
25	0.45	15 crores
26	0.30	7
27	0.20	3
28	0.05	-5 loss

By resorting to overtime the company will not meet the high magnitude of sales. it will be able to satisfy the medium magnitude only even if high magnitude of sales results. Solve the problem to suggest which option to select.

Ans: Work overtime and produce the new product with the expected monetary value of 4.10 lakhs.

Summary

This unit would have given you an insight into the models of dynamic programming and the pattern of decision making under risk and uncertainty.

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UNIT V

QUEUING THEORY AND REPLACEMENT MODELS

INTRODUCTION To The Unit

The problem of replacement is to decide the best policy in determining the time at which the replacement is most economical instead of continuing at an increase cost. The main objective of replacement is to direct the organization for maximizing its profit (or minimizing the cost). This unit will give an insight about the replacement models.

There are multiple situations where persons waiting to be served are to be modeled for optimizing their waiting time. One effective model is the Queuing model. We here discuss the elementary single and multi-channel model.

Learning Objectives

1. To solve problems involving single channel models
2. To solve problems involving multi-channel models
3. To compute replacement of items that deteriorate with time (Capital items) both with and without time value of money.
4. To compute replacement of items that fail suddenly (Group replacement).

5.1. QUEUEING THEORY:

INTRODUCTION

A flow of customers from finite/infinite population towards the service facility forms a queue (waiting line) on account of lack of capability to serve them all at time. In the absence of a perfect balance between the service facilities and the customers, waiting time is required either for the service facilities or for the customer's arrival.

The arriving unit that requires some service to be performed is called customer. The customer may be person, machines, vehicles etc. Queue (waiting line) stands for the number of customers waiting to be serviced. This does not include the customer being serviced. The process or system that performs services to the customer is termed by service channel or service facility.

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QUEUEING SYSTEM

A queueing system can be completely described by

1) The input (arrival pattern)
2) The service mechanism (service pattern)
3) The queue discipline , and
4) Customer's Behaviour.

Behavior of Customers

Patient Customer: Customer arrives at the service system, stays in the queue until served, no matter how much he has to wait for service.

Impatient Customer: Customer arrives at the service system, waits for a certain time in the queue and leaves the system without getting service due to some reasons like long queue before him.

Balking: Customer decides not to join the queue by seeing the number of customers already in service system.

Reneging: Customer after joining the queue, waits for some time and leaves the service system due to delay in service.

Jockeying: Customer moves from one queue to another thinking that he will get served faster by doing so.

THE INPUT (ARRIVAL PATTERN)

The input describes the way in which the customers arrive and join the system. Generally, customers arrive in a more or less random fashion which is not worth making the prediction. Thus the arrival pattern can be described in terms of probabilities and consequently the probability distribution for inter arrival times (the time between two successive arrivals) must be defined. We deal with those queueing systems in which the customers arrive in poison fashion. The mean arrival rate is denoted by ë.

THE SERVICE MECHANISM:

This means the arrangement of service facility to serve customers. If there is finite number of servers then all the customers are served instantaneously on arrival and there will be no queue.

If the number of servers is finite then the customers are served according to a specific order with service time a constant or a random variable. Distribution of service time which is important in practice is the negative exponential distribution. The mean service rate is denoted by m.

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THE QUEUE DISCIPLINE:

It is a rule according to which the customers are selected for service when a queue has been formed. The most common disciplines are

1) First come first served (FCFS)
2) First in first out (FIFO)
3) Last in first out (LIFO)
4) Selection for service in random order (SIRO)

There are various other disciplines according to which a customer is served in preference over the other. Under priority discipline, the service is of two types, namely pre-emptive and non pre-emptive. In pre-emptive system, the high priority customers are given service over the low priority customers; in non pre-emptive system, a customer of low priority is serviced before a customer of high priority is entertained for service. In the case of parallel channels "fastest server rule" is adopted.

Traffic intensity (or utilization factor): An important measure of a simple queue is its traffic intensity

Given by traffic intensity ñ = Mean arrival rate / Mean service rate = ë / µ

The unit of traffic intensity is Erlang.

Only one service point	—	Single channel model
More than one service point	—	Multi channel model
	(a/b/c): (d/e/f)	
	Kendall's Notation	

λ — Arrival rate

μ — Service rate

Always-in terms of 1minute and 1 second etc.

μ must always be greater than λ

We'll now have a quick look at just two models (single and multi-channel) and a selected set of formulae for your perusal.

This is shown as a scanned document in the next page. It is mandatory that you know the formulae thoroughly to solve problems in queueing theory.

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Characteristics of Queuing System – Kendall's Notations (a/b/c): (d/e/f)

a: Arrival Distribution or inter-arrival (λ)
b: Departure distribution or service time(μ)
c: Number of service channels
d: Service Discipline
e: Maximum number of customers in the system
f: Calling source or population

Model I : (M/M/1): (FCFS/α/α) when ($\lambda < \mu$) – Single channel model

1. Average (Expected) number of customers in the system (Ls) = $\lambda / (\mu - \lambda)$
2. Average (Expected) number of customers in the queue (Lq) = Ls.(λ / μ) = $\lambda^2 / (\mu (\mu - \lambda))$
3. Expected (Average) time a customer spends in the system (Ws) = $1/(\mu - \lambda)$
4. Expected (Average) time a customer waits in the queue (Wq) = $(\lambda / \mu) .(1/(\mu - \lambda))$
5. Probability of an empty or idle system (Po) = $1 - (\lambda / \mu)$
6. Probability that a service channel is busy (traffic density or utilization factor)(ρ) = λ / μ
7. Probability that there are 'n' customers in the system (Pn) = $(\lambda / \mu)^n$. Po

Model II : (M/M/C): (FCFS/α/α) when ($\lambda < \mu$) – Multi channel model

n: Number of customers in the system

Pn: Probability of 'n' customers in the system

c: number of parallel service channels (c>1)

Case 1 (when n=0) P1 = (λ/ μ) . Po

Case 2 (when $1 <= n <= c-1$) Pn = $(1/ n!).(\lambda/ \mu)^n$.Po

Case 3 (when n >=c) Pn = $(1 / (c^{n-c} .c!)).(\lambda/ \mu)^n$.Po

$$Po = 1/\{[(\sum_{n=0}^{c-1} ((\lambda/ \mu)^n/ n!)] + ((\lambda/ \mu)^c / c!) . (c\mu / (c\mu - \lambda))\}$$

1. Ls = $\{[(\lambda. \mu . (\lambda/ \mu)^c) / ((c-1)! (c\mu - \lambda)^2)]. Po\} + (\lambda/ \mu)$
2. Lq = Ls - (λ/ μ)
3. Ws = Ls /λ
4. Wq = Lq / λ
5. Probability that a customer has to wait (P (n>=c)) = $[(\mu . (\lambda/ \mu)^c) / ((c-1)! (c\mu - \lambda))]$. Po
6. Probability that a customer enters service without waiting = 1 - (P (n>=c))
7. Average number of idle servers = c – average number of customers served
8. Utilization factor (ρ) = $\lambda / c\mu$
9. Efficiency (η) of M/M/C model = Average number of customers served / Total number of customers.

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5.1.1 Single Channel Models

Example 5.1.1

A self-service store employs one cashier at its counter. Nine customers arrive on an average every 5 minutes while the cashier can serve 10 in 5 minutes. Assuming Poisson distribution in arrival rate and exponential distribution for service rate find:

a) Average number of customers in the system
b) Average number of customers in the queue
c) Average time customers in the system
d) Average time of waiting before service

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SOLUTION:

a) Average number of customers in the system Ls = $\lambda/(\mu-\lambda)$ = 9 customers.

b) Average number of customers in queue Lq = $\dfrac{\lambda^{\wedge}2}{\mu(\mu-\lambda)}$

= 81/ 10

= 8.1

= 8 customers.

c) Average time a customer spends in system Ws = $1/(\mu-\lambda)$

= 5 minutes.

d) Average waiting time in queue Wq = $(\lambda/\mu)\ (1/\mu-\lambda)$ = 0.9 time unit = 4.5 minutes.

Example 5.1.2:

A branch of PNB has only one typist. Since the typing work varies in length the typing rate is randomly distributed approximately a Poisson distribution with μ - 8 letters / hour and λ - 5/ hour, during the entire 8 hour workday. If the typewriter is valued at RS. 1.5/- per hour, determine:

a) Equipment utilization
b) Percentage time of waiting of arriving letter
c) Average system time
d) Average idle time cost of typewriter/day.

SOLUTION:

a) Equipment utilization = λ/μ = 5/8 = 0.625
b) Percentage time an arriving letter has to wait = (λ/μ) * 100 = 62.5%
c) Average system time = $1/(\mu-\lambda)$ = 1/3 = 0.33 = 20 minutes.
d) Average idle time cost of typewriter/day

Idle time= $1-(\lambda/\mu)$ = 3/8=0.375* 8 = 3 hours

Idle time cost= RS. 4.50/day

Example 5.1.3:

A person repairing radios finds that time spent on radio sets has an exponential distribution with mean 20 minutes. If radios repaired on order of arrival and arrival is Poisson with average rate of 15 minutes in an 8-hour day.

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a) What is the repairman's expected idle time each day
b) How many jobs are ahead of the set that has just been brought in

SOLUTION:

λ= 15/day, μ=24/day

a) Idle time =1–(λ/μ)= 3 hours

Ls= $\lambda/(\mu-\lambda)$= 15/9= 1.67= 2sets.

5.1.2 Multi Channel Models

In all multi channel model (where more than one service channel is present) problems the most important formula to be computed is the Po formula shown below. All other formula can be computed only based in Po.

$$Po = \frac{1}{\sum_{n=0}^{c-1} \left[(\lambda/\mu)^n/n! \right] + (\lambda/\mu)^c/c! * C\mu/(C\mu-\lambda)}$$

Example 5.1.4:

A tax-consulting firm has 3 counters in its office to receive people who have problems concerning their income, wealth and sales taxes. On an average 48 persons arrive in an 8 hour day. Each tax advisor spends 15 minutes on an average on an arrival. If the arrivals are Poisson distributed and service time is exponential find:

a) Average number of customers in system
b) Average number of customers waiting to be serviced
c) Average time spent in system
d) Average waiting time for a customer
e) Number of hours each week a tax advisor spends performing his job.
f) Probability a customer has to wait before he gets serviced
g) Expected number of idle tax advisors at any specified time.

SOLUTION:

λ= 6/hour, μ=4/hour, c=3

Po = 1/ (1+3/2+9/8+(36/64*12/6))

= 1/(4.625) =0.2105

NOTES

a) Average number of customers in system= {[λ∗μ(λ/μ)^c / ((c-1)! (Cμ–λ)^2)]∗Po}+ (λ/μ)

= 1.737 = 2 customers

b) Average number of customers waiting to be serviced = 0.237

c) Average time spent in system = 1.737/6= 0.2895=2.32 hours

d) Average waiting time for a customer = 0.237/6= 0.0395

e) Number of hours each week a tax advisor spends performing his job= (6/12)*5

= 20hours/week.

f) Probability a customer has to wait before he gets serviced

= [(μ∗(λ/μ)^c) / ((c-1)! (Cμ–λ)] Po

=0.2368

g) Expected number of idle tax advisors at any specified time= 3p0+2p1+p2

=1.5

p1= (1/n!)* (λ/μ)^n p0= 0.3157

p2= (1/n!)* (λ/μ)^n p0= 0.2368

Example 5.1.5:

Arrivals at a telephone booth are considered to be Poisson with an average time of 10 minutes between one arrival and the next. The length of a phone call is assumed to be distributed exponentially with mean 3 minutes.

a) What is the probability that a person arriving at the booth will have to wait?
b) What is the average length of the queue that forms from time to time?
c) The telephone department will install a second booth when convinced that an arrival would except to have to wait at least three minutes for the phone. By how much must the flow of arrivals be increased in order to justify second booth?

SOLUTION:

Given ë = 1/10; μ = 1/3

a) Prob (w>0) = 1- P_0 = λ / μ = 3/10 = .3

b) (L/L>0) = μ(μ-λ) = (1/3) ((1/3)-10) = 1.43 persons

c) Wq = λ/ (μ(μ-ë))

Since Wq = 3, μ=1/3 ë=ë' for second booth.

3 = λ' / ((1/3) ((1/3) - ë')) = .16

Hence, increase in the arrival rate = 0.16 – 0.10 =.06 arrival per minute

Example 5.1.6:

Customers arrive at a one window drive in bank according to Poisson distribution with mean 10 per hour. Service time per customer is exponential with mean 5 minutes. The

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space in front of the window including that for the serviced car can accommodate a maximum of 3 cars. Others can wait outside this space.

1) What is the probability that an arriving customer can drive directly to the space in front of the window?
2) What is the probability that an arriving customer will have to wait outside the indicated space?
3) How long is an arriving customer expected to wait before starting service?

SOLUTION:

λ = 10 per hour

μ = 60/5 = 12 per hour

ñ = λ / μ = 10/12

P_0= 1- λ / μ = (1-(10/12))

1) The probability that an arriving customer can drive directly to the space in front of the window

$$P_0+P_1+P_2 = P_0 + (\lambda / \mu)P_0 + (\lambda / \mu)^2P_0$$

$$= P_0(1+(\lambda / \mu)+(\lambda / \mu)^2)$$

$$= (1-(10/12))\ (1 + (10/12) +(100/144))$$

=0.42

2) Probability that an arriving customer will have to wait outside the indicted space = 1- 0.42 = .58

3) Average waiting time of a customer in a queue = (ë / µ)(1/(µ-ë))

= 10 / (12(12-10)) = .417 hours

Exa

On an average 96 patients per 24 hour day require the service of an emergency clinic. Also on an average a patient requires 10 minutes of active attention. Assume that the facility can handle only one emergency at a time. Suppose that it cost the clinic Rs.100 per patient treated to obtain an average servicing time of 10 minutes and thus each minute of decrease in this average time would have to be budgeted by the clinic to decrease the average size of the queue from 4/3 patients to ½ patients?

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SOLUTION:

Given $\lambda = 96/(24*60) = 1/15$ patient/minute

$\mu = 1/10$ patient/minute

Average number of patients in the queue

$Lq = (\lambda / \mu) - (\lambda/(\lambda-\mu)) = (1/15)^2 / ((1/10 - 1/15)(1/10)) = 4/3$ patients

But $Lq = 4/3$ is reduced to $Lq' = ½$

Substituting $Lq' = ½$, $\lambda=\lambda' = 1/15$ in the formula

$Lq' = \lambda'^2/(\mu'(\mu'-\lambda'))$

$½ = (1/15)^2/ (\mu'(\mu'-1/15))$

$\mu' = 2/15$ patients/minute

Hence the average rate of treatment required is $1/ \mu' = 7.5$ minutes. Decrease in the time required by each patient

$= 10 -(15/2) = 5/2$ minutes

The budget required for each patient = 100 + (5*10/2) = Rs. 125

So in order to get the required size of the queue the budget should be increased from Rs.100 to Rs.125 per patient.

Example 5.1.8

In a railway marshalling yard, goods train arrive at rate of 30 trains per day. Assuming that the inter arrival time and the service time distribution follows an exponential distribution with an average of 30 minutes. Calculate the following

1) The mean queue size
2) The probability that queue exceeds 10
3) If the input of the train increases to an average of 33 per day, what will be the changes in 1 and 2?

SOLUTION

Given $\lambda = 30/(60*24) = 1/48$ trains/minute

$\mu = 1/36$ trains/minute

$\tilde{n} = \lambda / \mu = 36/48 = .75$

1) $Ls = \tilde{n}/(1-\tilde{n}) = 0.75 / (1-0.75) = 3$ trains

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2) $P(>=10) = (0.75)^{10} = .056$

3) When input increases to 33 train per day

We have ë= 33 /(60*24) = 1/480 trains/minute

ñ = ë / µ = .825

Ls = ñ/(1-ñ) = 0.825 / (1-0.825) = 5 trains approximately

Also $ñ(>=10) = ñ^{10} = .825^{10} = .1460$

5.2 REPLACEMENT MODELS

INTRODUCTION

The replacement problems are concerned with the situations that arise when some items such as machines, electric light bulbs etc need replacement due to their decreased efficiency, failure or breakdown. This decreased efficiency or complete breakdown may be either gradual or sudden.

Following are some of the situations which demand the replacement of certain items.

1) The old items have become inefficient or require expensive maintenance.
2) The old item has failed due to accident or otherwise and does not work at all, or the old item is expected to fail shortly.
3) A better design of equipment has been developed or due to obsolescence.

The replacement situations may be divided into four categories.

1) Replacement of capital equipment that suffers heavy depreciation in the course of the time (e.g.) machines tool, buses in a transport organization, planes etc.
2) Group replacement of items that fail completely (e.g.) light bulbs, radio tubes etc.
3) Problems of mortality and staffing
4) Miscellaneous problems.

We will deal only with the first two cases, that is replacement of capital items and that replacement of items that fail suddenly.

5.2.1 Replacement of Items that deteriorate with time (Capital Items)

Generally the cost of maintenance and repair of certain items increases with time and at one stage these costs become so high that it is more economical to replace the item by a new one. At this point a replacement is justified. An example is explained here

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EXAMPLE 5.2.1:

A machine costs Rs. 10000. Its operating cost and resale values are given below:

Year	1	2	3	4	5	6	7	8
Operating cost	1000	1200	1400	1700	2000	2500	3000	3500
Resale Value	6000	4000	3200	2600	2500	2400	2000	1600

SOLUTION:

Given the cost of the equipment (C) = Rs. 10000

To determine the optimal time, n, when the equipment should be replaced, we calculate an average total cost per year as shown in the following table.

The first and the second columns are part of the data. The third column is the cumulative operating cost (that is operating cost added to every subsequent year) The fourth column (Scrap Value) is also part of the data. The fifth column is the capital cost – scrap value. The sixth column is the addition of column 3 and 5. The last column (7^{th} column) is the computation of the average cost based on Total cost divided by the corresponding year. That is, column 6 divided by column 1.

1 Year (n)	**2** Operating Cost	**3** Cumulative Operating Cost (COC)	**4** Scrap Value (S)	**5** Capital Cost (C) - Scrap Value (S)	**6=3+5** Total Cost (TC) = (C-S) + COC	**7=6/1** Average Cost (AC) = TC/n
1	1000	1000	6000	4000	5000	5000
2	1200	2200	4000	6000	8200	4100
3	1400	3600	3200	6800	10400	3466
4	1700	5300	2600	7400	12700	3175
5	2000	7300	2500	7500	14800	2960
6	2500	9800	2400	7600	17400	2900
7	3000	12800	2000	8000	20800	2971
8	3500	16300	1600	8400	24700	3087

From the table it is clear that the average annual cost is minimum at the end of the 6^{th} year (Rs. 2900), we conclude that the equipment should be replaced at the end of the 6^{th} year.

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EXAMPLE 5.2.2:

A machine is purchased for RS. 1 lakh. Running cost is RS.10, 000 each for 1st 2 years and increases by RS. 5000 for the next 2 years. After that it increases by Rs. 10,000 every year. If scrap value or resale price is Rs. 8,000 find when it should be replaced.

SOLUTION:

Initial Investment (I.I)– Rs 100000, Scrap Value (S.V) – Rs 8,000. Let us prepare the table just the same way we had prepared in the previous example.

Year (n)	Running Cost (RC)	Cumulative RC (CRC)	Total Cost (TC) = (Initial Investment) – (ScrapValue)	Average cost (AC) = TC / n
1	10000	10000	102000	102000
2	10000	20000	112000	56000
3	15000	35000	127000	42333
4	20000	55000	147000	36750
5	30000	85000	177000	35400
6	40000	125000	217000	36167
7	50000	175000	267000	38143

(The year after which average cost increases is replacement year.)

Since average cost declines till 5years and increases thereafter, the machine should be replaced in the 5th year.

REPLACEMENT OF CAPITAL EQUIPMENTS CONSIDERING TIME VALUE OF MONEY

While the first two examples did not take the time value of money into account we shall now look into an example which considers the decreasing value of money over a period of time.

EXAMPLE 5.2.3:

The cost of a new machine is Rs.5000. The running cost of the nth year is given by Rn= 500 (n-1) where n =1, 2,…..... . Suppose money is worth 5% per year, after how many years will it be economical to replace the machine by a new one?

NOTES

SOLUTION:

The rate at which the value of money decreases is given by the Discount Factor (D.f)

$D.f = 1 / [1+(r/100)]^{n}$

where ‘r’ is the rate at which the value of money decreases and ‘n’ is the year. For example, the discount factor at the end of second year will take the value of n =2. This is shown below.

$D.f = 1 / (1 + (5/100))^{n-1} = .9523$

Now the running cost at the end of the second year is discounted by this value.

The first column is the year. The second column is the running cost calculated from the relation given in the problem. Rn = 500 (n – 1) where ‘n’ is the subsequent years. For example for the first year the Running cost = 500 (1 -1) = 0. In the second year it is 500 (2 – 1) = 500. The running cost for all the years are calculated in a similar manner. The third column is the discount factor as explained above. The fourth column is the discounted running cost which is = the running cost x discount factor. The fifth column is the cumulative discounted running cost. The sixth column is the capital cost + the cumulative discounted running cost. The seventh column is the cumulative discount factor. And the computation of the average cost is dividing the total cost by the cumulative discount factor. That is column 6 divided by 7.

A small word of caution here. DO NOT DIVIDE THE TOTAL COST BY THE YEAR. <u>Divide the total cost by the cumulative discount factor.</u>

1	2	3	4	5	6	7	8 = 6/7
Year	Running Cost	Discount Factor	Discounted Running Cost (D.R.) = Running Cost x Discount Factor	Cumulative Discounted Running Cost	Capital Cost + Cumulative Discounted Running Cost	Cumulative Discount Factor	Average Cost
n	R (n-1)	D.f (n-1)	R (n-1) x D.f (n-1)	? R(n) D.f (n-1)	C + ? R(n) D.f (n-1)	D.f (n-1)	A.C (n) = 6 / 7
1	0	1	0	0	5000	1	5000
2	500	0.9523	476.15	476.15	5476.15	1.9523	2804
3	1000	0.9069	906.9	1383.05	6383.05	2.8592	2232
4	1500	0.8636	1295.4	2678.45	7678.45	3.7228	2062
5	2000	0.8224	1644.8	4323.25	9323.25	4.5452	2051
6	2500	0.7832	1958	6281.25	11281.25	5.3284	2117

Since the weighted average cost is minimum at the end of 5^{th} year, and begins to increase thereafter, it is economical to replace the machine by a new one at the end of 5 years.

NOTES

EXAMPLE 5.2.4:

A machine costs Rs. 6000. The running cost and the salvage value at the end of the year is given in the table below.

Year	1	2	3	4	5	6	7
Running cost	1200	1400	1600	1800	2000	2400	3000
Resale Value	4000	2666	2000	1500	1000	600	600

If the interest rate is 10% per year, find when the machine is to be replaced.

SOLUTION:

C =6000

The rate at which the value of money decreases is given by the Discount Factor (D.f)

D.f = $1 / [1+(r/100)]^n$

where 'r' is the rate at which the value of money decreases and 'n' is the year. For example, the discount factor at the end of second year will take the value of n =2. This is shown below.

D.f = $1 / (1 + (10/100))^{n-1}$ = .9091

Now the running cost at the end of the second year is discounted by this value.

Unlike in the previous example we will have to discount the scrap value also every year. The difference is the running cost is discounted from the second year but the scrap value is discounted from the first year onwards. All other calculations are similar to the previous example.

NOTES

1	2	3	4	5	6	7	8	9	10=9/8
Year n	Running Cost R (n-1)	Discount Factor D.f (n-1)	Discounted Running Cost R (n-1) D.f (n-1)	Cumulative Discounted Running Cost ? R(n) D.f (n-1)	Scrap Cost S(n)	Discounted Scrap Value S(n) D.f (n)	Cumulative Discount Factor D.f (n-1)	C - S(n)V(n)+ ? R(n) D.f (n-1)	Average Cost (n)
1	1200	1	1200	1200	4000	3636.4	1	3563.6	3563
2	1400	0.9091	1272.74	2472.74	2666	2203.449	1.9091	6269.291	3283
3	1600	0.8265	1322.4	3795.14	2000	1502.6	2.7356	8292.54	3031
4	1800	0.7513	1352.34	5147.48	1500	1024.5	3.4869	10122.98	2903
5	2000	0.683	1366	6513.48	1000	620.9	4.1699	11892.58	2852
6	2400	0.6209	1490.16	8003.64	600	338.64	4.7908	13665	2852
7	3000	0.5644	1693.2	9696.84	600	307.916	5.3552	15388.924	2873

NOTES

The machine is to be replaced at the end of the 6th year.

Another simple problem without scrap value is shown for your understanding.

EXAMPLE 5.2.5: A machine costs RS. 15,000, running cost for the years are given below:

Year	1	2	3	4	5	6	7
RC	2500	3000	4000	5000	6000	8000	10000

Find optimal replacement period if capital is worth 10% and machine has no scrap value.

SOLUTION:

The rate at which the value of money decreases is given by the Discount Factor (D.f)

$D.f = 1 / [1+(r/100)]^n$

where 'r' is the rate at which the value of money decreases and 'n' is the year. For example, the discount factor at the end of second year will take the value of n =2. This is shown below.

$D.f = 1 / (1 + (10/100))^{n-1} = .9091$

Year	**Running cost (a)**	**Discount factor (b)**	**(a)*(b) df RC**	**Cum df**	**Cum discounted RC**	**TC**	**AC**
1	2500	1	2500	1	2500	17500	17500
2	3000	0.909	2727	1.909	5227	20227	10596
3	4000	0.8264	3306	2.7354	8533	23533	8603
4	5000	0.7513	3757	3.4867	12290	27290	7827
5	6000	0.6830	4440	4.1697	16730	31730	7610
6	8000	0.6209	4967	4.7906	21697	36697	7660
7	10000	0.5645	5645	5.3551	27342	42342	7907

Since average cost decreases till the 5th year and increases from the 6th year, machine should be replaced at the end of the 5th year.

We'll next look into a problem relating to group replacement.

NOTES

5.2.2 Replacement of Items that fail suddenly (Group Replacement)

As we had discussed earlier group replacement is concerned with items that fail suddenly and these are small items where buying them one at a time may a little more expensive than replacing them in bulk. An example is illustrated for your understanding.

EXAMPLE 5.2.6

The following mortality rates have been resolved for a light bulb type: -

Week	1	2	3	4	5
% Of failure	10	25	50	80	100

There are 1000 bulbs and it costs Rs 2/- to replace an individual burned out bulb, if all the bulbs were replaced together it would cost 50p/ bulb. Should group replacement or individual replacement be followed?

SOLUTION:

We can interpret the data like this. Out of 1000 bulbs 10 % fail by end of the first week. That is 100 bulbs fail by the end of the first week. Similarly 250 (25%) bulbs will fail by the end of the second week. By this estimate it is clear by the end of the fifth week all bulbs (100%) would have failed.

Week	1	2	3	4	5
% Of failure	10	25	50	80	100

So the maximum possible life span of the bulbs is 5 weeks. The data shown above are the cumulative probabilities. For example, in the second week 25% of the bulbs fail which includes 10% in the first week and the remaining 15% in the second week. So in the second week only 15% of the bulbs fail. This computation is shown in the table below.

Week (i)	1	2	3	4	5
% Of failure	10	25	50	80	100
Combined probability	0.1	0.25	0.5	0.8	1
Individual probability (p_i)	0.1	0.15	0.25	0.3	0.2

First, the cost of Individual replacement is calculated.

NOTES

The average life span of the bulbs is computed from the relationship shown below. It is the summation of i * p_i where i represents the time (that is i=1,2,3,4,5 weeks) and p_i represents the probability of failure in each week.

$$\text{Average life span} = \sum_{i=1}^{5} i * p_i$$

=1*0.1+2*0.15+3*0.25+4*0.3+5*0.2

=3.35 weeks

Therefore the average no. of bulbs that fail= 1000/3.35= 299 bulbs/week

Cost of individual replacement =299 x Rs 2 / bulb = Rs.598/week.

No= 1000

N1= No*P1= 1000*0.1= 100 bulbs, similarly

N2 = N1* P1 + No * P2 = 100 * 0.1 + 1000 * 0.15 = 160 bulbs

N3 = N2 * P1 + N1 * P2 + No * P3 = 281 bulbs

N4 = N3 * P1 + N2 * P2 + N1 * P3 + No * P4 = 377 bulbs

N5 = N4 * P1 + N3 * P2 + N2 * P3 + N1 * P4 + No * P5 = 350

The number of bulbs likely to be replaced at the end of each week is shown in the table below.

Week	1	2	3	4	5
Number of bulbs likely to be replaced	100	160	281	377	350

The table showing the cost computations for group replacement is shown below.

NOTES

1	2	3	4	5	6 = 5 / 1
Week (i)	**No. of bulbs replaced Ni**	**Cumulative number of bulbs replaced (CN)**	**Cumulative Cost of number of bulbs replaced (CTN)= Rs 2/- * CN**	**TC =Initial cost (1000 * 0.5 =500) + CTN (cumulative cost of individual replacement)**	**AC=TC / i**
1	100	100	200	700	100
2	160	260	520	1020	510
3	281	541	1082	1582	527
4	377	918	1836	2336	584
5	350	1268	2536	3036	607

Since the average cost begins to increase after the second week we can replace all bulbs (group replacement) at end of every 2nd week, as individual average cost of replacement is higher at RS.598/ week.

Exercises:

Have you understood?

1) When capital items are to be replaced what type of replacement model is to be followed?
2) What replacement model is appropriate for small items that fail suddenly?
3) What is the significance of the discount factor?
4) What is the purpose of studying Queueing models.
5) What happens if the arrival rate is more than the service rate?

Solve

REPLACEMENT MODELS

1. A machine costs Rs.10, 000. Its operating cost and resale values are given below.

Year	1	2	3	4	5	6	7	8
Operating cost	1000	1200	1400	1700	2000	2500	3000	3500
Resale value	6000	4000	3200	2600	2500	2400	2000	1600

Determine when the machine should be replaced?

Ans: The machine should be replaced at the end of the 6th year.

NOTES

2. A machine costs Rs.6000. Its operating cost and resale values are given below.

Year	1	2	3	4	5	6	7	8
Operating cost	1000	1200	1400	1800	2300	2800	3400	4000
Resale value	3000	1500	750	375	200	200	200	200

Determine when the machine should be replaced.

Ans: The machine should be replaced at the end of the 5th year.

3. A machine costs Rs.15, 000. Its operating cost is given below.

Year	1	2	3	4	5	6	7
Operating cost	2500	3000	4000	5000	6500	8000	10000

Determine when the machine should be replaced if the capital is worth 10% and the machine has no salvage value.

Ans: The machine should be replaced at the end of the 5th year.

4.A machine costs Rs.6000. Its operating cost and salvage values are given below.

Year	1	2	3	4	5	6	7
Operating cost	1200	1400	1600	1800	2000	2400	3000
Salvage value	4000	2666	2000	1500	1000	600	600

Determine when the machine should be replaced if the interest rate in 10% per year.

Ans: The machine should be replaced at the end of the 4th year.

5. The mortality rate of a certain type of fuses is as given below.

Week	1	2	3	4	5
% Failing by the end of the week	5	15	35	75	100

There are totally 1000 fuses in use and it costs Rs.5 to replace an individual fuse. If all the fuses were replaced simultaneously it would cost Rs1.25 per fuse. Determine which is the optimal policy of replacing the fuses. If group replacement is optimal find out at what interval should it be carried on.

NOTES

Ans: Average cost of individual replacement is Rs.1350.

Average cost of group replacement by the end of 2nd week is Rs.1005.

QUEUING THEORY

1. Workers come to a tool storeroom to enquire about the tool for a particular job. The average time between the arrivals is 60 sec. The average service time is 40 sec.

Determine

a) Average queue length (1.33 workers)
b) Average no of workers in the system (2 workers)

2. Customers arrive to a bank at the rate of 30 per hour. The teller takes on an average a minute and a half to serve the customers.

a) Calculate the percentage of time the teller is busy. (75%)
b) Calculate the average waiting time of a customer. (6 minutes)

3. A super market has three sales girls .If the service time for each customer is 20 minutes and the arrival rate is 6 per hour, determine

a) Probability that the customer has to wait (0.444)
b) Average no of customers waiting in the queue (0.889)
c) Average time spent in the supermarket (28.9 min)

4. There are two clerks in a university to receive dues from the students. If the service time is 4minutes per students and the arrival rate I 10 per hour determine

a) Probability of having to wait for the service (0.167)
b) Percentage of idle time for each clerk (0.67)

Summary

This unit would have given an overview on the relevance of replacement models to present day applications and how queuing models are significant in modeling service based systems.

NOTES

Table VI : Random Digits

5294	6695	7471	9235	7132	2330
4339	0587	2009	2353	3545	1175
9821	6851	0854	6413	4368	7996
0393	7985	1709	5643	4697	3800
1933	2685	0093	6201	2533	6148
6780	7309	8118	2292	1642	7949
8016	5775	6688	0102	9684	6589
1129	8237	1451	1006	9988	1821
5635	8142	1143	3833	8731	2938
8527	7400	6440	5748	2444	1093
0652	1488	3884	8103	1157	3980
2825	5571	2717	2184	7843	6814
4398	3132	8710	1246	8319	2208
5395	9559	2227	2701	5367	4534
2920	4973	6841	1884	2137	1207
0111	8260	9023	1368	6122	3001
0272	4702	8030	9239	7092	2201
8414	5198	7896	7026	1111	1331
9005	8021	0205	6855	7342	1548
4600	0560	8892	6515	3237	7916
7631	7361	9031	9749	7000	6032
4114	2228	4595	0277	7193	6515
2478	8899	1901	2176	4140	4482
8522	9041	4748	5044	3897	5606
7755	2031	1191	7745	0124	6341
0456	9977	6923	2539	6678	9906
7196	7275	4971	0110	0220	6817
8252	8006	3957	7149	3576	6591
3026	0014	9368	7262	8134	4141
8245	4972	9200	8898	5225	6855