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CLASS XII

MATHEMATICS

	Units	Weightage (Marks)
(i)	Relations and Functions	10
(ii)	Algebra	13
(iii)	Calculus	44
(iv)	Vector and Three Dimensional Geometry	17
(v)	Linear Programming	06
(vi)	Probability	10
		Total : 100

Unit I : RELATIONS AND FUNCTIONS

1. Relations and Functions **(10 Periods)**

Types of Relations : Reflexive, symmetric, transitive and equivalence relations. One to one and onto functions, composite functions, inverse of a function. Binary operations.

2. Inverse Trigonometric Functions **(12 Periods)**

Definition, range, domain, principal value branches. Graphs of inverse trigonometric functions. Elementary properties of inverse trigonometric functions.

Unit II : ALGEBRA

1. Matrices **(18 Periods)**

Concept, notation, order, equality, types of matrices, zero matrix, transpose of a matrix, symmetric and skew symmetric matrices. Addition, multiplication and scalar multiplication of matrices, simple properties of addition, multiplication and scalar multiplication. Non-commutativity of multiplication of matrices and existence of non-zero matrices whose product is the zero matrix (restrict to square matrices of order 2). Concept of elementary row and column operations. Invertible matrices and proof of the uniqueness of inverse, if it exists; (Here all matrices will have real entries).

2. Determinants **(20 Periods)**

Determinant of a square matrix (up to 3 × 3 matrices), properties of determinants, minors, cofactors and applications of determinants in finding the area of a triangle. adjoint and inverse of a square matrix. Consistency, inconsistency and number of solutions of system of linear equations by examples, solving system of linear equations in two or three variables (having unique solution) using inverse of a matrix.

Unit III : CALCULUS

1. Continuity and Differentiability **(18 Periods)**

Continuity and differentiability, derivative of composite functions, chain rule, derivatives of inverse trigonometric functions, derivative of implicit function. Concept of exponential and logarithmic functions and their derivatives. Logarithmic differentiation. Derivative of functions expressed in parametric forms. Second order derivatives. Rolle's and Lagrange's mean Value Theorems (without proof) and their geometric interpretations.

2. Applications of Derivatives **(10 Periods)**

Applications of Derivatives : Rate of change, increasing/decreasing functions, tangents and normals, approximation, maxima and minima (first derivative test motivated geometrically and second derivative test given as a provable tool). Sample problems (that illustrate basic principles and understanding of the subject as well as real-life situations).

3. Integrals **(20 Periods)**

Integration as inverse process of differentiation. Integration of a variety of functions by substitution, by partial fractions and by parts, only simple integrals of the type to be evaluated.

$$\int \frac{dx}{x^2 \pm a^2}, \int \frac{dx}{\sqrt{x^2 \pm a^2}}, \int \frac{dx}{\sqrt{a^2 - x^2}}, \int \frac{dx}{ax^2 + bx + c}, \int \frac{dx}{\sqrt{ax^2 + bx + c}}$$

$$\int \frac{(px+q)}{ax^2+bx+c}dx, \int \frac{(px+q)}{\sqrt{ax^2+bx+c}}dx, \int \sqrt{a^2 \pm x^2}\, dx \text{ and } \int \sqrt{x^2 - a^2}\, dx$$

Definite integrals as a limit of a sum, Fundamental Theorem of Calculus (without proof). Basic properties of definite integrals and evaluation of definite integrals.

4. Applications of the Integrals (10 Periods)

Application in finding the area under simple curves, especially lines, area of circles/parabolas/ellipses (in standard form only), area between the two above said curves (the region should be clearly identifiable).

5. Differential Equations (10 Periods)

Definition, order and degree, general and particular solutions of a differential equation. Formation of differential equation whose general solution is given. Solution of differential equations by method of separation of variables, homogeneous differential equations of first order and first degree. Solutions of linear differential equation of the type :

$$\frac{dy}{dx}+p(x)y=q(x), \text{ where } p(x) \text{ and } q(x) \text{ are function of } x.$$

Unit IV : VECTORS AND THREE-DIMENSIONAL GEOMETRY

1. Vectors (12 Periods)

Vectors and scalars, magnitude and direction of a vector. Direction cosines/ ratios of vectors. Types of vectors (equal, unit, zero, parallel and collinear vectors), position vector of a point, negative of a vector, components of a vector, addition of vectors, multiplication of a vector by a scalar, position vector of a point dividing a line segment in a given ratio. Scalar (dot) product of vectors, projection of a vector on a line. Vector (cross) product of vectors.

2. Three-Dimensional Geometry (12 Periods)

Direction cosines/ratios of a line joining two points. Cartesian and vector equation of a line, coplanar and skew lines, shortest distance between two lines. Cartesian and vector equation of a plane. Angle between (*i*) two lines, (*ii*) two planes, (*iii*) a line and a plane. Distance of a point from a plane.

Unit V : LINEAR PROGRAMMING

(12 Periods)

1. **Linear Programming :** Introduction, definition of related terminology such as constraints, objective function, optimization, different types of linear programming (L.P.) problems, mathematical formulation of L.P. problems, graphical method of solution for problems in two variables, feasible and infeasible regions, feasible and infeasible solutions, optimal feasible solutions (up to three non-trivial constraints).

Unit VI : PROBABILITY

1. **Probability** **(18 Periods)**

   Multiplication theorem on probability. Conditional probability, independent events, total probability, Baye's theorem, Random variable and its probability distribution, mean and variance of haphazard variable. Repeated independent (Bernoulli) trials and Binomial distribution.

CONTENTS

CHAPTER 1

RELATIONS AND FUNCTIONS

IMPORTANT POINTS TO REMEMBER

- Relation R from a set A to a set B is subset of $A \times B$.
- $A \times B = \{(a, b) : a \in A, b \in B\}$.
- If $n(A) = r$, $n(B) = s$ from set A to set B then $n(A \times B) = rs$.

  and no. of relations $= 2^{rs}$
- ϕ is also a relation defined on set A, called the void (empty) relation.
- $R = A \times A$ is called universal relation.
- **Reflexive Relation :** Relation R defined on set A is said to be reflexive iff $(a, a) \in R \ \forall \ a \in A$
- **Symmetric Relation :** Relation R defined on set A is said to be symmetric iff $(a, b) \in R \Rightarrow (b, a) \in R \ \forall \ a, b, \in A$
- **Transitive Relation :** Relation R defined on set A is said to be transitive if $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R \ \forall \ a, b, c \in R$
- **Equivalence Relation :** A relation defined on set A is said to be equivalence relation iff it is reflexive, symmetric and transitive.
- **One-One Function :** $f : A \rightarrow B$ is said to be one-one if distinct elements in A has distinct images in B. *i.e.* $\forall \ x_1, x_2 \in A$ *s.t.* $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

  OR

  $\forall \ x_1, x_2 \in A$ s.t. $f(x_1) = f(x_2)$

  $\Rightarrow \ x_1 = x_2$

  One-one function is also called injective function.

- Onto function (surjective) : A function $f : A \to B$ is said to be onto iff $R_f = B$ *i.e.* $\forall\ b \in B$, there exist $a \in A$ s.t. $f(a) = b$
- A function which is not one-one is called many-one function.
- A function which is not onto is called into.
- **Bijective Function :** A function which is both injective and surjective is called bijective.
- **Composition of Two Function :** If $f : A \to B$, $g : B \to C$ are two functions, then composition of f and g denoted by gof is a function from A to C given by, $(gof)\ (x) = g\ (f\ (x))\ \forall\ x \in A$

  Clearly gof is defined if Range of $f \subset$ domain of g. Similarly fog can be defined.
- **Invertible Function :** A function $f : X \to Y$ is invertible iff it is bijective.

  If $f : X \to Y$ is bijective function, then function $g : Y \to X$ is said to be inverse of f iff $fog = I_y$ and $gof = I_x$

  when I_x, I_y are identity functions.
- g is inverse of f and is denoted by f^{-1}.
- **Binary Operation :** A binary operation '*' defined on set A is a function from $A \times A \to A$. $*\ (a, b)$ is denoted by $a * b$.
- Binary operation * defined on set A is said to be commutative iff

  $$a * b = b * a\ \forall\ a, b \in A.$$
- Binary operation * defined on set A is called associative iff $a * (b * c) = (a * b) * c\ \forall\ a, b, c \in A$
- If * is Binary operation on A, then an element $e \in A$ is said to be the identity element iff $a * e = e * a\ \forall\ a \in A$
- Identity element is unique.
- If * is Binary operation on set A, then an element b is said to be inverse of $a \in A$ iff $a * b = b * a = e$
- Inverse of an element, if it exists, is unique.

VERY SHORT ANSWER TYPE QUESTIONS

1. If A is the set of students of a school then write, which of following relations are. (Universal, Empty or neither of the two).

   $R_1 = \{(a, b) : a, b$ are ages of students and $|a - b| \geq 0\}$

   $R_2 = \{(a, b) : a, b$ are weights of students, and $|a - b| < 0\}$

   $R_3 = \{(a, b) : a, b$ are students studying in same class$\}$

   $R_4 = \{(a, b) : a, b$ are age of students and $a > b\}$

2. Is the relation R in the set $A = \{1, 2, 3, 4, 5\}$ defined as $R = \{(a, b) : b = a + 1\}$ reflexive?

3. If R, is a relation in set N given by

   $R = \{(a, b) : a = b - 3, b > 5\}$,

   then does elements $(5, 7) \in R$?

4. If $f : \{1, 3\} \rightarrow \{1, 2, 5\}$ and $g : \{1, 2, 5\} \rightarrow \{1, 2, 3, 4\}$ be given by

   $f = \{(1, 2), (3, 5)\}$, $g = \{(1, 3), (2, 3), (5, 1)\}$

   Write down gof.

5. Let $g, f : R \rightarrow R$ be defined by

   $g(x) = \dfrac{x+2}{3}, f(x) = 3x - 2$. Write fog.

6. If $f : R \rightarrow R$ defined by

   $f(x) = \dfrac{2x-1}{5}$

   be an invertible function, write $f^{-1}(x)$.

7. If $f(x) = \dfrac{x}{x+1} \forall x \neq -1$, Write $fo\, f(x)$.

8. Let * is a Binary operation defined on R, then if

   (i) $a * b = a + b + ab$, write $3 * 2$

(ii) $a * b = \frac{(a+b)^2}{3}$, Write $(2 * 3) * 4$.

(iii) $a * b = 4a - 9b^2$, Write $(1 * 2) * 3$.

9. If $n(A) = n(B) = 3$, Then how many bijective functions from A to B can be formed?

10. If $f(x) = x + 1$, $g(x) = x - 1$, Then (gof) (3) = ?

11. Is $f : N \to N$ given by $f(x) = x^2$ is one-one? Give reason.

12. If $f : R \to A$, given by

    $f(x) = x^2 - 2x + 2$ is onto function, find set A.

13. If $f : A \to B$ is bijective function such that $n(A) = 10$, then $n(B)$ = ?

14. If $n(A) = 5$, then write the number of one-one functions from A to A.

15. $R = \{(a, b) : a, b \in N, a \neq b$ and a divides $b\}$. Is R reflexive? Give reason?

16. Is $f : R \to R$, given by $f(x) = |x - 1|$ is one-one? Give reason?

17. $f : R \to B$ given by $f(x) = \sin x$ is onto function, then write set B.

18. Is $f : R \to R$, $f(x) = x^3$ is bijective function?

19. If '*' is a binary operation on set Q of rational numbers given by $a * b = \frac{ab}{5}$ then write the identity element in Q.

20. If * is Binary operation on N defined by $a * b = a + ab \ \forall \ a, b \in N$. Write the identity element in N if it exists.

SHORT ANSWER TYPE QUESTIONS (4 Marks)

21. Check the following functions for one-one and onto.

    (a) $f : R \to R$, $f(x) = \frac{2x - 3}{7}$

    (b) $f : R \to R$, $f(x) = x^2 + 2$

(c) $f : R \to R$, $f(x) = |x + 1|$

(d) $f : R - \{2\} \to R$, $f(x) = \dfrac{3x-1}{x-2}$

(e) $f : R \to R$, $f(x) = \sin x$

(f) $f : R \to [-1, 1]$, $f(x) = \sin^2 x$

(g) $f : R \to R$, $f(x) = x^2 - 2x + 3$

22. Show $f : R \to R$ given by $f(x) = \dfrac{3x-1}{5}$ is bijective. Also find f^{-1}.

23. See $f : R - \left\{\dfrac{-4}{3}\right\} \to R - \left\{\dfrac{4}{3}\right\}$ be a function given by $f(x) = \dfrac{4x}{3x+4}$. Show that f is invertible with $f^{-1}(x) = \dfrac{4x}{4-3x}$.

24. Let R be the relation on set $A = \{x : x \in Z, 0 \le x \le 10\}$ given by $R = \{(a, b) : (a - b)$ is multiple of $4\}$, is an equivalence relation. Also, write all elements related to 4.

25. Show that function $f : A \to B$ defined as $f(x) = \dfrac{3x + 4}{5x - 7}$ where $A = R - \left\{\dfrac{7}{5}\right\}$, $B = R - \left\{\dfrac{3}{5}\right\}$ is invertible and hence find f^{-1}.

26. Let * be a binary operation on Q. Such that $a * b = a + b - ab$.

(i) Prove that * is commutative and associative.

(ii) Find identify element of * in Q (if it exists).

27. If * is a binary operation defined on $R - \{0\}$ defined by $a * b = \dfrac{2a}{b^2}$, then check * for commutativity and associativity.

28. If $A = N \times N$ and binary operation * is defined on A as

$(a, b) * (c, d) = (ac, bd)$.

(i) Check * for commutativity and associativity.

(ii) Find the identity element for * in A (If it exists).

29. Show that the relation R defined by $(a, b)\ R(c, d) \Leftrightarrow a + d = b + c$ on the set $N \times N$ is an equivalence relation.

30. Let * be a binary operation on set Q defined by $a * b = \frac{ab}{4}$, *show that*

   (i) 4 is the identity element of * on Q.

   (ii) Every non zero element of Q is invertible with

   $a^{-1} = \frac{16}{a}, \quad a \in Q - \{0\}.$

31. Show that $f : R_+ \rightarrow R_+$ defined by $f(x) = \frac{1}{2x}$ is bijective where R_+ is the set of all non-zero positive real numbers.

32. Consider $f : R_+ \rightarrow [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$ show that f is invertible with $f^{-1} = \frac{\sqrt{x+6}-1}{3}$.

33. If '*' is a binary operation on R defined by $a * b = a + b + ab.$ Prove that * is commutative and associative. Find the identify element. Also show that every element of R is invertible except –1.

34. If $f, g : R \rightarrow R$ defined by $f(x) = x^2 - x$ and $g(x) = x + 1$ find (fog) (x) and (gof) (x). Are they equal?

35. Prove that composition of two one-one functions is also one-one?

36. $f : R \rightarrow R,\ g : R \rightarrow R$ given by $f(x) = [x],\ g(x) = |x|$ then find

   $(fog)\left(\frac{-2}{3}\right)$ and $(gof)\left(\frac{-2}{3}\right)$.

H.O.T.S.

37. $f : [1, \infty) \rightarrow [2, \infty)$ is given by $f(x) = x + \frac{1}{x}$, then find $f^{-1}(x)$.

38. $f(x) = \left\{4 - (x-7)^3\right\}^{1/4}$, find $f^{-1}(x)$

39. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$ show that $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$.

ANSWERS

VERY SHORT ANSWER TYPE QUESTIONS

1. R_1 : is universal relation.

   R_2 : is empty relation.

   R_3 : is neither universal nor empty.

   R_4 : is neither universal nor empty.

2. No, R is not reflexive.

3. $(5, 7) \notin R$

4. $gof = \{(1, 3), (3, 1)\}$

5. $(fog)(x) = x \ \forall \ x \in R$

6. $f^{-1}(x) = \frac{5x+1}{2}$

7. $(fof)(x) = \frac{x}{2x+1}, x \neq -\frac{1}{2}$

8. (i) 3 * 2 = 11

   (ii) $\frac{1369}{27}$

   (iii) –209

9. 6

10. 3

11. Yes, f is one-one $\because \forall\ x_1, x_2 \in N \Rightarrow x_1^2 = x_2^2$.

12. $A = [1, \infty)$ because $R_f = [1, \infty)$

13. $n(B) = 10$

14. 120.

15. No, R is not reflexive $\because (a, a) \notin R \ \forall \ a \in N$

16. f is not one-one functions

$\because$ $f(3) = f(-1) = 2$

$3 \neq -1$ *i.e.* distinct element has same images.

17. $B = [-1, 1]$

18. Yes

19. $e = 5$

20. Identity element does not exist.

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

21. (a) Bijective
    (b) Neither one-one nor onto.
    (c) Neither one-one nor onto.
    (d) One-one, but not onto.
    (e) Neither one-one nor onto.
    (f) Neither one-one nor onto.
    (g) Neither one-one nor onto.

22. $f^{-1}(x) = \dfrac{5x+1}{3}$

24. Elements related to 4 are 0, 4, 8.

25. $f^{-1}(x) = \dfrac{7x+4}{5x-3}$

26. 0 is the identity element.

27. Neither commutative nor associative.

28. (i) Commutative and associative.
    (ii) (1, 1) is identity in $N \times N$

33. *0* is the identity element.

34. $(fog)\ (x) = x^2 + x$

$(gof)\ (x) = x^2 - x + 1$

Clearly, they are unequal.

35. **Hint :** Let $x_1, x_2 \in R$ s.t.

$(\text{fog})\ (x_1) = (\text{fog})\ (x_2)$

$\Rightarrow \quad f(g\ (x)) = f\ (g(x_2))$

$\Rightarrow \quad g(x_1) = g\ (x_2)$ $\quad (\because\ f$ is one-one)

$x_1 = x_2$ $\quad (\because\ g$ is one-one).

Hence $(fog)\ (x)$ is one-one function.

Similarly, $(gof)\ (x)$ is also one-one function.

36. $(fog)\left(\frac{-2}{3}\right) = 0$

$(gof)\left(\frac{-2}{3}\right) = 1$

37. $f^{-1}(x) = \frac{x + \sqrt{x^2 - 4}}{2}$

38. $f^{-1}(x) = 7 + (4 - x^4)^{1/3}$

CHAPTER 2

INVERSE TRIGONOMETRIC FUNCTIONS

IMPORTANT POINTS

- $\sin^{-1} x$, $\cos^{-1} x$, ... etc., are angles.
- If $\sin\theta = x$ and $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ then $\theta = \sin^{-1}x$ etc.
-

Function	***Domain***	***Range (Principal Value Branch)***
$\sin^{-1}x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}x$	R	$(0, \pi)$
$\sec^{-1}x$	$R - (-1, 1)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$\text{cosec}^{-1}x$	$R - (-1, 1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

- $\sin^{-1} (\sin x) = x \ \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

  $\cos^{-1} (\cos x) = x \ \forall \ x \in [0, \pi]$ etc.
- $\sin (\sin^{-1}x) = x \ \forall \ x \in [-1, 1]$

  $\cos (\cos^{-1}x) = x \ \forall \ x \in [-1, 1]$ etc.

- $\sin^{-1}x = \text{cosec}^{-1}\left(\frac{1}{x}\right) \forall x \in [-1, 1]$

  $\tan^{-1}x = \cot^{-1}(1/x) \ \forall \ x > 0$

  $\sec^{-1}x = \cos^{-1}(1/x), \ \forall \ |x| \geq 1$

- $\sin^{-1}(-x) = -\sin^{-1}x \ \forall \ x \in [-1, 1]$

  $\tan^{-1}(-x) = -\tan^{-1}x \ \forall \ x \in R$

  $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x \ \forall \ |x| \geq 1$

- $\cos^{-1}(-x) = \pi - \cos^{-1}x \ \forall \ \times \in [-1, 1]$

  $\cot^{-1}(-x) = \pi - \cot^{-1}x \ \forall \ x \in -R$

  $\sec^{-1}(-x) = \pi - \sec^{-1}x \ \forall \ |x| \geq 1$

- $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, \quad x \in [-1, 1]$

  $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2} \ \forall x \in R$

  $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2} \ \forall |x| \geq 1$

- $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right); \quad xy < 1.$

- $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right); \quad xy > -1.$

- $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right), |x| < 1$

VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

1. Write the principal value of

   (i) $\sin^{-1}\left(-\sqrt{3}/2\right)$ (ii) $\sin^{-1}\left(\sqrt{3}/2\right)$.

(iii) $\cos^{-1}\left(-\sqrt{3}/2\right)$ (iv) $\cos^{-1}\left(\sqrt{3}/2\right)$.

(v) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ (vi) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

(vii) cosec^{-1} (– 2). (viii) cosec^{-1} (2)

(ix) $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ (x) $\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

(xi) $\sec^{-1}$ (– 2). (xii) $\sec^{-1}$ (2).

(xiii) $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) + \cos^{-1}\left(\frac{-1}{2}\right) + \tan^{-1}\left(-1/\sqrt{3}\right)$

2. What is value of the following functions (using principal value).

(i) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$. (ii) $\sin^{-1}\left(-\frac{1}{2}\right) - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

(iii) $\tan^{-1}$ (1) – $\cot^{-1}$ (–1). (iv) $\cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$.

(v) $\tan^{-1}\left(\sqrt{3}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$. (vi) $\text{cosec}^{-1}\left(\sqrt{2}\right) + \sec^{-1}\left(\sqrt{2}\right)$.

(vii) $\tan^{-1}$ (1) + $\cot^{-1}$ (1) + $\sin^{-1}$ (1).

(viii) $\cot^{-1}\left(\sqrt{3}\right) - \sin^{-1}\left(-\frac{1}{2}\right)$.

(ix) $\sin^{-1}\left(\sin\frac{4\pi}{5}\right)$. (x) $\cos^{-1}\left(\cos\frac{7\pi}{5}\right)$.

(xi) $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)$. (xii) $\text{cosec}^{-1}\left(\text{cosec}\frac{3\pi}{4}\right)$.

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

3. Show that $\tan^{-1}\left(\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right)=\frac{\pi}{4}+\frac{x}{2}.$ $\quad x \in [0, \pi]$

4. Prove

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)-\cot^{-1}\left(\sqrt{\frac{1+\cos x}{1-\cos x}}\right)=\frac{\pi}{4} \qquad x \in (0, \pi/2).$$

5. Prove $\quad \tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)=\sin^{-1}\frac{x}{a}=\cos^{-1}\left(\frac{\sqrt{a^2-x^2}}{a}\right).$

6. Prove

$$\cot^{-1}\left[2\tan\left(\cos^{-1}\frac{8}{17}\right)\right]+\tan^{-1}\left[2\tan\left(\sin^{-1}\frac{8}{17}\right)\right]=\tan^{-1}\left(\frac{300}{161}\right).$$

7. Prove $\quad \tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2.$

8. Solve $\quad \cot^{-1}2x+\cot^{-1}3x=\frac{\pi}{4}.$

9. Prove that $\tan^{-1}\left(\frac{m}{n}\right)-\tan^{-1}\left(\frac{m-n}{m+n}\right)=\frac{\pi}{4}, m, n>0$

10. Prove that $\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right]=\frac{x+y}{1-xy}$

11. Solve for x, $\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)+\frac{1}{2}\tan^{-1}\left(\frac{-2x}{1-x^2}\right)=\frac{2\pi}{3}$

12. Prove that $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

13. Solve for x, $\tan(\cos^{-1} x) = \sin(\tan^{-1} 2)$; $x > 0$

14. Prove that $2\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{4}\right) = \tan^{-1}\left(\frac{32}{43}\right)$

15. Evaluate $\tan\left[\frac{1}{2}\cos^{-1}\left(\frac{3}{\sqrt{11}}\right)\right]$

H.O.T.S.

16. Prove that $\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right) = \tan^{-1}\left(\frac{a}{b}\right) - x$

17. Prove that

$$\cot\left\{\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)\right\} + \cos^{-1}(1 - 2x^2) + \cos^{-1}(2x^2 - 1) = \pi, \ x > 0$$

18. Prove that $\tan^{-1}\left(\frac{a-b}{1+ab}\right) + \tan^{-1}\left(\frac{b-c}{1+bc}\right) + \tan^{-1}\left(\frac{c-a}{1+ca}\right) = 0$ where $a, b, c > 0$

19. Solve for x, $2\tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$

20. Express $\sin^{-1}\left(x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right)$ in simplest form.

21. If $\tan^{-1}a + \tan^{-1}b + \tan^{-1}c = \pi$, then

    prove that $a + b + c = abc$

22. If $\sin^{-1}x > \cos^{-1}x$, then x belongs to which interval?

ANSWERS

1. (i) $-\frac{\pi}{3}$ (ii) $\frac{\pi}{3}$ (iii) $\frac{5\pi}{6}$ (iv) $\frac{\pi}{6}$

   (v) $\frac{-\pi}{6}$ (vi) $\frac{\pi}{6}$ (vii) $\frac{-\pi}{6}$ (viii) $\frac{\pi}{6}$

(ix) $\frac{2\pi}{3}$ (x) $\frac{\pi}{3}$ (xi) $\frac{2\pi}{3}$ (xii) $\frac{\pi}{3}$

(xiii) $\frac{\pi}{6}$.

2. (i) 0 (ii) $\frac{-\pi}{3}$ (iii) $-\frac{\pi}{2}$ (iv) $\frac{\pi}{2}$

(v) $\frac{2\pi}{3}$ (vi) $\frac{\pi}{2}$ (vii) π (viii) $\frac{\pi}{3}$

(ix) $\frac{\pi}{5}$ (x) $\frac{3\pi}{5}$ (xi) $\frac{-\pi}{6}$ (xii) $\frac{\pi}{4}$.

8. 1

11. $\tan\frac{\pi}{12} = 2 - \sqrt{3}$

13. $\frac{\sqrt{5}}{3}$

15. $\sqrt{\frac{\sqrt{11}-3}{3+\sqrt{11}}}$

19. $x = \frac{\pi}{4}$.

20 $\sin^{-1} x - \sin^{-1} \sqrt{x}$.

22. $\left(\frac{1}{\sqrt{2}}, 1\right)$

CHAPTER 3 & 4

MATRICES AND DETERMINANTS

POINTS TO REMEMBER

- **Matrix :** A matrix is an ordered rectangular array of numbers or functions. The numbers or functions are called the elements of the matrix.
- **Order of Matrix :** A matrix having '*m*' rows and '*n*' columns is called the matrix of order *mxn.*
- **Zero Matrix :** A matrix having all the elements zero is called zero matrix or null matrix.
- **Diagonal Matrix :** A square matrix is called a diagonal matrix if all its non diagonal elements are zero.
- **Scalar Matrix :** A diagonal matrix in which all diagonal elements are equal is called a scalar matrix.
- **Identity Matrix :** A scalar matrix in which each diagonal element is 1, is called an identity matrix or a unit matrix. It is denoted by I.

$\therefore \quad I = [e_{ij}]_{n \times n}$

where, $e_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

- **Transpose of a Matrix :** If $A = [a_{ij}]_{m \times n}$ be an $m \times n$ matrix then the matrix obtained by interchanging the rows and columns of A is called the transpose of the matrix. Transpose of A is denoted by A' or A^T.

  Properties of the transpose of a matrix.

  (i) $(A')' = A$ (ii) $(A + B)' = A' + B'$

  (iii) $(kA)' = kA'$, k is a scalar (iv) $(AB)' = B'A'$

- **Symmetrix Matrix :** A square matrix $A = [a_{ij}]$ is symmetrix if $a_{ij} = a_{ji}$ $\forall$ i, j. Also a square matrix A is symmetrix if $A' = A$.
- **Skew Symmetrix Matrix :** A square matrix $A = [a_{ij}]$ is skew-symmetrix, if $a_{ij} = -a_{ji}$ $\forall$ i, j. Also a square matrix A is skew - symmetrix, if $A' = -A$.
- **Determinant :** To every square matrix $A = [a_{ij}]$ of order $n \times n$, we can associate a number (real or complex) called determinant of A. It is denoted by det A or $|A|$.

Properties

(i) $|AB| = |A|\ |B|$

(ii) $|kA|_{n \times n} = k^n\ |A|_{n \times n}$ where k is a scalar.

Area of triangles with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

The points (x_1, y_1), (x_2, y_2), (x_3, y_3) are collinear $\Leftrightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Adjoint of a square matrix A is the transpose of the matrix whose elements have been replaced by their cofactors and is denoted as *adj A*.

Let $A = [a_{ij}]_{n \times n}$

$adj\ A = [A_{ji}]_{n \times n}$

Properties

(i) $A(adj\ A) = (adj\ A)\ A = |A|\ I$

(ii) If A is a square matrix of order n then $|adj\ A| = |A|^{n-1}$

(iii) $adj\ (AB) = (adj\ B)\ (adj\ A)$.

Singular Matrix : A square matrix is called singular if $|A| = 0$, otherwise it will be called a non-singular matrix.

Inverse of a Matrix : A square matrix whose inverse exists, is called invertible matrix. Inverse of only a non-singular matrix exists. Inverse of a matrix A is denoted by A^{-1} and is given by

$$A^{-1} = \frac{1}{|A|} . adj. A$$

Properties

(i) $AA^{-1} = A^{-1}A = I$

(ii) $(A^{-1})^{-1} = A$

(iii) $(AB)^{-1} = B^{-1}A^{-1}$

(iv) $(A^{T})^{-1} = (A^{-1})^{T}$

- **Solution of system of equations using matrix :**

If $AX = B$ is a matrix equation then its solution is $X = A^{-1}B$.

(i) If $|A| \neq 0$, system is consistent and has a unique solution.

(ii) If $|A| = 0$ and $(adj\ A)\ B \neq 0$ then system is inconsistent and has no solution.

(iii) If $|A| = 0$ and $(adj\ A)\ B = 0$ then system is consistent and has infinite solution.

VERY SHORT ANSWER TYPE QUESTIONS (1 Mark)

1. If $\begin{bmatrix} x+3 & 4 \\ y-4 & x+y \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 9 \end{bmatrix}$, find x and y.

2. If $A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ and $B = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, find AB.

3. Find the value of $a_{23} + a_{32}$ in the matrix $A = [a_{ij}]_{3 \times 3}$

   where $a_{ij} = \begin{cases} |2i - j| & \text{if } i > j \\ -i + 2j + 3 & \text{if } i \leq j \end{cases}$.

4. If B be a 4 × 5 type matrix, then what is the number of elements in the third column.

5. If $A = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$ find $3A - 2B$.

6. If $A = \begin{bmatrix} 2 & -3 \\ -7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & -6 \end{bmatrix}$ find $(A + B)'$.

7. If $A = [1 \quad 0 \quad 4]$ and $B = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ find AB.

8. If $A = \begin{bmatrix} 4 & x+2 \\ 2x-3 & x+1 \end{bmatrix}$ is symmetrix matrix, then find x.

9. For what value of x the matrix $\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -4 \\ 3 & 4 & x+5 \end{bmatrix}$ is skew symmetrix matrix.

10. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} = P + Q$ where P is symmetric and Q is skew-symmetric matrix, then find the matrix Q.

11. Find the value of $\begin{vmatrix} a + ib & c + id \\ -c + id & a - ib \end{vmatrix}$

12. If $\begin{vmatrix} 2x + 5 & 3 \\ 5x + 2 & 9 \end{vmatrix} = 0$, find x.

13. For what value of k, the matrix $\begin{bmatrix} k & 2 \\ 3 & 4 \end{bmatrix}$ has no inverse.

14. If $A = \begin{bmatrix} \sin 30^\circ & \cos 30^\circ \\ -\sin 60^\circ & \cos 60^\circ \end{bmatrix}$, what is $|A|$.

15. Find the cofactor of a_{12} in $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$.

16. Find the minor of a_{23} in $\begin{vmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ 3 & 5 & 2 \end{vmatrix}$.

17. Find the value of P, such that the matrix $\begin{bmatrix} -1 & 2 \\ 4 & P \end{bmatrix}$ is singular.

18. Find the value of x such that the points (0, 2), (1, x) and (3, 1) are collinear.

19. Area of a triangle with vertices (k, 0), (1, 1) and (0, 3) is 5 unit. Find the value (s) of k.

20. If A is a square matrix of order 3 and $|A| = -2$, find the value of $|-3A|$.

21. If $A = 2B$ where A and B are square matrices of order 3×3 and $|B| = 5$, what is $|A|$?

22. What is the condition that a system of equation $AX = B$ has no solution.

23. Find the area of the triangle with vertices (0, 0), (6, 0) and (4, 3).

24. If $\begin{vmatrix} 2x & 4 \\ -1 & x \end{vmatrix} = \begin{vmatrix} 6 & -3 \\ 2 & 1 \end{vmatrix}$, find x.

25. If $A = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$, write the value of det A.

26. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ such that $|A| = -15$, find $a_{11} C_{21} + a_{12}C_{22}$ where C_{ij} is cofactors of a_{ij} in $A = [a_{ij}]$.

27. If A is a non-singular matrix of order 3 and $|A| = -3$ find $|adj\ A|$.

28. If $A = \begin{bmatrix} 5 & -3 \\ 6 & 8 \end{bmatrix}$ find $(adj\ A)$

29. Given a square matrix A of order 3×3 such that $|A| = 12$ find the value of $|A \; adj \; A|$.

30. If A is a square matrix of order 3 such that $|adj \; A| = 8$ find $|A|$.

31. Let A be a non-singular square matrix of order 3×3 find $|adj \; A|$ if $|A| = 10$.

32. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ find $\left|(A^{-1})^{-1}\right|$.

33. If $A = \begin{bmatrix} -1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$ find $|AB|$.

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

34. Find x, y, z and w if $\begin{bmatrix} x - y & 2x + z \\ 2x - y & 3x + w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$.

35. Construct a 3×3 matrix $A = [a_{ij}]$ whose elements are given by

$$a_{ij} = \begin{cases} 1 + i + j & \text{if } i \geq j \\ \dfrac{|i - 2j|}{2} & \text{if } i < j \end{cases}$$

36. Find A and B if $2A + 3B = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & -1 \end{bmatrix}$ and $A - 2B = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 6 & 2 \end{bmatrix}$.

37. If $A = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -1 & -4 \end{bmatrix}$, verify that $(AB)' = B'A'$.

38. Express the matrix $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} = P + Q$ where P is a symmetric and Q is a skew-symmetric matrix.

39. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$ where n is a natural number.

40. Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$, find a matrix D such that $CD - AB = O$.

41. Find the value of x such that $\begin{bmatrix} 1 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = 0$

42. Prove that the product of the matrices

$$\begin{bmatrix} \cos^2\theta & \cos\theta\ \sin\theta \\ \cos\theta\ \sin\theta & \sin^2\theta \end{bmatrix} \text{ and } \begin{bmatrix} \cos^2\phi & \cos\phi\ \sin\phi \\ \cos\phi\ \sin\phi & \sin^2\phi \end{bmatrix}$$

is the null matrix, when θ and ϕ differ by an odd multiple of $\frac{\pi}{2}$.

43. If $A = \begin{bmatrix} 5 & 3 \\ 12 & 7 \end{bmatrix}$ show that $A^2 - 12A - I = 0$. Hence find A^{-1}.

44. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$ find $f(A)$ where $f(x) = x^2 - 5x - 2$.

45. If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, find x and y such that $A^2 - xA + yI = 0$.

46. Find the matrix X so that $X \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$.

47. If $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$ then show that $(AB)^{-1} = B^{-1}A^{-1}$.

48. Test the consistency of the following system of equations by matrix method :

$3x - y = 5;\ 6x - 2y = 3$

49. Using elementary row transformations, find the inverse of the matrix $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$, if possible.

50. By using elementary column transformation, find the inverse of $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$.

51. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ and $A + A' = \mathrm{I}$, then find the general value of α.

Using properties of determinants, prove the following : *Q* 52 to *Q* 59.

52. $$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

53. $$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix} = 0 \text{ if } a, b, c \text{ are in } A.P.$$

54. $$\begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix} = 0$$

55. $$\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2.$$

56. $$\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}.$$

57. $$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2.$$

58. $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = x^2(x+a+b+c)$.

59. Show that :

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = (y-z)(z-x)(x-y)(yz+zx+xy).$$

60. (i) If the points (a, b) (a', b') and $(a - a', b - b')$ are collinear. Show that $ab' = a'b$.

(ii) If $A = \begin{bmatrix} 2 & 5 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -3 \\ 2 & 5 \end{bmatrix}$ verity that $|AB| = |A||B|$.

61. Given $A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$. Find the product AB and also find $(AB)^{-1}$.

62. Solve the following equation for x.

$$\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0.$$

63. Verify that $(AB)^{-1} = B^{-1}A^{-1}$ for the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}.$$

64. Use matrix method to solve the following system of equations : $5x - 7y = 2$, $7x - 5y = 3$.

LONG ANSWER TYPE QUESTIONS (6 MARKS)

65. Obtain the inverse of the following matrix using elementary row operations

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}.$$

66. Use product $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$ to solve the system of equations $x - y + 2z = 1,\ 2y - 3z = 1,\ 3x - 2y + 4z = 2.$

67. Solve the following system of equations by matrix method, where $x \neq 0$, $y \neq 0$, $z \neq 0$

$$\frac{2}{x} - \frac{3}{y} + \frac{3}{z} = 10,\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 10,\ \frac{3}{x} - \frac{1}{y} + \frac{2}{z} = 13.$$

68. Find A^{-1}, where $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}$, hence solve the system of linear equations :

$$x + 2y - 3z = -4$$
$$2x + 3y + 2z = 2$$
$$3x - 3y - 4z = 11$$

69. The sum of three numbers is 2. If we subtract the second number from twice the first number, we get 3. By adding double the second number and the third number we get 0. Represent it algebraically and find the numbers using matrix method.

70. Compute the inverse of the matrix.

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 5 \end{bmatrix} \text{ and verify that } A^{-1} A = I_3.$$

71. If the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 2 \end{bmatrix}$, then compute $(AB)^{-1}$.

72. Using matrix method, solve the following system of linear equations :

$$2x - y = 4,\ 2y + z = 5,\ z + 2x = 7.$$

73. Find A^{-1} if $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Also show that $A^{-1} = \dfrac{A^2 - 3I}{2}$.

74. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ by using elementary column transformations.

75. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ and $f(x) = x^2 - 4x + 7$. Show that $f(A) = 0$. Use this result to find A^5.

76. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, verify that $A \cdot (adj\ A) = (adj\ A) \cdot A = |A|\ I_3$.

77. For the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, verify that $A^3 - 6A^2 + 9A - 4I = 0$, hence find A^{-1}.

78. Find the matrix X for which

$$\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} . X . \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}$$

79. By using properties of determinants prove the following :

$$\begin{vmatrix} 1 + a^2 - b^2 & 2ab & -2b \\ 2ab & 1 - a^2 + b^2 & 2a \\ 2b & -2a & 1 - a^2 - b^2 \end{vmatrix} = \left(1 + a^2 + b^2\right)^3.$$

80. $\begin{vmatrix} (y+z)^2 & xy & zx \\ xy & (x+z)^2 & yz \\ xz & yz & (x+y)^2 \end{vmatrix} = 2xyz(x+y+z)^3.$

81. $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3.$

82. If x, y, z are different and $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0.$ Show that $xyz = -1$.

H.O.T.S.

83. If a matrix A has 11 elements, what is its possible order?

84. Given a square matrix A of order 3×3, such that $|A| = -5$, find the value of $|A \,.\, adj\, A|$.

85. If $|A| = 3$ and $A = [a_{ij}]_{3 \times 3}$ and C_{ij} the cofactors of a_{ij} then what is the value of $a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{23}$.

86. What is the number of all possible matrices of order 2×3 with each entry 0, 1 or 2.

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

87. If $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and I is the identity matrix of order 2, show that

$$I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

88. If $F(\theta)=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$, show that $F(\theta)\ F(\phi) = F(\theta + \phi)$.

89. Let $A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$, show that $(aI + bA)^n = a^nI + na^{n-1}\ bA$, where I is the identity matrix of order 2 and $n \in N$.

90. If x, y, z are the 10th, 13th and 15th terms of a G.P. find the value of $\Delta=\begin{vmatrix}\log x & 10 & 1\\ \log y & 13 & 1\\ \log z & 15 & 1\end{vmatrix}$.

91. Using properties of determinants, show that

$$\begin{vmatrix}(b+c)^2 & a^2 & bc\\ (c+a)^2 & b^2 & ca\\ (a+b)^2 & c^2 & ab\end{vmatrix}=(a-b)(b-c)(c-a)(a+b+c)\left(a^2+b^2+c^2\right).$$

LONG ANSWER TYPE QUESTIONS (6 MARKS)

92. Using properties of determinants prove that

$$\begin{vmatrix}-bc & b^2+bc & c^2+bc\\ a^2+ac & -ac & c^2+ac\\ a^2+ab & b^2+ab & -ab\end{vmatrix}=(ab+bc+ca)^3$$

93. If $A=\begin{vmatrix}3 & 2 & 1\\ 4 & -1 & 2\\ 7 & 3 & -3\end{vmatrix}$, find A^{-1} and hence solve the system of equations $3x + 4y + 7z = 14,\ 2x - y + 3z = 4,\ x + 2y - 3z = 0.$

ANSWERS

1. $x = 2,\ y = 7$

2. $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$

3. 11.

4. 4

5. $\begin{bmatrix} 9 & -6 \\ 0 & 29 \end{bmatrix}$.

6. $\begin{bmatrix} 3 & -5 \\ -3 & -1 \end{bmatrix}$.

7. $AB = [26]$.

8. $x = 5$

9. $x = -5$

10. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

11. $a^2 + b^2 + c^2 + d^2$.

12. $x = -13$

13. $k = \frac{3}{2}$

14. $|A| = 1$.

15. 46

16. –4

17. $P = -8$

18. $x = \frac{5}{3}$.

19. $k = \frac{10}{3}$.

20. 54.

21. 40.

22. $|A| = 0$, $(adj\ A) \cdot B \neq 0$

23. 9 sq. units

24. $x = \pm 2$

25. 0

26. 0

27. 9

28. $\begin{bmatrix} 8 & 3 \\ -6 & 5 \end{bmatrix}$.

29. 1728

30. $|A| = 9$

31. 100

32. 11

33. $|AB| = -11$

34. $x = 1$, $y = 2$, $z = 3$, $w = 4$

35. $\begin{bmatrix} 3 & 3/2 & 5/2 \\ 4 & 5 & 2 \\ 5 & 6 & 7 \end{bmatrix}$.

36. $A = \begin{bmatrix} \frac{11}{7} & -\frac{9}{7} & \frac{9}{7} \\ \frac{1}{7} & \frac{18}{7} & \frac{4}{7} \end{bmatrix}$, $B = \begin{bmatrix} -\frac{5}{7} & -\frac{2}{7} & \frac{1}{7} \\ \frac{4}{7} & -\frac{12}{7} & -\frac{5}{7} \end{bmatrix}$

40. $D = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$.

41. $x = -2$ or -14

43. $A^{-1} = \begin{bmatrix} -7 & 3 \\ 12 & -5 \end{bmatrix}$.

44. $f(A) = 0$

45. $x = 9$, $y = 14$

46. $x = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$.

48. Inconsistent

49. Inverse does not exist.

50. $A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$.

51. $\alpha = 2n\pi \pm \frac{\pi}{3}, \; n \in z$

61. $AB = \begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix}$, $(AB)^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -2 \\ 2 & -1 \end{bmatrix}$.

62 0, $3a$

64. $x = \frac{11}{24}$, $y = \frac{1}{24}$.

65. $A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$.

66. $x = 0$, $y = 5$, $z = 3$

67. $x = \frac{1}{2},\ y = \frac{1}{3},\ z = \frac{1}{5}$

68. $A^{-1} = -\frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$

69. $x = 1,\ y = -2,\ z = 2$

70. $A^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

71. $(AB)^{-1} = \frac{1}{19}\begin{bmatrix} 16 & 12 & 1 \\ 21 & 11 & -7 \\ 10 & -2 & 3 \end{bmatrix}.$

72. $x = 3,\ y = 2,\ z = 1.$

73. $A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$

74. $A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

75. $A^5 = \begin{bmatrix} -118 & -93 \\ 31 & -118 \end{bmatrix}.$

77. $A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}.$

78. $x = \begin{bmatrix} -16 & 3 \\ 24 & -5 \end{bmatrix}.$

83. $11 \times 1,\ 1 \times 11$

84. –125

85. 3

86. 729

90. 0

93. $x = 1,\ y = 1,\ z = 1.$

CHAPTER 5

CONTINUITY AND DIFFERENTIATION

POINTS TO REMEMBER

- A function $f(x)$ is said to be continuous at $x = c$ iff $\lim\limits_{x \to c} f(x) = f(c)$

  i.e., $\lim\limits_{x \to c^-} f(x) = \lim\limits_{x \to c^+} f(x) = f(c)$

- $f(x)$ is continuous in $]a, b[$ iff it is continuous at $x = c \; \forall \; c \in (a, b)$.

- $f(x)$ is continuous in $[a, b]$ iff

  (i) $f(x)$ is continuous in (a, b)

  (ii) $\lim\limits_{x \to a^+} f(x) = f(a)$,

  (iii) $\lim\limits_{x \to b^-} f(x) = f(b)$

  Trigonometric function are continous in their resp. domains.

- Every polynomial function is continuous on R.

- If $f(x)$ and $g(x)$ are two continuous functions and $c \in R$ then at $x = a$

  (i) $f(x) \pm g(x)$ are also continuous functions at $x = a$.

  (ii) $g(x) \, . \, f(x)$, $f(x) + c$, $cf(x)$, $|f(x)|$ are also continuous at $x = a$.

  (iii) $\dfrac{f(x)}{g(x)}$ is continuous at $x = a$ provided $g(a) \neq 0$.

- $f(x)$ is derivable at $x = c$ iff

  L.H.D. (c) = R.H.D. (c)

i.e. $\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$

and value of above limit is denoted by $f'(c)$ and is called the derivative of $f(x)$ at $x = c$.

- $\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$

- $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$

- If $y = f(u)$, $x = g(u)$ then

  $\frac{dy}{dx} = \frac{f'(u)}{g'(u)}$.

- If $y = f(u)$ and $u = g(t)$ then $\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt} = f'(u).g'(t)$ (Chain Rule)

- $f(x) = [x]$ is discontinuous at all integral points and continuous for all $x \in R - Z$.

- **Rolle's theorem :** If $f(x)$ is continuous in $[a, b]$, derivable in (a, b) and $f(a) = f(b)$ then there exists atleast one real number $c \in (a, b)$ such that $f'(c) = 0$.

- **Mean Value Theorem :** If $f(x)$ is continuous in $[a, b]$ and derivable in (a, b) then there exists atleast one real number $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

- $f(x) = \log_e x$, $(x > 0)$ is continuous function.

VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

1. For what value of x, $f(x) = |2x - 7|$ is not derivable.
2. Write the set of points of continuity of $g(x) = |x - 1| + |x + 1|$.

3. What is derivative of $|x - 3|$ at $x = -1$.

4. What are the points of discontinuity of $f(x) = \dfrac{(x-1)+(x+1)}{(x-7)(x-6)}$.

5. Write the number of points of discontinuity of $f(x) = [x]$ in [3, 7].

6. The function, $f(x) = \begin{cases} \lambda x - 3 & \text{if } x < 2 \\ 4 & \text{if } x = 2 \\ 2x & \text{if } x > 2 \end{cases}$ is a continuous function for all $x \in R$, find λ.

7. For what value of K, $f(x) = \begin{cases} \dfrac{\tan 3x}{\sin 2x}, & x \neq 0 \\ 2K, & x = 0 \end{cases}$ is continuous $\forall\, x \in R$.

8. Write derivative of $\sin x$ *w.r.t.* $\cos x$.

9. If $f(x) = x^2 g(x)$ and $g(1) = 6$, $g'(1) = 3$ find value of $f'(1)$.

10. Write the derivatives of the following functions :

    (i) $\log_3 (3x + 5)$ (ii) $e^{\log_2 x}$

    (iii) $e^{6 \log_e (x-1)}, x > 1$

    (iv) $\sec^{-1}\sqrt{x} + \operatorname{cosec}^{-1}\sqrt{x}, x \geq 1.$

    (v) $\sin^{-1}\left(x^{7/2}\right)$ (vi) $\log_x 5$, $x > 0$.

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

11. Discuss the continuity of following functions at the indicated points.

    (i) $f(x) = \begin{cases} \dfrac{x - |x|}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$ at $x = 0$.

(ii) $g(x) = \begin{cases} \dfrac{\sin 2x}{3x}, & x \neq 0 \\ \dfrac{3}{2} & x = 0 \end{cases}$ at $x = 0$.

(iii) $f(x) = \begin{cases} x^2 \cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ at $x = 0$.

(iv) $f(x) = |x| + |x - 1|$ at $x = 1$.

(v) $f(x) = \begin{cases} x - [x], & x \neq 1 \\ 0 & x = 1 \end{cases}$ at $x = 1$.

12. For what value of k, $f(x) = \begin{bmatrix} 3x^2 - kx + 5, & 0 \leq x < 2 \\ 1 - 3x & 2 \leq x \leq 3 \end{bmatrix}$ is continuous $\forall\ x \in [0, 3]$.

13. For what values of a and b

$$f(x) = \begin{cases} \dfrac{x+2}{|x+2|} + a & \text{if } x < -2 \\ a + b & \text{if } x = -2 \\ \dfrac{x+2}{|x+2|} + 2b & \text{if } x > -2 \end{cases}$$ is continuous at $x = 2$.

14. Prove that $f(x) = |x + 1|$ is continuous at $x = -1$, but not derivable at $x = -1$.

15. For what value of p,

$$f(x) = \begin{cases} x^p \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$ is derivable at $x = 0$.

16. If $y = \dfrac{1}{2}\left[\tan^{-1}\left(\dfrac{2x}{1 - x^2}\right) + 2\tan^{-1}\left(\dfrac{1}{x}\right)\right]$, $0 < x < 1$ find $\dfrac{dy}{dx}$.

17. If $y = \sin\left[2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right]$ then $\frac{dy}{dx} = ?$

18. If $5^x + 5^y = 5^{x+y}$ then prove that $\frac{dy}{dx} + 5^{y-x} = 0$.

19. If $x\sqrt{1-y^2} + y\sqrt{1-x^2} = a$ then show that $\frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}$.

20. If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$ then show that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

21. If $(x + y)^{m+n} = x^m \cdot y^n$ then prove that $\frac{dy}{dx} = \frac{y}{x}$.

22. Find the derivative of $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ w.r.t. $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

23. Find the derivative of $\log_e(\sin x)$ w.r.t. $\log_a(\cos x)$.

24. If $x^y + y^x + x^x = m^n$, then find the value of $\frac{dy}{dx}$.

25. If $x = a\cos^3\theta$, $y = a\sin^3\theta$ then find $\frac{d^2y}{dx^2}$.

26. If $x = ae^t(\sin t - \cos t)$

$y = ae^t(\sin t + \cos t)$ then show that $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$ is 1.

27. If $y = \sin^{-1}\left[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right]$ then find $-\frac{dy}{dx}$.

28. If $y = x^{\log_e x} + (\log_e x)^x$ then find $\frac{dy}{dx}$.

29. Differentiate x^{x^x} w.r.t. x.

30. Differentiate $e^{x^{e^x}}$ w.r.t. x.

H.O.T.S.

31. If $y = \tan^{-1}\left(\dfrac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right)$ where $\dfrac{\pi}{2} < x < \pi$ find $\dfrac{dy}{dx}$.

32. If $x = \sin\left(\dfrac{1}{a}\log_e y\right)$ then show that $(1 - x^2)\, y'' - xy' - a^2y = 0$.

33. If $y = \sin^{-1}\left(\dfrac{12x+5\sqrt{1-x^2}}{13}\right), \dfrac{dy}{dx} = ?$

34. If $\sin y = x \sin (a + y)$ then show that $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$.

35. If $y = \sin^{-1}x$, find $\dfrac{d^2y}{dx^2}$ in terms of y.

36. If $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, then show that $\dfrac{d^2y}{dx^2}=\dfrac{-b^4}{a^2y^3}$.

37. If $ax^2 + 2hxy + by^2 = 1$ then prove that $\dfrac{d^2y}{dx^2}=\dfrac{h^2-ab}{(hx+by)^3}$.

38. If $y^3 = 3ax^2 - x^3$ then prove that $\dfrac{d^2y}{dx^2}=\dfrac{-2a^2x^2}{y^5}$.

ANSWERS

VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

1. $x = -7/2$
2. R
3. -1
4. $x = 6, 7$

5. Points of discontinuity of $f(x)$ are 4, 5, 6, 7 *i.e.* four points.

Note : At $x = 3$, $f(x) = [x]$ is continuous. because $\lim_{x\to 3^+} f(x) = 3 = f(3)$.

6. $\lambda = \frac{7}{2}$.

7. $k = \frac{3}{4}$.

8. $-\cot x$

9. 15

10. (i) $\frac{3}{3x + 5}\log_3 e$

(ii) $e^{\log_2 x}\frac{1}{x}.\log_2 e$.

(iii) $6(x - 1)^5$

(iv) 0

(v) $\frac{7}{2}\frac{x^2\sqrt{x}}{\sqrt{1-x^7}}$.

(vi) $\frac{-\log_e 5}{x(\log_e x)^2}$.

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

11. (i) Discontinuous

(ii) Discontinuous

(iii) Continuous

(iv) continuous

(v) Discontinuous

12. k = 11

13. $a = 0$, $b = -1$.

15. $p > 1$.

16. 0

17. $\frac{-x}{\sqrt{1-x^2}}$.

22. 1

23. $-\cot^2 x \log_e a$

24. $\frac{dy}{dx} = \frac{-x^x(1+\log x) - yx^{y-1} - y^x \log y}{x^y \log x + xy^{x-1}}$.

25. $\frac{d^2y}{dx^2} = \frac{1}{3a}\operatorname{cosec}\theta \sec^4\theta$.

27. $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x}\sqrt{1-x}}$.

28. $x^{\log x}\dfrac{2\log x}{x}+(\log x)^{x}\left[\dfrac{1}{\log x}+\log(\log x)\right].$

29. $\dfrac{dy}{dx}=x^{x^{x}}.\,x^{x}\log x\left(1+\log x+\dfrac{1}{x\log x}\right).$

30. $e^{x^{e^{x}}}.x^{e^{x}}e^{x}\left(\dfrac{1}{x}+\log x\right).$

H.O.T.S.

31. $\dfrac{dy}{dx}=-\dfrac{1}{2}.$

Hint. : $\sin\dfrac{x}{2}>\cos\dfrac{x}{2}$ *for* $x\in\left(\dfrac{\pi}{2},\pi\right).$

33. $\dfrac{1}{\sqrt{1-x^{2}}}.$

35. $\sec^{2}y$ tany.

CHAPTER 6

APPLICATIONS OF DERIVATIVES

POINTS TO REMEMBER

- **Rate of Change :** Let $y = f(x)$ be a function then the rate of change of y with respect to x is given by $\frac{dy}{dx} = f'(x)$ where a quantity y varies with another quantity x.

  $\left.\frac{dy}{dx}\right]_{x=x_0}$ or $f'(x_0)$ represents the rate of change of y w.r.t. x at $x = x_0$.

- If $x = f(t)$ and $y = g(t)$

  By chain rule

  $$\frac{dy}{dx} = \frac{dy}{dt} \bigg/ \frac{dx}{dt} \text{ if } \frac{dx}{dt} \neq 0.$$

- (i) A function $f(x)$ is said to be non-decreasing on an interval (a, b) if $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) \leq f(x_2)$ $\forall$ $x_1, x_2 \in (a, b)$. Alternatively if $f'(x_0) \geq 0 \; \forall \; x \in (a, b)$, then $f(x)$ is increasing function in (a, b).

  (ii) A function $f(x)$ is said to be non-increasing on an interval (a, b). If $x_1 \leq x_2$ in $(a, b) \Rightarrow f(x_1) \geq f(x_2)$ $\forall$ $x_1, x_2 \in (a, b)$. Alternatively if $f'(x) \leq 0$ $\forall$ $x \in (a, b)$, then $f(x)$ is decreasing function in (a, b).

- The equation of tangent at the point (x_0, y_0) to a curve $y = f(x)$ is given by

  $$y - y_0 = \left.\frac{dy}{dx}\right]_{(x_0, y_0)} (x - x_0).$$

where $\left.\frac{dy}{dx}\right]_{(x_0, y_0)}$ = slope of the tangent of the point (x_0, y_0).

- Slope of the normal to the curve at the point (x_0, y_0) is given by $-\frac{1}{\left.\frac{dy}{dx}\right]_{x=x_0}}$.

- Equation of the normal to the curve $y = f(x)$ at a point (x_0, y_0) is given by

$$y - y_0 = -\frac{1}{\left.\frac{dy}{dx}\right]_{(x_0, y_0)}}(x - x_0).$$

- If $\left.\frac{dy}{dx}\right]_{(x_0, y_0)} = 0$. Then the tangent is parallel to x-axis at (x_0, y_0). Equation of the normal is $x = x_0$.

- If $\left.\frac{dy}{dx}\right]_{(x_0, y_0)}$ does not exit, then the normal is parallel to x-axis and the equation of the normal is $y = y_0$.

- Let $y = f(x)$

  Δx = the small increment in x and

  Δy be the increment in y corresponding to the increment in x

  Then approximate change in y is given by

$$dy = \left(\frac{dy}{dx}\right)\Delta x \qquad \text{or} \qquad dy = f'(x)\,\Delta x$$

  The approximate change in the value of f is given by

$$f(x + \Delta x) = f(x) + f'(x)\,\Delta x$$

- Let f be a function. Let point c be in the domain of the function f at which either $f'(x) = 0$ or f is not derivable is called a critical point of f.

- **First Derivative Test :** Let f be a function defined on an open interval I. Let f be continuous at a critical point $c \in I$. Then

(i) $f'(x)$ changes sign from positive to negative as we pass through c, then c is called the point of the local maxima.

(ii) If $f'(x)$ changes sign from negative to positive as we pass through c, then c is a point of *local maxima*.

(iii) If $f'(x)$ does not change sign as we pass through c, then c is neither a point of *local maxima* nor a point of *local minima*. Such a point is called a point of *inflexion.*

- **Second Derivative Test :** Let f be a functions defined on an interval I and let $c \in I$.

(i) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$.

Then $f(c)$ is the local maximum value of f.

(ii) $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$. Then $f(c)$ is the local minimum value of f.

(iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$.

VERY SHORT ANSWER TYPE QUESTIONS

1. The side of a square is increasing at a rate of 0.2 cm/sec. Find the rate of increase of perimeter of the square.

2. The radius of the circle is increasing at the rate of 0.7 cm/sec. What is the rate of increase of its circumference?

3. If the radius of a soap bubble is increasing at the rate of $\frac{1}{2}\text{cm/sec}$. At what rate its volume is increasing when the radius is 1 cm.

4. A stone is dropped into a quiet lake and waves move in circles at a speed of 4 cm/sec. At the instant when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?

5. The total revenue in rupees received from the sale of x units of a product is given by

   $R(x) = 13x^2 + 26x + 15$. Find the marginal revenue when $x = 7$.

6. Find the maximum and minimum values of function $f(x) = \sin 2x + 5$.

7. Find the maximum and minimum values (if any) of the function

$$f(x) = -|x-1| + 7 \quad \forall x \in R.$$

8. Find the value of a for which the function $f(x) = x^2 - 2ax + 6$, $x > 0$ is strictly increasing.

9. Write the interval for which the function $f(x) = \cos x$, $0 \le x \le 2\pi$ is decreasing.

10. What is the interval on which the function $f(x) = \frac{\log x}{x}$, $x \in (0, \infty)$ is increasing?

11. For which values of x, the functions $y = x^4 - \frac{4}{3}x^3$ is increasing?

12. Write the interval for which the function $f(x) = \frac{1}{x}$ is strictly decreasing.

13. Find the sub-interval of the interval $(0, \pi/2)$ in which the function $f(x) = \sin 3x$ is increasing.

14. Without using derivatives, find the maximum and minimum value of $y = |3 \sin x + 1|$.

15. If $f(x) = ax + \cos x$ is strictly increasing on R, find a.

16. Write the interval in which the function $f(x) = x^9 + 3x^7 + 64$ is increasing.

17. What is the slope of the tangent to the curve $f = x^3 - 5x + 3$ at the point whose x co-ordinate is 2?

18. At what point on the curve $y = x^2$ does the tangent make an angle of 45° with positive direction of the x-axis?

19. Find the point on the curve $y = 3x^2 - 12x + 9$ at which the tangent is parallel to x-axis.

20. What is the slope of the normal to the curve $y = 5x^2 - 4 \sin x$ at $x = 0$.

21. Find the point on the curve $y = 3x^2 + 4$ at which the tangent is perpendicular to the line with slope $-\frac{1}{6}$.

22. Find the point on the curve $y = x^2$ where the slope of the tangent is equal to the y – co-ordinate.

23. If the curves $y = 2e^x$ and $y = ae^{-x}$ intersect orthogonally (cut at right angles), what is the value of a?

24. Find the slope of the normal to the curve $y = 8x^2 - 3$ at $x = \frac{1}{4}$.

25. Find the rate of change of the total surface area of a cylinder of radius r and height h with respect to radius when height is equal to the radius of the base of cylinder.

26. Find the rate of change of the area of a circle with respect to its radius. How fast is the area changing w.r.t. its radius when its radius is 3 cm?

27. For the curve $y = (2x + 1)^3$ find the rate of change of slope at $x = 1$.

28. Find the slope of the normal to the curve

$$x = 1 - a \sin \theta \quad ; \quad y = b \cos^2\theta \quad \text{at} \quad \theta = \frac{\pi}{2}$$

29. If a manufacturer's total cost function is $C(x) = 1000 + 40x + x^2$, where x is the out put, find the marginal cost for producing 20 units.

30. Find a for which $f(x) = (x + \sin x) + a$ is increasing (strictly).

ANSWERS

1. 0.8 cm/sec.
2. 4.4 cm/sec.
3. 2π cm^3/sec.
4. 80π cm^2/sec.
5. Rs. 208.
6. Minimum value = 4, maximum value = 6.
7. Maximum value = 7, minimum value does not exist.
8. $a \leq 0$.
9. $(0, \pi]$
10. $(0, e]$
11. $x \geq 1$
12. $(-\infty, 0) \cup (0, \infty)$
13. $\left(0, \frac{\pi}{6}\right)$.

14. Maximum value = 4, minimum valve = 0.

15. $a > 1$.

16. R

17. 7

18. $\left(\frac{1}{2}, \frac{1}{4}\right)$.

19. (2, – 3)

20. $\frac{1}{4}$

21. (1, 7)

22. (0, 0), (2, 4)

23. $\frac{1}{2}$.

24. $-\frac{1}{4}$.

25. $8 \pi r$

26. $2 \pi \text{ cm}^2/\text{cm}$

27. 72

28. $-\frac{a}{2b}$.

29. Rs. 80.

30. $a > 0$.

VERY SHORT ANSWER TYPE QUESTIONS

1. A particle cover along the curve $6y = x^3 + 2$. Find the points on the curve at which the y co-ordinate is changing 8 times as fast as the x co-ordinate.

2. A ladder 5 metres long is leaning against a wall. The bottom of the ladder is pulled along the ground away from the wall as the rate of 2 cm/sec. How fast is its height on the wall decreasing when the foot of the ladder is 4 metres away from the wall?

3. A balloon which always remain spherical is being inflated by pumping in 900 cubic cm of a gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.

4. A man 2 meters high walks at a uniform speed of 5 km/hr away from a lamp post 6 metres high. Find the rate at which the length of his shadow increases.

5. Water is running out of a conical funnel at the rate of 5 cm^3/sec. If the

radius of the base of the funnel is 10 cm and attitude is 20 cm, find the rate at which the water level is dropping when it is 5 cm from the top.

6. The length x of a rectangle is decreasing at the rate of 5 cm/sec and the width y is increasing as the rate of 4 cm/sec when $x = 8$ cm and $y = 6$ cm. Find the rate of change of

   (a) Perimeter (b) Area of the rectangle.

7. Sand is pouring from a pipe at the rate of 12c.c/sec. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when height is 4 cm?

8. The area of an expanding rectangle is increasing at the rate of 48 cm^2/sec. The length of the rectangle is always equal to the square of the breadth. At what rate is the length increasing at the instant when the breadth is 4.5 cm?

9. Find a point on the curve $y = (x - 3)^2$ where the tangent is parallel to the line joining the points (4, 1) and (3, 0).

10. Find the equation of all lines having slope zero which are tangents to the curve $y = \dfrac{1}{x^2 - 2x + 3}$.

11. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.

12. Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.

13. Show that the curves $4x = y^2$ and $4xy = k$ cut as right angles if $k^2 = 512$.

14. Find the equation of the tangent to the curve $y = \sqrt{3x - 2}$ which is parallel to the line $4x - y + 5 = 0$.

15. Find the equation of the tangent to the curve $\sqrt{x} + \sqrt{y} = a$ at the point $\left(\dfrac{a^2}{4}, \dfrac{a^2}{4}\right)$.

16. Find the points on the curve $4y = x^3$ where slope of the tangent is $\dfrac{16}{3}$.

17. Show that $\frac{x}{a}+\frac{y}{b}=1$ touches the curve $y = be^{-x/a}$ at the point where the curve crosses the y-axis.

18. Find the equation of the tangent to the curve given by $x = a \sin^3 t$, $y = b \cos^3 t$ at a point where $t=\frac{\pi}{2}$.

19. Find the intervals in which the function $f(x) = \log(1 + x) - \frac{x}{1+x}, x > -1$ is increasing or decreasing.

20. Find the intervals in which the function $f(x) = x^3 - 12x^2 + 36x + 17$ is

    (a) Increasing (b) Decreasing.

21. Prove that the function $f(x) = x^2 - x + 1$ is neither increasing nor decreasing in [0, 1].

22. Find the intervals on which the function $f(x)=\frac{x}{x^2+1}$ is decreasing.

23. Prove that $f(x)=\frac{x^3}{3}-x^2+9x,\ x\in[1,2]$ is strictly increasing. Hence find the minimum value of $f(x)$.

24. Find the intervals on which the function $f(x)=\frac{\log x}{x},\ x\in(0,\infty)$ is increasing or decreasing.

25. Find the intervals in which the function $f(x) = \sin^4 x + \cos^4 x,\ 0\le x\le\frac{\pi}{2}$ is increasing or decreasing.

26. Find the least value of a such that the function $f(x) = x^2 + ax + 1$ is strictly increasing on (1, 2).

27. Find the interval in which the function $f(x)=5x^{\frac{3}{2}}-3x^{\frac{5}{2}},\ x>0$ is strictly decreasing.

28. Show that the function $f(x) = \tan^{-1}(\sin x + \cos x)$, is strictly increasing on the interval $\left(0, \frac{\pi}{4}\right)$.

29. Show that the function $f(x) = \cos\left(2x + \frac{\pi}{4}\right)$ is strictly increasing on $\left(\frac{3\pi}{8}, \frac{7\pi}{8}\right)$.

30. Show that the function $f(x) = \frac{\sin x}{x}$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$.

Using differentials, find the approximate value of (Q. No. 31 to 37).

31. $(0.009)^{\frac{1}{3}}$.

32. $(255)^{\frac{1}{4}}$.

33. $(0.0037)^{\frac{1}{2}}$.

34. $\sqrt{0.037}$.

35. $(66)^{\frac{1}{3}}$.

36. $\sqrt{25.3}$.

37. $\sqrt{24}$.

38. Find the approximate value of $f(5.001)$ where $f(x) = x^3 - 7x^2 + 15$.

39. Find the approximate value of $f(3.02)$ where $f(x) = 3x^2 + 5x + 3$.

40. Find the approximate value of $f(2.998)$ where $f(x) = 5x^2 - 3x + 4$.

ANSWERS

1. $(4, 11)$ and $\left(-4, -\frac{31}{3}\right)$.

2. $-\frac{8}{3}$ cm/sec.

3. $\frac{1}{\pi}$ cm/sec.

4. 2.5 km/hr.

5. $\frac{4}{45\pi}$ cm/sec.

6. (a) –2 cm/min, (b) 2 cm^2/min

7. $\frac{1}{48\pi}$ cm/sec.

8. 7.11 cm/sec.

9. $\left(\frac{7}{2}, \frac{1}{4}\right)$.

10. $y = \frac{1}{2}$.

12. $2x + 3my = am^2 (2 + 3m^2)$

14. $48x - 24y = 23$

15. $2x + 2y = a^2$

16. $\left(\frac{8}{3}, \frac{128}{27}\right), \left(\frac{-8}{3}, -\frac{128}{27}\right)$.

18. $y = 0$

19. Increasing in $(0, \infty)$, decreasing in $(-1, 0)$.

20. Increasing in $(-\infty, 2) \cup (6, \infty)$, Decreasing in $(2, 6)$.

22. $(-\infty, -1)$ and $(1, \infty)$.

23. $\frac{25}{3}$.

24. Increasing in $(0, e]$ and Decreasing in $[e, \infty)$.

25. Increasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ Decreasing in $\left(0, \frac{\pi}{4}\right)$.

26. $a = -2$.

27. Strictly decreasing in $(1, \infty)$.

31. 0.2083

32. 3.9961

33. 0.06083

34. 0.1925

35. 4.04

36. 5.03

37. 4.9

38. –34.995

39. 45.46

40. 39.946

LONG ANSWER TYPE QUESTION (6 MARKS)

1. Show that of all rectangles inscribed in a given fixed circle, the square has the maximum area.

2. Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is maximum.

3. Show that of all the rectangles of given area, the square has the smallest perimeter.

4. Show that the right circular cone of least curved surface area and given volume has an altitude equal to $\sqrt{2}$ times the radium of the base.

5. Show that the semi vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$.

6. A point on the hypotenuse of a triangle is at a distance *a* and *b* from the sides of the triangle. Show that the minimum length of the hypotenuse is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

7. Prove that the volume of the largest cone that can be inscribed in a sphere of radius *R* is $\frac{8}{27}$ of the volume of the sphere.

8. Find the interval in which the function *f* given by $f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$ is strictly increasing or strictly decreasing.

9. Find the intervals in which the function $f(x) = (x + 1)^3 (x - 3)^3$ is strictly increasing or strictly decreasing.

10. Find the local maximum and local minimum of $f(x) = \sin 2x - x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

11. Find the intervals in which the function $f(x) = 2x^3 - 15x^2 + 36x + 1$ is strictly increasing or decreasing. Also find the points on which the tangents are parallel to *x*-axis.

12. A solid is formed by a cylinder of radius *r* and height *h* together with two hemisphere of radius *r* attached at each end. It the volume of the solid is constant but radius *r* is increasing at the rate of $\frac{1}{2\pi}$ metre/min. How fast must *h* (height) be changing when *r* and *h* are 10 metres.

13. Find the equation of the normal to the curve

    $x = a(\cos\theta + \theta\sin\theta)$; $y = a(\sin\theta - \theta\cos\theta)$ at the point θ and show that its distance from the origin is a.

14. For the curve $y = 4x^3 - 2x^5$, find all the points at which the tangent passes through the origin.

15. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point (1, 2).

16. Find the equation of the tangents at the points where the curve $2y = 3x^2 - 2x - 8$ cuts the x-axis and show that they make supplementary angles with the x-axis.

17. Find the equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0).

18. A window is in the form of a rectangle surmounted by an equilateral triangle. Given that the perimeter is 16 metres. Find the width of the window in order that the maximum amount of light may be admitted.

19. A jet of an enemy is flying along the curve $y = x^2 + 2$. A soldier is placed at the point (3, 2). What is the nearest distance between the soldier and the jet?

20. Find a point on the parabola $y^2 = 4x$ which is nearest to the point (2, –8).

21. A square piece of tin of side 18 cm is to be made into a box without top by cutting a square from each cover and folding up the flaps to form the box. What should be the side of the square to be cut off so that the valume of the box is the maximum.

22. A window in the form of a rectangle is surmounted by a semi circular opening. The total perimeter of the window is 30 metres. Find the dimensions of the rectangular part of the window to admit maximum light through the whole opening.

23. An open box with square base is to be made out of a given iron sheet of area 27 sq. meter, show that the maximum value of the box is 13.5 cubic metres.

24. A wire of length 28 cm is to be cut into two pieces. One of the two pieces is to be made into a square and other in to a circle. What should be the length of two pieces so that the combined area of the square and the circle is minimum?

25. Show that the height of the cylinder of maximum volume which can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

26. Show that the altitude of the right circular cone of maximum volume that can be inscribed is a sphere of radius r is $\frac{4r}{3}$.

27. Prove that the surface area of solid cuboid of a square base and given volume is minimum, when it is a cube.

28. Show that the volume of the greatest cylinder which can be inscribed in a right circular cone of height h and semi-vertical angle α is $\frac{4}{27}\pi h^3 \tan^2 \alpha$.

29. Show that the right triangle of maximum area that can be inscribed in a circle is an isosceles triangle.

30. A given quantity of metal is to be cast half cylinder with a rectangular box and semicircular ends. Show that the total surface area is minimum when the ratio of the length of cylinder to the diameter of its semicircular ends is $\pi : (\pi + 2)$.

ANSWERS

2. 25, 10

8. Strictly increasing in $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right)$

   Strictly decreasing is $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

9. Strictly increasing in $(1, 3) \cup (3, \infty)$

   Strictly decreasing in $(-\infty, -1) \cup (-1, 1)$.

10. Local maxima at $x = \frac{\pi}{6}$

    Local max. value $= \frac{\sqrt{3}}{2} - \frac{\pi}{6}$

    Local minima at $x = -\frac{\pi}{6}$

Local minimum value $= \frac{-\sqrt{3}}{2} + \frac{\pi}{6}$

11. Strictly increasing in $(-\infty, 2] \cup [3, \infty)$

    Strictly decreasing in (2, 3).

    Points are (2, 29) and (3, 28).

12. $-\frac{3}{\pi}$ metres/min.

13. $x + y \tan\theta - a \sec\theta = 0$.

14. (0, 0), (–1, –2) and (1, 2).

15. $x + y = 3$

16. $5x - y - 10 = 0$ and $15x + 3y + 20 = 0$

17. $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1, \quad \frac{y - y_0}{a^2 y_0} + \frac{x - x_0}{b^2 x_0} = 0.$

18. $\frac{16}{6 - \sqrt{3}}$

19. $\sqrt{5}$

20. (4, –4)

21. 3cm

22. $\frac{60}{\pi + 4}, \frac{30}{\pi + 4}$.

24. $\frac{112}{\pi + 4}$ cm, $\frac{28\pi}{\pi + 4}$ cm.

CHAPTER 7

INTEGRALS

POINTS TO REMEMBER

- Integration is the reverse process of Differentiation.
- Let $\frac{d}{dx}F(x) = f(x)$ then we write $\int f(x)dx = F(x) + c$.
- These integrals are called indefinite integrals and c is called constant of integrations.
- From geometrical point of view an indefinite integral is collection of family of curves each of which is obtained by translating one of the curves parallel to itself upwards or downwards along with y-axis.

STANDARD FORMULAE

1. $\int x^n dx = \begin{cases} \dfrac{x^{n+1}}{n+1} + c & n \neq -1 \\ \log|x| + c & n = -1 \end{cases}$

2. $\int (ax + b)^n \, dx = \begin{cases} \dfrac{(ax + b)^{n+1}}{(n + 1)\, a} + c & n \neq -1 \\ \dfrac{1}{a} \log|ax + b| + c & n = -1 \end{cases}$

3. $\int \sin x \, dx = -\cos x + c.$

4. $\int \cos x \, dx = \sin x + c.$

5. $\int \tan x \, . \, dx = -\log|\cos x| + c = \log|\sec x| + c.$

6. $\int \cot x \; dx = \log|\sin x| + c.$ 7. $\int \sec^2 x \, . \, dx = \tan x + c.$

8. $\int \text{cosec}^2 x \, . \, dx = -\cot x + c.$ 9. $\int \sec x \, . \tan x \, . \, dx = \sec x + c.$

10. $\int \text{cosec} \; x \cot x \; dx = - \; \text{cosec} \; x + c.$

11. $\int \sec x \; dx = \log|\sec x + \tan x| + c.$

12. $\int \text{cosec} \; x \; dx = \log|\text{cosec} \; x - \cot x| + c.$

13. $\int e^x dx = e^x + c.$ 14. $\int a^x dx = \dfrac{a^x}{\log a} + c$

15. $\int \dfrac{1}{\sqrt{1 - x^2}} dx = \sin^{-1} x + c, \; |x < 1|.$

16. $\int \dfrac{1}{1 + x^2} dx = \tan^{-1} x + c.$

17. $\int \dfrac{1}{x\sqrt{x^2 - 1}} dx = \sec^{-1} x + c, \; |x| > 1.$

18. $\int \dfrac{1}{a^2 - x^2} dx = \dfrac{1}{2a} \log\left|\dfrac{a + x}{a - x}\right| + c.$

19. $\int \dfrac{1}{x^2 - a^2} dx = \dfrac{1}{2a} \log\left|\dfrac{x - a}{x + a}\right| + c.$

20. $\int \dfrac{1}{a^2 + x^2} dx = \dfrac{1}{a} \tan^{-1} \dfrac{x}{a} + c.$

21. $\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1} \frac{x}{a} + c.$

22. $\int \frac{1}{\sqrt{a^2 + x^2}}\, dx = \log\left|x + \sqrt{a^2 + x^2}\right| + c.$

23. $\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \log\left|x + \sqrt{x^2 - a^2}\right| + c.$

24. $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c.$

25. $\int \sqrt{a^2 + x^2}\, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2} \log\left|x + \sqrt{a^2 + x^2}\right| + c.$

26. $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log\left|x + \sqrt{x^2 - a^2}\right| + c.$

RULES OF INTEGRATION

1. $\int k.f(x)dx = k\int f(x)dx.$

2. $\int k\{f(x) \pm g(x)\}dx = k\int f(x)dx \pm k\int g(x)dx.$

INTEGRATION BY SUBSTITUTION

1. $\int \frac{f'(x)}{f(x)}\, dx = \log|f(x)| + c.$

2. $\int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c.$

3. $\int \frac{f'(x)}{[f(x)]^n}\,dx = \frac{(f(x))^{-n+1}}{-n+1} + c.$

INTEGRATION BY PARTS

$$\int f(x).\, g(x)\,dx = f(x).\left[\int g(x)\,dx\right] - \int f'(x).\left[\int g(x)\,dx\right]dx.$$

DEFINITE INTEGRALS

$$\int_a^b f(x)\,dx = F(b) - F(a), \text{ where } F(x) = \int f(x)\,dx.$$

DEFINITE INTEGRAL AS A LIMIT OF SUMS.

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h \left[\begin{array}{l} f(a) + f(a+h) + f(a+2h) \\ \qquad\qquad + \ldots.. + f\left(a + \overline{n-1}\,h\right) \end{array}\right]$$

where $h = \frac{b-a}{h}$. or $\int_a^b f(x)\,dx = \lim_{h \to 0}\left[h \sum_{r=1}^{n} f(a + rh)\right]$

PROPERTIES OF DEFINITE INTEGRAL

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

2. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt.$

3. $\int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$

4. (i) $\int_a^b f(x)dx = \int_a^b f(a+b-x)\,dx.$ (ii) $\int_0^a f(x)dx = \int_0^a f(a-x)\,dx.$

5. $\int_{-a}^{a} f(x) = 0$; if $f(x)$ is odd function.

6. $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$, if $f(x)$ is even function.

7. $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x)=f(x) \\ 0, & \text{if } f(2a-x)=-f(x) \end{cases}$

VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

Evaluate the following integrals

1. $\int \left(\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}\right)dx.$

2. $\int_{-1}^{1} e^{|x|}dx.$

3. $\int \frac{1}{1-\sin^2 x}dx.$

4. $\int \left(8^x + x^8 + \frac{8}{x} + \frac{x}{8}\right)dx.$

5. $\int_{-1}^{1} x^{99}\cos^4 x\,dx.$

6. $\int \frac{1}{x\log x\log(\log x)}dx.$

7. $\int_0^{\pi/2}\log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx.$

8. $\int \left(e^{a\log x} + e^{x\log a}\right)dx.$

9. $\int \left(\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}\right)dx.$

10. $\int_{-\frac{\pi}{2}}^{\pi/2} \sin^7 x\,dx.$

11. $\int (x^c + c^x)\,dx.$

12. $\frac{d}{dx}\left[\int f(x)\,dx\right].$

13. $\int \frac{1}{\sin^2 x \cos^2 x}\,dx.$

14. $\int \frac{1}{\sqrt{x} + \sqrt{x-1}}\,dx.$

15. $\int e^{-\log e^x}\,dx.$

16. $\int \frac{e^x}{a^x}\,dx.$

17. $\int 2^x e^x\,dx.$

18. $\int \frac{x}{\sqrt{x+1}}\,dx.$

19. $\int \frac{x}{(x+1)^2}\,dx.$

20. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx.$

21. $\int \cos^2 \alpha\,dx.$

22. $\int \frac{1}{x\cos\alpha + 1}\,dx.$

23. $\int \sec x.\log(\sec x + \tan x)\,dx.$

24. $\int \frac{1}{\cos\alpha + x\sin\alpha}\,dx.$

25. $\int \cot x.\log \sin x\,dx.$

26. $\int \left(x - \frac{1}{2}\right)^3 dx.$

27. $\int \frac{1}{x(2 + 3\log x)}\,dx.$

28. $\int \frac{1 - \sin x}{x + \cos x}\,dx.$

29. $\int \frac{1 - \cos x}{\sin x}\,dx.$

30. $\int \frac{x^{e-1} + e^{x-1}}{x^e + e^x}\,dx.$

31. $\int \frac{(x+1)}{x}(x + \log x)\,dx.$

32. $\int \left(\sqrt{ax} - \frac{1}{\sqrt{ax}}\right)^2 dx.$

33. $\int_0^{\pi} |\cos x|\,dx.$

34. $\int_0^2 [x]dx$ where [ ] is greatest integer function.

35. $\int_0^{\sqrt{2}} [x^2]dx$ where [ ] is greatest integer function.

36. $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx.$

37. $\int_{-2}^{1} \frac{|x|}{x}dx.$

38. $\int_{-1}^{1} x|x|dx.$

39. If $\int_0^a \frac{1}{1+x^2} = \frac{\pi}{4}$, then what is value of a.

40. $\int_a^b f(x)dx + \int_b^a f(x)dx.$

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

41. (i) $\int \frac{x \text{ cosec } \left(\tan^{-1} x^2\right)}{1 + x^4} dx.$ (ii) $\int \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} dx.$

(iii) $\int \frac{1}{\sin(x-a)\sin(x-b)} dx.$ (iv) $\int \frac{\cos(x+a)}{\cos(x-a)} dx.$

(v) $\int \cos x \cos 2x \cos 3x \; dx.$ (vi) $\int \cos^5 x \; dx.$

(vii) $\int \sin^2 x \cos^4 x \; dx.$ (viii) $\int \cot^3 x \text{ cosec}^4 x \; dx.$

(ix) $\int \frac{\sin x \cos x}{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}} dx.$ (x) $\int \frac{1}{\sqrt{\cos^3 x \cos(x+a)}} dx.$

(xi) $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx.$ (xii) $\int \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx.$

42. Evaluate :

(i) $\int \frac{x}{x^4 + x^2 + 1} dx.$

*(ii) $\int \frac{1}{x\left[6(\log x)^2 + 7\log x + 2\right]} dx.$

(iii) $\int \frac{dx}{1 + x - x^2}.$

(iv) $\int \frac{1}{\sqrt{9 + 8x - x^2}} dx.$

(v) $\int \frac{1}{\sqrt{(x - a)(x - b)}} dx.$

(vi) $\int \sqrt{\frac{\sin(x - \alpha)}{\sin(x + \alpha)}} dx.$

(vii) $\int \frac{5x - 2}{3x^2 + 2x + 1} dx.$

(viii) $\int \frac{x^2}{x^2 + 6x + 12} dx.$

(ix) $\int \frac{x + 2}{\sqrt{4x - x^2}} dx.$

(x) $\int x\sqrt{1 + x - x^2} dx.$

(xii) $\int (3x - 2)\sqrt{x^2 + x + 1} dx.$

(xiii) $\int \sqrt{\sec x + 1} dx.$

43. Evaluate :

(i) $\int \frac{dx}{x(x^7 + 1)}.$

(ii) $\int \frac{\sin x}{(1 + \cos x)(2 + 3\cos x)} dx.$

(iii) $\int \frac{\sin\theta\cos\theta}{\cos^2\theta - \cos\theta - 2} d\theta.$

(iv) $\displaystyle\int \frac{x-1}{(x+1)(x-2)(x+3)}\,dx.$

(v) $\displaystyle\int \frac{x^2+x+2}{(x-2)(x-1)}\,dx.$

(vi) $\displaystyle\int \frac{\left(x^2+1\right)\left(x^2+2\right)}{\left(x^3+3\right)\left(x^2+4\right)}\,dx.$

(vii) $\displaystyle\int \frac{dx}{(2x+1)\left(x^2+4\right)}.$

(viii) $\displaystyle\int \frac{dx}{\sin x\,(1-2\cos x)}.$

(ix) $\displaystyle\int \frac{\sin x}{\sin 4x}\,dx.$

(x) $\displaystyle\int \frac{x^2-1}{x^4+x^2+1}\,dx.$

(xi) $\displaystyle\int \sqrt{\tan x}\;dx.$

(xii) $\displaystyle\int \frac{x^2+9}{x^4+81}\,dx.$

44. Evaluate :

(i) $\displaystyle\int x^5 \sin x^3 dx.$

(ii) $\displaystyle\int \sec^3 x\,dx.$

(iii) $\displaystyle\int e^{ax}\cos(bx+c)\,dx.$

(iv) $\displaystyle\int \sin^{-1}\frac{6x}{1+9x^2}\,dx.$

(v) $\displaystyle\int \cos\sqrt{x}\;dx.$

(vi) $\displaystyle\int x^3\tan^{-1}x\;dx.$

(vii) $\displaystyle\int e^{2x}\left(\frac{1+\sin 2x}{1+\cos 2x}\right)dx.$

(viii) $\displaystyle\int e^{x}\left(\frac{x-1}{2x^2}\right)dx.$

(ix) $\displaystyle\int e^{x}\left(\frac{1-x}{1+x^2}\right)^2 dx.$

(x) $\displaystyle\int e^{x}\frac{\left(x^2+1\right)}{(x+1)^2}\,dx.$

(xi) $\displaystyle\int e^{x}\frac{(2+\sin 2x)}{(1+\cos 2x)}\,dx.$

(xii) $\displaystyle\int \left\{\log(\log x)+\frac{1}{(\log x)^2}\right\}dx.$

45. Evaluate the following definite integrals :

(i) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x}\, dx.$ (ii) $\displaystyle\int_0^{\frac{\pi}{2}} \cos 2x \log \sin x \, dx.$

(iii) $\displaystyle\int_0^{1} x\sqrt{\frac{1 - x^2}{1 + x^2}}\, dx.$ (iv) $\displaystyle\int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{\left(1 - x^2\right)^{3/2}}\, dx.$

(v) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\sin^4 x + \cos^4 x}\, dx.$ (vi) $\displaystyle\int_1^{2} \frac{5x^2}{x^2 + 4x + 3}\, dx.$

(vii) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{x + \sin x}{1 + \cos x}\, dx.$

46. Evaluate :

(i) $\displaystyle\int_1^{3} \left\{|x - 1| + |x - 2| + |x - 3|\right\} dx.$ (ii) $\displaystyle\int_0^{\pi} \frac{x}{1 + \sin x}\, dx.$

(iii) $\displaystyle\int_0^{\frac{\pi}{4}} \log\left(1 + \tan x\right) dx.$ (iv) $\displaystyle\int_0^{\frac{\pi}{2}} \log \sin x \, dx.$

(v) $\displaystyle\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx.$

(vi) $\displaystyle\int_{-2}^{2} f(x)\, dx$ where $f(x) = \begin{cases} 2x - x^3 & \text{when } -2 \le x < 1 \\ x^3 - 3x + 2 & \text{when } -1 \le x < 1 \\ 3x - 2 & \text{when } 1 \le x < 2. \end{cases}$

(vii) $\int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx.$

(viii) $\int_0^{\pi} \frac{x}{a^2 \cos^2 x + b^2 \sin^2 x} dx.$

47. Evaluate the following integrals

(i) $\int_1^3 \left|x^2 - 2x\right| dx.$

(ii) $\int_0^1 \sin^{-1}\left(\frac{2x}{1 + x^2}\right) dx.$

(iii) $\int_{-1}^1 \log\left(\frac{1 + \sin x}{1 - \sin x}\right) dx.$

(iv) $\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx.$

LONG ANSWER TYPE QUESTIONS (6 MARKS)

48. Evaluate the following integrals :

(i) $\int \frac{x^5 + 4}{x^5 - x} dx.$

(ii) $\int \frac{dx}{(x - 1)(x^2 + 4)} dx$

(iii) $\int \frac{2x^3}{(x + 1)(x - 3)^2} dx$

(iv) $\int \frac{x^4}{x^4 - 16} dx$

(v) $\int_0^{\frac{\pi}{2}} \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx.$

(vi) $\int \frac{1}{x^4 + 1} dx.$

(vii) $\int_0^{\infty} \frac{x \tan^{-1} x}{\left(1 + x^2\right)^2} dx.$

49. Evaluate the following integrals as limit of sums :

(i) $\int_{2}^{4} (2x + 1)\, dx.$ (ii) $\int_{0}^{2} \left(x^2 + 3\right) dx.$

(iii) $\int_{1}^{3} \left(3x^2 - 2x + 4\right) dx.$ (iv) $\int_{0}^{4} \left(3x^2 + e^{2x}\right) dx.$

(v) $\int_{2}^{5} \left(x^2 + 3x\right) dx.$

H.O.T.S.

VERY SHORT ANSWER TYPE QUESTIONS (1 MARK EACH)

50. $\int_{0}^{1} [2x]dx$ where [ ] is greatest integer function.

51. $\int e^{\log x + \log \sin x} dx.$

52. $\int e^{\log(x+1) - \log x} dx.$

53. $\int \frac{\sin x}{\sin 2x}\, dx.$

54. $\int \sin x \sin 2x\, dx.$

55. $\int_{-\frac{\pi}{4}}^{\pi/4} |\sin x|\, dx.$

56. $\int_{a}^{b} f(x)\,dx + \int_{b}^{a} f(a+b-x)\, dx.$

57. $\int \frac{1}{\sec x + \tan x} dx.$

58. $\int \frac{\sin^2 x}{1 + \cos x} dx.$

59. $\int \frac{1 - \tan x}{1 + \tan x} dx.$

60. $\int \frac{a^x + b^x}{c^x} dx.$

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

61. Evaluate

(i) $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\, dx,\ x \in [0, 1]$

(ii) $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\, dx$

(iii) $\int \frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right) - 2\log x\right]}{x^4}\, dx$

(iv) $\int \frac{x^2}{(x\sin x + \cos x)^2}\, dx$

(v) $\int \sin^{-1}\sqrt{\frac{x}{a+x}}\, dx$

(vi) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\, dx$

(vii) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\sin|x| - \cos|x|\right) dx$

(viii) $\int_{1}^{2} \left[x^2\right] dx$, where [x] is greatest integer function

(ix) $\int_{-1}^{\frac{3}{2}} |x \sin \pi x|\, dx$.

LONG ANSWER TYPE QUESTIONS (6 MARKS)

62. Evaluate

(i) $\int_0^1 \cot^{-1}\left(1 - x + x^2\right) dx$

(ii) $\int \frac{dx}{(\sin x - 2\cos x)(2\sin x + \cos x)}$

(iii) $\int_0^1 \frac{\log(1 + x)}{1 + x^2} dx$

(iv) $\int_0^{\frac{\pi}{2}} (2\log \sin x - \log \sin 2x)\, dx.$

63. $\int \frac{1}{\sin x + \sin 2x} dx.$

64. $\int \frac{(3\sin\theta - 2)\cos\theta}{5 - \cos^2\theta - 4\sin\theta} d\theta.$

65. $\int \sec^3 x\, dx.$

66. $\int e^{2x} \cos 3x\, dx.$

ANSWERS

1. $\frac{\pi}{2} x + c.$

2. $2e - 2$

3. $\tan x + c.$

4. $\frac{8^x}{\log 8} + \frac{x^9}{9} + 8\log|x| + \frac{x^2}{16} + c.$

5. 0

6. $\log |\log (\log x)| + c$

7. 0

8. $\frac{x^{a+1}}{a+1} + \frac{a^x}{\log a} + c$

9. $\tan x + c$

10. 0

11. $\frac{x^{c+1}}{c+1}+\frac{c^x}{\log c}+c$

12. $f(x) + c$

13. $\tan x - \cot x + c$

14. $\frac{2}{3}x^{3/2}-\frac{2}{3}(x-1)^{3/2}+c$

15. $\log |x| + c$

16. $\left(\frac{e}{a}\right)^x \Big/ \log(e/a)+c$

17. $\frac{2^x e^x}{\log(2e)}+c$

18. $\frac{2}{3}(x+1)^{3/2}-2(x+1)^{1/2}+c.$

19. $\log|x+1|+\frac{1}{x+1}+c.$

20. $2e^{\sqrt{x}}+c$

21. $x \cos^2 \alpha + c$

22. $\frac{\log|x\cos\alpha+1|}{\cos\alpha}+c.$

23. $\frac{(\log|\sec x+\tan x|)^2}{2}+c$

24. $\frac{\log|\cos\alpha+x\sin\alpha|}{\sin\alpha}+c$

25. $\frac{(\log\sin x)^2}{2}+c$

26. $\frac{x^4}{4}+\frac{1}{2x^2}-\frac{3x^2}{2}+3|\log x|+c.$

27. $\frac{1}{3}\log|2+3\log x|+c.$

28. $\log |x + \cos x|\ c$

29. $2 \log |\sec x/2| + c.$

30. $\frac{1}{e}\log|x^e+e^x|+c.$

31. $\frac{(x+\log x)^2}{2}+c$

32. $a\frac{x^2}{2}+\frac{\log|ax|}{a}-2x+c.$

33. 0

34. 1

35. $\left(\sqrt{2}-1\right)$

36. $\dfrac{b-a}{2}$

37. –1

38. 0

39. 1

40. 0

41. (i) $\dfrac{1}{2}\log\left[\operatorname{cosec}\left(\tan^{-1}x^{2}\right)-\dfrac{1}{x^{2}}\right]+c.$

(ii) $\dfrac{1}{2}\left(x^{2}-x\sqrt{x^{2}-1}\right)+\dfrac{1}{2}\log\left|x+\sqrt{x^{2}-1}\right|+c.$

(iii) $\dfrac{1}{\sin(a-b)}\log\left|\dfrac{\sin(x-a)}{\sin(x-b)}\right|+c$

(iv) $x\cos 2a-\sin 2a\log|\sec(x-a)|+c.$

(v) $\dfrac{1}{48}[12x+6\sin 2x+3\sin 4x+2\sin 6x]+c.$

(vi) $\sin x-\dfrac{2}{3}\sin^{3}x+\dfrac{1}{5}\sin^{5}x+c.$

(vii) $\dfrac{1}{32}\left[2x+\dfrac{1}{2}\sin 2x-\dfrac{1}{2}\sin 4x-\dfrac{1}{6}\sin 6x\right]+c.$

(viii) $-\left(\dfrac{\cot^{6}x}{6}+\dfrac{\cot^{4}x}{4}\right)+c.$

(ix) $\dfrac{1}{\left(a^{2}-b^{2}\right)\sqrt{a^{2}\sin^{2}x+b^{2}\cos^{2}x}}+c.$

[**Hint. :** put $a^2\sin^2 x+b^2\cos^2 x=t$]

(x) $-2\operatorname{cosec} a\sqrt{\cos a-\tan x\,.\,\sin a}+c.$

[**Hint. :** Take $\sec^2 x$ as numerator]

(xi) $\tan x - \cot x - 3x + c.$

(xii) $\sin^{-1}(\sin x - \cos x) + c.$

42. (i) $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + c.$ [**Hint :** put $x^2 = t$]

(ii) $\log\left|\frac{2\log x + 1}{3\log x + 2}\right| + C$ [**Hint :** put $\log x = t$]

(iii) $\frac{1}{\sqrt{5}}\log\left|\frac{\sqrt{5} - 1 + 2x}{\sqrt{5} + 1 - 2x}\right| + c$ (iv) $\sin^{-1}\left(\frac{x-4}{5}\right) + c.$

(v) $2\log\left|\sqrt{x-a} + \sqrt{x-b}\right| + c$

(vi)

$$-\cos\alpha\sin^{-1}\left(\frac{\cos x}{\cos\alpha}\right) - \sin\alpha.\log\left|\sin x + \sqrt{\sin^2 x - \sin^2\alpha}\right| + c$$

$$\left[\textbf{Hint} : \sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)}} = \frac{\sin(x-\alpha)}{\sin^2 x - \sin^2\alpha}\right]$$

(vii) $\frac{5}{6}\log\left|3x^2 + 2x + 1\right| + \frac{(-11)}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + c$

(viii) $x - 3\log\left|x^2 + 6x + 12\right| + 2\sqrt{3}\tan^{-1}\left(\frac{x+3}{\sqrt{3}}\right) + c$

(ix) $-\sqrt{4x - x^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + c$

(x) $\frac{-1}{3}\left(1 + x - x^2\right)^{\frac{3}{2}} + \frac{1}{8}(2x-1)\sqrt{1 + x - x^2}$

$$+\frac{5}{16}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + c$$

(xi) $\left(x^2 + x + 1\right)^{\frac{3}{2}} - \frac{7}{2}\left[\begin{array}{l}\left(x + \frac{1}{2}\right)\sqrt{x^2 + x + 1} \\ + \frac{3}{8}\log\left|x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right|\end{array}\right] + c$

(xii) $-\log\left|\cos x + \frac{1}{2} + \sqrt{\cos^2 x + \cos x}\right| + c$

[**Hint. :** Multiply and divide by $\sqrt{\sec x + 1}$]

43. (i) $\frac{1}{7}\log\left|\frac{x^7}{x^7 + 1}\right| + c$

(ii) $\log\left|\frac{1 + \cos x}{2 + 3\cos x}\right| + c$

(iii) $\frac{-2}{3}\log|\cos\theta - 2| - \frac{1}{3}\log|1 + \cos\theta| + c.$

(iv) $\frac{9}{10}\log|x + 3| + \frac{4}{15}\log|x - 2| - \frac{1}{6}|x + 1| + c$

(v) $x + 4\log\left|\frac{(x - 2)^2}{x - 1}\right| + c$

(vi) $x + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - 3\tan^{-1}\left(\frac{x}{2}\right) + c$

[**Hint. :** put $x^2 = t$]

(vii) $\frac{2}{17}\log|2x + 1| - \frac{1}{17}\log\left|x^2 + 4\right| + \frac{1}{34}\tan^{-1}\frac{x}{2} + c$

(viii) $-\frac{1}{2}\log|1 - \cos x| - \frac{1}{6}\log|1 + \cos x| + \frac{2}{3}\log|1 - 2\cos x| + c$

[**Hint. :** Multiply N^r and D^r by sin x and put cos $x = t$]

(ix) $\frac{-1}{8}\log\left|\frac{1+\sin x}{1-\sin x}\right| + \frac{1}{4\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right| + c$

(x) $\frac{1}{2}\log\left|\frac{x^2 - x + 1}{x^2 + x + 1}\right| + c$

(xi) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right) + \frac{1}{2\sqrt{2}}\log\left|\frac{\tan x - \sqrt{2\tan x} + 1}{\tan x + \sqrt{2\tan x} + 1}\right| + c$

(xii) $\frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{x^2 - 9}{3\sqrt{2}}\right) + c$

44. (i) $\frac{1}{3}\left[-x^3\cos x^3 + \sin x^3\right] + c$

(ii) $\frac{1}{2}\left[\sec x\tan x + \log|\sec x + \tan x|\right] + c$

[**Hint. :** Write $\sec^3 x = \sec x \cdot \sec^2 x$ and take $\sec x$ as first function]

(iii) $\frac{e^{ax}}{a^2 + b^2}\left[a\cos(bx + c) + b\sin(bx + c)\right] + c_1$

(iv) $2x\tan^{-1}3x - \frac{1}{3}\log\left|1 + 9x^2\right| + c$ [**Hint. :** put $3x = \tan\theta$]

(v) $2\left[\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right] + c$

(vi) $\left(\frac{x^4 - 1}{4}\right)\tan^{-1}x - \frac{x^3}{12} + \frac{x}{4} + c.$

(vii) $\frac{1}{2}e^{2x}\tan x + c.$ (viii) $\frac{e^x}{2x} + c.$

(ix) $\dfrac{e^x}{1+x^2}+c.$ (x) $e^x\left(\dfrac{x-1}{x+1}\right)+c.$

(xi) $e^x \tan x + c.$

(xii) $x\log|\log x| - \dfrac{x}{\log x} + c.$ [**Hint. :** put $\log x = t \Rightarrow x = e^t$]

45. (i) $\dfrac{1}{20}\log 3.$ (ii) $-\dfrac{\pi}{4}$

(iii) $\dfrac{\pi}{4}-\dfrac{1}{2}.$ [**Hint. :** put $x^2 = t$] (iv) $\dfrac{\pi}{4}-\dfrac{1}{2}\log 2.$

(v) $\dfrac{\pi}{2}.$

(vi) $5 - 10\log\dfrac{15}{8} + \dfrac{25}{2}\log\left(\dfrac{6}{5}\right).$

(vii) $\pi/2.$ $\left[\textbf{Hint. :} \left(\dfrac{x}{1+\cos x}+\dfrac{\sin x}{1+\cos x}\right)dx.\right]$

46. (i) 8. (ii) $\pi.$

(iii) $\dfrac{\pi}{8}\log 2.$ (iv) $\dfrac{-\pi}{2}\log 2.$

(v) $\dfrac{1}{4}\pi^2.$

(vi) 95/12.

$$\left[\textbf{Hint. :} \int_{-2}^{2} f(x)\,dx = \int_{-2}^{-1} f(x)\,dx + \int_{-1}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx\right]$$

(vii) $\frac{\pi^2}{16}$.

(viii) $\frac{\pi^2}{2ab}$. $\left[\textbf{Hint. : } \text{Use } \int_0^a f(x) = \int_0^a f(a-x)\right]$

47. (i) 2. (ii) $\frac{\pi}{2} - \log 2$.

(iii) 0. (iv) $\pi/2$.

48. (i) $x - 4\log|x| + \frac{5}{4}\log|x-1| + \frac{3}{4}\log|x+1|$

$+ \log\left|x^2+1\right| - \frac{1}{2}\tan^{-1} x + c.$

$x + \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1} x + \log\left|\frac{x^2-1}{x^4+1}\right| + c.$

(ii) $\frac{1}{5}\log|x-1| - \frac{1}{10}\log\left|x^2+4\right| - \frac{1}{10}\tan^{-1}\left(\frac{x}{2}\right) + c.$

(iii) $2x - \frac{1}{8}\log|x+1| + \frac{81}{8}\log|x-3| - \frac{27}{2(x-3)} + c.$

(iv) $x + \frac{1}{2}\log\left|\frac{x-2}{x+2}\right| - \tan^{-1}\left(\frac{x}{2}\right) + c.$

(v) $\pi/\sqrt{2}$.

(vi) $\frac{1}{2\sqrt{2}}\tan^{-1}\frac{\left(x^2-1\right)}{\sqrt{2x}} - \frac{1}{4\sqrt{2}}\log\left|\frac{x^2-\sqrt{2x}+1}{x^2+\sqrt{2x}+1}\right| + c$

(vii) $\pi/8$.

49. (i) 14. (ii) $\frac{26}{3}$.

(iii) 26. (iv) $\frac{1}{2}\left(127 + e^{8}\right)$.

(v) $\frac{141}{2}$.

50. $\frac{1}{2}$

51. $-x \cos x + \sin x + c$.

52. $x + \log x + c$.

53. $\frac{1}{2}\log|\sec x + \tan x| + c$.

54. $-\frac{1}{2}\left(\frac{\sin 3x}{3} - \sin x\right)$

55. $2 - \sqrt{2}$

56. 0

57. $\log|1 + \sin x| + c$

58. $x - \sin x + c$

59. $\log|\cos x + \sin x| + c$

60. $\frac{(a/c)^{x}}{\log(a/c)} + \frac{(b/c)^{x}}{\log(b/c)} + C$.

61. (i) $\frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x} + \frac{2\sqrt{x - x^{2}}}{\pi} - x + c$

(ii) $-2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x - x^2} + c$

(iii) $-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\left[\log\left(1+\frac{1}{x^2}\right)-\frac{2}{3}\right]+c$

(iv) $\frac{\sin x - x\cos x}{x\sin x + \cos x} + c$

(v) $(x+a)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + c$

(vi) $2\sin^{-1}\frac{\sqrt{3}-1}{2}$

(vii) 0

(viii) $-\sqrt{2} - \sqrt{3} + 5$

(ix) $\frac{3}{\pi} + \frac{1}{\pi^2}$.

62. (i) $\frac{\pi}{2} - \log 2$

(ii) $-\frac{1}{5}\log\left|\frac{\tan x - x}{2\tan x + 1}\right| + c$

(iii) $\frac{\pi}{8}\log 2$.

(iv) $\frac{\pi}{2}\log\left(\frac{1}{2}\right)$.

63. $\frac{1}{6}\log|1-\cos x| + \frac{1}{2}\log(1+\cos x) - \frac{2}{3}\log|1+2\cos x| + c.$

64. $3\log|(2-\sin\theta)| + \frac{4}{2-\sin\theta} + c.$

65. $\frac{1}{2}\sec x + \tan x + \frac{1}{2}\log|\sec x + \tan x| + c.$

66. $\frac{e^{2x}}{13}(2\cos 3x + 3\sin 3x) + c.$

CHAPTER 8

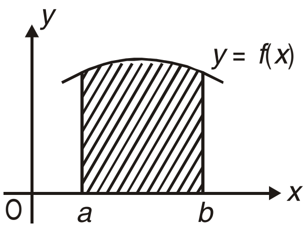
APPLICATIONS OF INTEGRALS

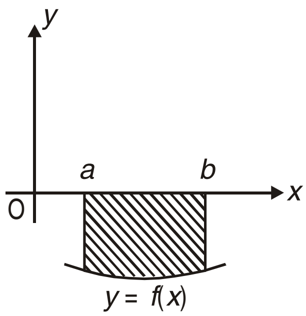
POINTS TO REMEMBER

AREA OF BOUNDED REGION

- Area bounded by the curve $y = f(x)$, the x axis and between the ordinates, $x = a$ and $x = b$ is given by

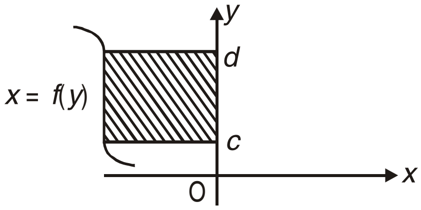
$$\text{Area} = \int_a^b |f(x)|\, dx$$

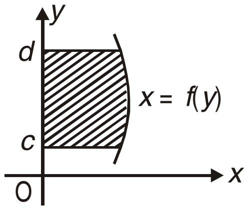




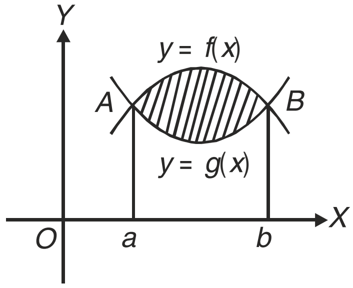
- Area bounded by the curve $x = f(y)$ the y-axis and between abscissas, $y = c$ and $y = d$ is given by

$$\text{Area} = \int_c^d |f(y)|\, dy$$





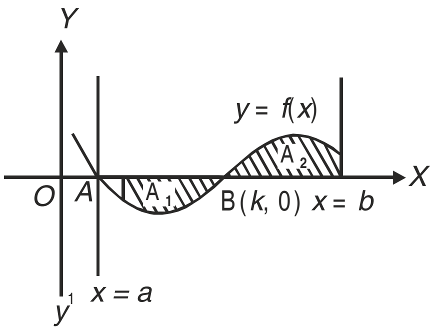
- Area bounded by two curves $y = f(x)$ and $y = g(x)$ such that $0 \le g(x) \le f(x)$ for all $x \in [a, b]$ and between the ordinate at $x = a$ and $x = b$ is given by



$$\text{Area} = \int_a^b [f(x) - g(x)]\, dx$$

- Required Area

$$= \left| \int_a^k f(x)dx \right| + \int_k^b f(x)dx.$$



LONG ANSWER TYPE QUESTION (6 MARKS)

1. Find the area enclosed by circle $x^2 + y^2 = a^2$.

2. Find the area of region bounded by $\left\{(x, y) : |x - 1| \le y \le \sqrt{25 - x^2}\right\}$.

3. Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

4. Find the area of region in the first quadrant enclosed by x–axis, the line $y = x$ and the circle $x^2 + y^2 = 32$.

5. Find the area of region $\{(x, y) : y^2 \le 4x, 4x^2 + 4y^2 \le 9\}$

6. Prove that the curve $y = x^2$ and, $x = y^2$ divide the square bounded by $x = 0$, $y = 0$, $x = 1$, $y = 1$ into three equal parts.

7. Find smaller of the two areas enclosed between the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the line

   $bx + ay = ab.$

8. Find the common area bounded by the circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$.

9. Using integration, find the area of the region bounded by the triangle whose vertices are

   (a) $(-1, 0)$, $(1, 3)$ and $(3, 2)$ (b) $(-2, 2)$ $(0, 5)$ and $(3, 2)$

10. Using integration, find the area bounded by the lines.

    (i) $x + 2y = 2$, $y - x = 1$ and $2x + y - 7 = 0$

    (ii) $y = 4x + 5$, $y = 5 - x$ and $4y - x = 5$.

11. Find the area of the region $\{(x, y) : x^2 + y^2 \le 1 \le x + y\}$.

12. Find the area of the region bounded by

    $y = |x - 1|$ and $y = 1$.

13. Find the area enclosed by the curve $y = \sin x$ between $x = 0$ and $x = \frac{3\pi}{2}$ and x-axis.

14. Find the area bounded by semi circle $y = \sqrt{25 - x^2}$ and x-axis.

15. Find area of region given by $\{(x, y) : x^2 \le y \le |x|\}$.

16. Find area of smaller region bounded by ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and straight line $2x + 3y = 6$.

17. Find the area of region bounded by the curve $x^2 = 4y$ and line $x = 4y - 2$.

18. Using integration find the area of region in first quadrant enclosed by x-axis, the line $x = \sqrt{3}y$ and the circle $x^2 + y^2 = 4$.

19. Find smaller of two areas bounded by the curve $y = |x|$ and $x^2 + y^2 = 8$.

H.O.T.S.

20. Find the area lying above x-axis and included between the circle $x^2 + y^2 = 8x$ and the parabola $y^2 = 4x$.

21. Using integration, find the area enclosed by the curve $y = \cos x$, $y = \sin x$ and x-axis in the interval $\left(0, \frac{\pi}{2}\right)$.

22. Sketch the graph $y = |x - 5|$. Evaluate $\int_0^6 |x - 5|\, dx$.

23. Find area enclosed between the curves, $y = 4x$ and $x^2 = 6y$.

ANSWERS

1. πa^2 sq. units.

2. $\left(25\frac{\pi}{4} - \frac{1}{2}\right)$ *sq. units.*

3. πab sq. units

4. $(4\pi - 8)$ sq. units

5. $\frac{\sqrt{2}}{6} + \frac{9\pi}{8} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)$ sq. units

7. $\frac{(\pi - 2)\, ab}{4}$ sq. units

8. $\left(\frac{8\pi}{3} - 2\sqrt{3}\right)$ sq. units

9. (a) 4 sq. units (b) 2 sq. units

10. (a) 6 sq. unit [**Hint.** Coordinate of vertices are (0, 1) (2, 3) (4, – 1)]

(b) $\frac{15}{2}$ sq. [**Hint.** Coordinate of vertices are (– 1, 1) (0, 5) (3, 2)]

11. $\left(\frac{\pi}{4}-\frac{1}{2}\right)$ sq. units

12. 1 sq. units

13. 3 sq. units

14. $\frac{25}{2}\pi$ sq. units

15. $\frac{1}{3}$ sq. units

16. $\frac{3}{2}(\pi - 2)$ sq. units

17. $\frac{9}{8}$ sq. units

18. $\frac{\pi}{3}$ sq. unit

19. 2π sq. unit.

20. $\frac{4}{3}(8 + 3\pi)$ sq. units

21. $\left(2-\sqrt{2}\right)$ sq. units.

22. 13 sq. units.

23. 8 sq. units.

CHAPTER 9

DIFFERENTIAL EQUATION

POINTS TO REMEMBER

- **Differential Equation :** Equation containing derivatives of a dependant variable with respect to an independent variable is called differential equation.
- **Order of a Differential Equation :** The order of a differential equation is defined to be the order of the highest order derivative occurring in the differential equation.
- **Degree of a Differential Equation :** Highest power of highest order derivative involved in the equation is called degree of differential equation where equation is a polynomial equation in differential coefficients.
- **Formation of a Differential Equation :** We differentiate the family of curves as many times as the number of arbitrary constant in the given family of curves. Now eliminate the arbitrary constants from these equations. After elimination the equation obtained is differential equation.
- **Solution of Differential Equation**

  (i) **Variable Separable Method**

  $$\frac{dy}{dx} = f(x, y)$$

  We separate the variables and get

  $$f(x)dx = g(y)dy$$

  Then $\int f(x)\,dx = \int g(y)\,dy + c$ is the required solutions.

  (ii) ***Homogenous Differential Equation :*** A differential equation of the form $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$ where $f(x, y)$ and $g(x, y)$ are both homogeneous functions of the same degree in x and y *i.e.,* of the form $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ is called a homogeneous differential equation.

For solving this type of equations we substitute $y = vx$ and then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. The equation can be solved by variable separable method.

(iii) ***Linear Differential Equation :*** An equation of the from $\frac{dy}{dx} + Py = Q$ where P and Q are constant or functions of x only is called a linear differential equation. For finding solution of this type of equations, we find integrating factor ($I.F.$) $= e^{\int P\,dx}$.

Solution is $y\,(I.F.) = \int Q.\,(I.F.)\,dx + c$

VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

1. Write the order and degree of the following differential equations.

(i) $\frac{dy}{dx} + \cos y = 0.$

(ii) $\left(\frac{dy}{dx}\right)^2 + 3\frac{d^2y}{dx^2} = 4.$

(iii) $\frac{d^4y}{dx^4} + \sin x = \left(\frac{d^2y}{dx^2}\right)^5.$

(iv) $\frac{d^5y}{dx^5} + \log\left(\frac{dy}{dx}\right) = 0.$

(v) $\sqrt{1 + \frac{dy}{dx}} = \left(\frac{d^2y}{dx^2}\right)^{1/3}.$

(vi) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = k\,\frac{d^2y}{dx^2}.$

(vii) $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^3 = \sin x.$

2. Write the general solution of following differential equations.

(i) $\frac{dy}{dx} = x^5 + x^2 - \frac{2}{x}.$ (ii) $(e^x + e^{-x})\, dy = (e^x - e^{-x})dx$

(iii) $\frac{dy}{dx} = x^3 + e^x + x^e.$ (iv) $\frac{dy}{dx} = 5^{x+y}.$

(v) $\frac{dy}{dx} = \frac{1 - \cos 2x}{1 + \cos 2y}.$ (vi) $\frac{dy}{dx} = \frac{1 - 2y}{3x + 1}.$

3. Write integrating factor of the following differential equation

(i) $\frac{dy}{dx} + y \cos x = \sin x$

(ii) $\frac{dy}{dx} + y \sec^2 x = \sec x + \tan x$

(iii) $x^2 \frac{dy}{dx} + y = x^4.$ (iv) $x \frac{dy}{dx} + y \log x = x + y$

(v) $x \frac{dy}{dx} - 3y = x^3$ (vi) $\frac{dy}{dx} + y \tan x = \sec x$

4. Write order of the differential equation of the family of following curves

(i) $y = Ae^x + Be^{x+c}$ (ii) $Ay = Bx^2$

(iii) $(x - a)^2 + (y - b)^2 = 9$ (iv) $Ax + By^2 = Bx^2 - Ay$

(v) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

5. (i) Show that $y = e^{m \sin^{-1} x}$ is a solution of

$$\left(1 - x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2 y = 0.$$

(ii) Show that $y = \sin(\sin x)$ is a solution of differential equation

$$\frac{d^2y}{dx^2} + (\tan x)\frac{dy}{dx} + y\cos^2 x = 0.$$

(iii) Show that $y = Ax + \frac{B}{x}$ is a solution of $\frac{x^2 d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$

(iv) Show that $y = a \cos(\log x) + b \sin(\log x)$ is a solution of

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

(v) Find the differential equation of the family of curves

$y = e^x (A \cos x + B \sin x)$, where A and B are arbitrary constants.

(vi) Find the differential equation of an ellipse with major and minor axes $2a$ and $2b$ respectively.

(vii) Form the differential equation representing the family of curves $(y - b)^2 = 4(x - a)$.

6. Solve the following diff. equations.

(i) $\frac{dy}{dx} + y\cot x = \sin 2x.$ (ii) $x\frac{dy}{dx} + 2y = x^2 \log x.$

(iii) $\frac{dx}{dy} + \frac{1}{x}.y = \cos x + \frac{\sin x}{x}, \quad x > 0.$

(iv) $\cos^3 x \frac{dy}{dx} + \cos x = \sin x.$

7. Solve each of the following differential equations :

(i) $y - x\frac{dy}{dx} = 2\left(y^2 + \frac{dy}{dx}\right).$

(ii) $\cos y\, dx + (1 + 2e^{-x}) \sin y\, dy = 0.$

(iii) $x\sqrt{1-y^2}\,dy + y\sqrt{1-x^2}\,dx = 0.$

(iv) $\sqrt{(1-x^2)(1-y^2)}\;dy + xy\;dx = 0.$

(v) $(xy^2 + x)\,dx + (yx^2 + y)\,dy = 0;\ y(0) = 1.$

(vi) $\dfrac{dy}{dx} = y\sin^3 x\cos^3 x + xy\,e^x.$

(vii) $\tan x\ \tan y\,dx + \sec^2 x\sec^2 y\,dy = 0$

8. Solve the following differential equations :

(i) $x^2 y\,dx - (x^3 + y^3)\,dy = 0.$

(ii) $x^2\dfrac{dy}{dx} = x^2 + xy + y^2.$

(iii) $\left(x^2 - y^2\right)dx + 2xy\;dy = 0,\quad y(1) = 1.$

(iv) $\left(y\sin\dfrac{x}{y}\right)dx = \left(x\sin\dfrac{x}{y} - y\right)dy.$ (v) $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan\left(\dfrac{y}{x}\right).$

(vi) $\dfrac{dy}{dx} = \dfrac{2xy}{x^2 + y^2}$ (vii) $\dfrac{dy}{dx} = e^{x+y} + x^2e^y.$

(viii) $\dfrac{dy}{dx} = \sqrt{\dfrac{1-y^2}{1-x^2}}.$

9. (i) Form the differential equation of the family of circles touching y-axis at (0, 0).

(ii) Form the differential equation of family of parabolas having vertex at (0, 0) and axis along the (i) positive y-axis (ii) positive x-axis.

(iii) Form differential equation of family of circles passing through origin and whose centre lie on x-axis.

10. Show that the differential equation $\frac{dy}{dx} = \frac{x + 2y}{x - 2y}$ is homogeneous and solve it.

11. Show that the differential equation :

$(x^2 + 2xy - y^2)\ dx + (y^2 + 2xy - x^2)\ dy = 0$ is homogeneous and solve it.

12. Solve the following differential equations :

(i) $\frac{dy}{dx} - 2y = \cos 3x.$

(ii) $\sin x \frac{dy}{dx} + y \cos x = 2 \sin^2 x \cos x$ if $y\left(\frac{\pi}{2}\right) = 1$

LONG ANSWER TYPE QUESTIONS (6 MARKS)

13. Solve the following differential equations :

(i) $(x^3 + y^3)\ dx = (x^2y + xy^2)dy.$

(ii) $x\ dy - y\ dx = \sqrt{x^2 + y^2}dx.$

(iii) $y\left\{x \cos\left(\frac{y}{x}\right) + y \sin\left(\frac{y}{x}\right)\right\} dx$

$- x\left\{y \sin\left(\frac{y}{x}\right) - x \cos\left(\frac{y}{x}\right)\right\} dy = 0.$

(iv) $x^2dy + y(x + y)\ dx = 0$ given that $y = 1$ when $x = 1$.

(v) $xe^{\frac{y}{x}} - y + x\frac{dy}{dx} = 0$ if $y(e) = 0$

(vi) $(x^3 - 3xy^2)\ dx = (y^3 - 3x^2y)dy.$

VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

14. (i) Write the order and degree of the differential equation

$$\frac{dy}{dx} + \tan\left(\frac{dy}{dx}\right) = 0.$$

(ii) What will be the order of the differential equation, corresponding to the family of curves $y = a \cos (x + b)$, where a is arbitrary constant.

(iii) What will be the order of the differential education $y = a + be^{x + c}$ where a, b, c are arbitrary constant.

(iv) Find the integrating factor for solving the differential education $\frac{dy}{dx} + y \tan x = \cos x.$

(v) Find the integrating factor for solving the differential equation $\frac{dy}{dx} + \frac{1}{1 + x^2} y = \sin x.$

15. (i) Form the differential equation of the family of circles in the first quadrant and touching the coordinate axes.

(ii) Verify that $y = \log\left(x + \sqrt{x^2 + a^2}\right)$ satisfies the differential equation :

$$\left(a^2 + x^2\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$$

16. Solve the following differential equations :

(i) $\cos^2 \frac{dy}{dx} = \tan x - y.$

(ii) $x \cos x \frac{dy}{dx} + y\,(x \sin x + \cos x) = 1.$

(iii) $\left(1 + e^{\frac{x}{y}}\right) dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right) dy = 0.$

(iv) $(y - \sin x)\ dx + \tan x\ dy = 0,\ y(0) = 0.$

LONG ANSWER TYPE QUESTIONS (6 MARKS EACH)

17. Solve the following differential equations :

(i) $(x\,dy - y\,dx)\,y \sin\left(\frac{y}{x}\right) = (y\,dx + x\,dy)\,x \cos\left(\frac{y}{x}\right)$

(ii) $3e^x \tan y\,dx + (1 - e^x)\sec^2 y\,dy = 0$ given that $y = \frac{\pi}{4}$, when $x = 1$.

(iii) $\frac{dy}{dx} + y \cot x = 2x + x^2 \cot x$ given that $y(0) = 0$.

ANSWERS

1.(i)	order = 1, degree = 1	(ii)	order = 2, degree = 1
(iii)	order = 4, degree = 1	(iv)	order = 5, degree is not defined.
(v)	order = 2, degree = 2	(vi)	order = 2, degree = 2
(vii)	order = 3, degree = 2		

2.(i) $y = \frac{x^6}{6} + \frac{x^3}{6} - 2\log|x| + c$ (ii) $y = \log_e\left|e^x + e^{-x}\right| + c$

(iii) $y = \frac{x^4}{4} + e^x + \frac{x^{e+1}}{e+1} + c.$ (iv) $5^x + 5^{-y} = c$

(v) $2(y - x) + \sin 2y + \sin 2x = c.$ (vi) $2\log|3x + 1| + 3\log|1 - 2y| = c.$

3.(i) $e^{\sin x}$ (ii) $e^{\tan x}$

(iii) $e^{-1/x}$ (iv) $e^{\frac{(\log x)^2}{2}}$

(v) $\frac{1}{x^3}$ (vi) $\sec x$

4.(i) 1 (ii) 1

(iii) 2 (iv) 1

(v) 1

5.(v) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ (vi) $x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2} = y\frac{dy}{dx}$

(vii) $2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$

6.(i) $y \sin x = \frac{2\sin^3 x}{3} + c$ (ii) $y = \frac{x^2(4\log_e x - 1)}{16} + \frac{c}{x^2}$

(iii) $y = \sin x + \frac{c}{x},\ x > 0$ (iv) $y = \tan x - 1 + ce^{-\tan x}$

7.(i) $cy = (x + 2)(1 - 2y)$ (ii) $(e^x + 2)\sec y = c$

(iii) $\sqrt{1 - x^2} + \sqrt{1 - y^2} = c$

(iv) $\frac{1}{2}\log\left|\frac{\sqrt{1 - y^2} - 1}{\sqrt{1 - y^2} + 1}\right| = \sqrt{1 - x^2} - \sqrt{1 - y^2} + c$

(v) $(x^2 + 1)(y^2 + 1) = 2$

(vi) $\log y = -\frac{1}{4}\cos^4 x + \frac{1}{6}\cos^6 x + xe^x - e^x + c$

$$= \frac{1}{16}\left[\frac{\cos^3 2x}{3} - \cos 2x\right] + (x - 1)e^x + c$$

(vii) $\log|\tan y| - \frac{\cos 2x}{y} = c$

8.(i) $\frac{-x^3}{3y^3} + \log|y| = c$ (ii) $\tan^{-1}\left(\frac{y}{x}\right) = \log|x| + c$

(iii) $x^2 + y^2 = 2x$

(iv) $y = ce^{\cos(x/y)}$ [**Hint. :** Put $\frac{1}{x} = v$]

(v) $\sin\left(\frac{y}{x}\right) = cx$ (vi) $c\left(x^2 - y^2\right) = y$

(vii) $-e^{-y} = e^x + \frac{x^3}{3} + c$ (viii) $\sin^{-1} y = \sin^{-1} x + c$

9.(i) $x^2 - y^2 + 2xy\frac{dy}{dx} = 0$ (ii) $2y = x\frac{dy}{dx}$, $y = 2x\frac{dy}{dx}$

(iii) $x^2 - y^2 + 2xy\frac{dy}{dx} = 0$

10. $\log\left|x^2 + xy + y^2\right| = 2\sqrt{3}\tan^{-1}\left(\frac{x + 2y}{\sqrt{3x}}\right) + c$

11. $\frac{x^3}{x^2 + y^2} = \frac{c}{x}(x + y)$

12.(i) $y = \frac{3\sin 3x}{13} - \frac{2\cos 3x}{13} + ce^{2x}$ (ii) $y = \frac{2}{3}\sin^2 x + \frac{1}{3}\operatorname{cosec} x$

13.(i) $-y = x\log\{c(x - y)\}$ (ii) $cx^2 = y + \sqrt{x^2 + y^2}$

(iii) $xy\cos\left(\frac{y}{x}\right) = c$ (iv) $3x^2y = y + 2x$

(v) $y = -x\log(\log|x|),\ x \neq 0$ (vi) $c(x^2 + y^2) = \sqrt{x^2 - y^2}$.

14. (i) Order = 1, Degree = not define

(ii) Order = 1 (iii) Order = 2

(iv) sec x (v) $e^{\tan^{-1} x}$

15. (i) $(x - y)^2\ (1 + y')^2 = (x + y\ y')^2$

16. (i) $y = \tan x - 1 + ce^{\tan^{-1} y}$ (ii) $y = \dfrac{\sin x}{x} + c\dfrac{\cos x}{x}$

(iii) $x + ye^{\frac{x}{y}} = c$ (iv) $2y = \sin x$

17. (i) $C\ xy = \sec\left(\dfrac{y}{x}\right)$ (ii) $(1 - e)^3 \tan y = (1 - e^x)^3$

(iii) $y = x^2$.

CHAPTER 10

VECTORS

POINTS TO REMEMBER

- A quantity that has magnitude as well as direction is called a *vector*. It is denoted by a directed line segment.
- Two or more vectors which are parallel to same line are called *collinear vectors*.
- *Position vector* of a point $P(a, b, c)$ w.r.t. origin (0, 0, 0) is denoted by $\overrightarrow{OP}$, where $\overrightarrow{OP} = a\hat{i} + b\hat{j} + c\hat{k}$ and $|\overrightarrow{OP}| = \sqrt{a^2 + b^2 + c^2}$.
- If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points in space, then $\overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$ and
$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$
- If two vectors $\vec{a}$ and $\vec{b}$ are represented in magnitude and direction by the two sides of a triangle taken in order, then their sum $\vec{a} + \vec{b}$ is represented in magnitude and direction by third side of triangle taken in opposite order. This is called triangle *law of addition of vectors.*
- If $\vec{a}$ is any vector and λ is a scalar, then $\lambda \vec{a}$ is a vector collinear with $\vec{a}$ and $|\lambda \vec{a}| = |\lambda||\vec{a}|$.
- If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then $\vec{a} = \lambda \vec{b}$ where λ is some scalar.
- Any vector $\vec{a}$ can be written as $\vec{a} = |\vec{a}|\hat{a}$, where $\hat{a}$ is a unit vector in the direction of $\vec{a}$.

- If $\vec{a}$ and $\vec{b}$ be the position vectors of points *A* and *B,* and *C* is any point which divides $\overrightarrow{AB}$ in ratio *m* : *n* internally then position vector $\vec{c}$ of point *C* is given as $\vec{C} = \dfrac{m\vec{b} + n\vec{a}}{m + n}$. If *C* divides $\overrightarrow{AB}$ in ratio *m* : *n* externally, then $\vec{C} = \dfrac{m\vec{b} - n\vec{a}}{m - n}$.
- The angles α, β and γ made by $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ with positive direction of x, *y* and *z*-axis are called direction angles and cosines of these angles are called *direction cosines* of $\vec{r}$ usually denoted as $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

  Also $l = \dfrac{a}{|\vec{r}|}$, $m = \dfrac{b}{|\vec{r}|}$, $n = \dfrac{c}{|\vec{r}|}$ and $l^2 + m^2 + n^2 = 1$.
- The numbers *a, b, c* proportional to *l, m, n* are called *direction ratios*.
- *Scalar product* of two vectors $\vec{a}$ and $\vec{b}$ is directed as $\vec{a}.\vec{b}$ and is defined as $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$.
- *Dot product* of two vectors is commutative *i.e.* $\vec{a} \cdot \vec{b} =$ and $\vec{b} \cdot \vec{a}$.
- $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} = \vec{o}, \vec{b} = \vec{o}$ or $\vec{a} \perp \vec{b}$.
- $\vec{a} \cdot \vec{a} = |\vec{a}|^2$, so $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$.
- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

  $\vec{a} \cdot \vec{b} = a_1a_2 + b_1b_2 + c_1c_2$.
- *Projection of* $\vec{a}$ on $\vec{b} = \left|\dfrac{\vec{a} \,.\, \vec{b}}{|\vec{b}|}\right|$ and projection vector of $\vec{a}$ along $\vec{b} = \left|\dfrac{(\vec{a} \,.\, \vec{b})}{|\vec{b}|}\right|\hat{b}$.
- *Cross product*/vector product of two vectors $\vec{a}$ and $\vec{b}$ is devoted as $\vec{a} \times \vec{b}$ and is defined as $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$. were θ is the angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$

such that $\vec{a}, \vec{b}$ *and* $\hat{n}$ form a right handed system.

- Cross product of two vectors is not commutative i.e., $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$, but $\vec{a} \times \vec{b} = -\left(\vec{b} \times \vec{a}\right)$.
- $\vec{a} \times \vec{b} = \vec{o} \Leftrightarrow \vec{a} = \vec{o},\ \vec{b} = \vec{o}$ *or* $\vec{a} \parallel \vec{b}$.
- $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{o}$.
- $\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}$ and $\hat{j} \times i = -\hat{k},\ \hat{k} \times \hat{j} = -\hat{i},\ \hat{i} \times \hat{k} = -\hat{j}$
- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- Unit vector perpendicular to both $\vec{a}$ and $\vec{b} = \pm\left(\dfrac{\left(\vec{a} \times \vec{b}\right)}{\left|\vec{a} \times \vec{b}\right|}\right)$.
- $\left|\vec{a} \times \vec{b}\right|$ is the *area of parallelogram* whose adjacent sides are $\vec{a}$ and $\vec{b}$.
- $\dfrac{1}{2}\left|\vec{a} \times \vec{b}\right|$ is the area of parallelogram where diagonals are $\vec{a}$ and $\vec{b}$.
- If $\vec{a}, \vec{b}$ and $\vec{c}$ forms a triangle, then *area of the triangle.*

$$= \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}\left|\vec{b} \times \vec{c}\right| = \frac{1}{2}\left|\vec{c} \times \vec{a}\right|.$$

VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

1. What is the horizontal and vertical components of a vector $\vec{a}$ of magnitude 5 making an angle of 150° with the direction of *x*-axis.

2. What is $a \in R$ such that $|a \overrightarrow{x}| = 1$, where $x = \hat{i} - 2\hat{j} + 2\hat{k}$?

3. Write when $|\overrightarrow{x} + \overrightarrow{y}| = |\overrightarrow{x}| + |\overrightarrow{y}|$.

4. What is the area of a parallelogram whose sides are given by $2\hat{i} - \hat{j}$ and $\hat{i} + 5\hat{k}$?

5. What is the angle between $\overrightarrow{a}$ *and* $\overrightarrow{b}$, If

$$\overrightarrow{a} \cdot \overrightarrow{b} = 3 \text{ and } |\overrightarrow{a} \times \overrightarrow{b}| = 3\sqrt{3}.$$

6. Write a unit vector which makes an angle of $\frac{\pi}{4}$ with *x*-axis and $\frac{\pi}{3}$ with *z*-axis and an acute angle with *y*-axis.

7. If *A* is the point (4, 5) and vector $\overrightarrow{AB}$ has components 2 and 6 along *x*-axis and *y*-axis respectively then write point *B*.

8. What is the point of trisection of *PQ* nearer to *P* if position of *P* and *Q* are $3\hat{i} + 3\hat{j} - 4\hat{k}$ and $9\hat{i} + 8\hat{j} - 10\hat{k}$.

9. What is the vector in the direction of $2\hat{i} + 3\hat{j} + 2\sqrt{3}\hat{k}$, whose magnitude is 10 units?

10. What are the direction cosines of a vector equiangular with co-ordinate axes?

11. What is the angles which $3\hat{i} - 6\hat{j} + 2\hat{k}$ makes with *x*-axis?

12. Write a unit vector perpendicular to both the vectors $3\hat{i} - 2\hat{j} + \hat{k}$ and $-2\hat{i} + \hat{j} - 2\hat{k}$.

13. What is the projection of the vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$?

14. If $|\overrightarrow{a}| = 2$, $|\overrightarrow{b}| = 2\sqrt{3}$ and $\overrightarrow{a} \perp \overrightarrow{b}$, what is the value of $|\overrightarrow{a} + \overrightarrow{b}|$?

15. For what value of λ, $\overrightarrow{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$ is perpendicular to $\overrightarrow{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$?

16. What is $|\overrightarrow{a}|$, if $(\overrightarrow{a} + \overrightarrow{b}).(\overrightarrow{a} - \overrightarrow{b}) = 3$ and $2|\overrightarrow{b}| = |\overrightarrow{a}|$?

17. What is the angle between $\vec{a}$ and $\vec{b}$, if $|\vec{a} - \vec{b}| = |\vec{a} + \vec{b}|$?

18. In a parallelogram *ABCD*, $\overrightarrow{AB} = 2\hat{i} - \hat{j} + 4\hat{k}$ and $\overrightarrow{AC} = \hat{i} + \hat{j} + 4\hat{k}$. What is the length of side $\overrightarrow{BC}$?

19. What is the area of a parallelogram whose diagonals are given by vectors $2\hat{i} + \hat{j} - 2\hat{k}$ and $-\hat{i} + 2\hat{k}$?

20. Find $|\vec{x}|$ if for a unit vector $\hat{a}$, $(\vec{x} - \hat{a}).(\vec{x} + \hat{a}) = 12$.

21. If $\vec{a}$ and $\vec{b}$ are two unit vectors and $\vec{a} + \vec{b}$ is also a unit vector then what is the angle between $\vec{a}$ and $\vec{b}$?

22. If $\hat{i}, \hat{j}, \hat{k}$ are the usual three mutually perpendicular unit vectors then what is the value of $\hat{i}.(\hat{j} \times \hat{k}) + \hat{j}.(\hat{i} \times \hat{k}) + \vec{k}.(\hat{j} \times \hat{i})$?

23. What is the angle between $\vec{x}$ and $\vec{y}$ if $\vec{x} . \vec{y} = |\vec{x} \times \vec{y}|$?

24. Write a unit vector in *xy*-plane, making an angle of 30° with the +ve direction of *x*–axis.

25. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors with $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then what is the value of $\vec{a} . \vec{b} + \vec{b} . \vec{c} + \vec{c} . \vec{a}$?

26. If $\vec{a}$ and $\vec{b}$ are unit vectors such that $(\vec{a} + 2\vec{b})$ is perpendicular to $(5\vec{a} - 4\vec{b})$, then what is the angle between $\vec{a}$ and $\vec{b}$?

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

1. If ABCDEF is a regular hexagon then using triangle law of addition prove that :

$$\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = 3\overrightarrow{AD} = 6\overrightarrow{AO}$$

O being the centre of hexagon.

2. Points *L, M, N* divides the sides *BC, CA, AB* of a ΔABC in the ratios 1 : 4, 3 : 2, 3 : 7 respectively. Prove that $\overrightarrow{AL} + \overrightarrow{BM} + \overrightarrow{CN}$ is a vector parallel to $\overrightarrow{CK}$ where *K* divides *AB* in ratio 1 : 3.

3. The scalar product of vector $\hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of the vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ *and* $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to 1. Find the value of λ.

4. $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors of equal magnitude. Show that $\vec{a} + \vec{b} + \vec{c}$ makes equal angles with $\vec{a}$, $\vec{b}$ and $\vec{c}$ with each angle as $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

5. If $\vec{\alpha} = 3\hat{i} - \hat{j}$ and $\vec{\beta} = 2\hat{i} + \hat{j} + 3\hat{k}$ then express $\vec{\beta}$ in the form of $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

6. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ then prove that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.

7. If $|\vec{a}| = 3$, $|\vec{b}| = 5$, $|\vec{c}| = 7$ and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, find the angle between $\vec{a}$ and $\vec{b}$.

8. Let $\vec{a} = \hat{i} - \hat{j}$, $\vec{b} = 3\hat{j} - \hat{k}$ and $\vec{c} = 7\hat{i} - \hat{k}$, find a vector $\vec{d}$ which is perpendicular to $\vec{a}$ and $\vec{b}$ and $\vec{c} \cdot \vec{d} = 1$.

9. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{c} = \hat{j} - \hat{k}$ are the given vectors then find a vector $\vec{b}$ satisfying the equation $\vec{a} \times \vec{b} = \vec{c}$, $\vec{a} \cdot \vec{b} = 3$.

10. Find a unit vector perpendicular to plane *ABC,* when position vectors of *A, B, C* are $3\hat{i} - \hat{j} + 2\hat{k}$, $\hat{i} - \hat{j} - 3\hat{k}$ and $4\hat{i} - 3\hat{j} + \hat{k}$ respectively.

11. For any two vector, show that $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$.

12. Evaluate $(\vec{a} \times \hat{i})^2 + (\vec{a} \times \hat{j})^2 + (\vec{a} \times \hat{k})^2$.

13. If $\hat{a}$ and $\hat{b}$ are unit vector inclined at an angle θ than prove that :

    (i) $\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a} - \hat{b}|$. (ii) $\tan\frac{\theta}{2} = \left|\frac{\hat{a} - \hat{b}}{\hat{a} + \hat{b}}\right|$.

14. For any two vectors, show that $\left|\vec{a} \times \vec{b}\right| = \sqrt{a^2b^2 - (\vec{a} \cdot \vec{b})^2}$.

15. $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + 2\hat{j}$ and $\vec{c} = x\hat{i} + (x-2)\hat{j} - \hat{k}$. If $\vec{c}$ lies in the plane of $\vec{a}$ and $\vec{b}$, then find the value of x.

16. Prove that angle between any two diagonals of *a* cube is $\cos^{-1}\left(\frac{1}{3}\right)$.

17. Let $\hat{a}$, $\hat{b}$ and $\hat{c}$ are unit vectors such that $\hat{a} \cdot \hat{b} = \hat{a} \cdot \hat{c} = 0$ and the angle between $\hat{b}$ and $\hat{c}$ is $\frac{\pi}{6}$, then prove that $\hat{a} = \pm 2(\hat{b} \times \hat{c})$.

18. Prove that the normal vector to the plane containing three points with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ lies in the direction of vector $\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}$.

19. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are position vectors of the vertices *A, B, C* of a triangle *ABC* then show that the area of ΔABC is $\frac{1}{2}\left|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right|$.

20. If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, then prove that $\vec{a} - \vec{d}$ is parallel to $\vec{b} - \vec{c}$ provided $\vec{a} \neq \vec{d}$ and $\vec{b} \neq \vec{c}$.

21. Dot product of a vector with vectors $\hat{i} + \hat{j} - 3\hat{k}$, $\hat{i} + 3\hat{j} - 2\hat{k}$ and $2\hat{i} + \hat{j} + 4\hat{k}$ is 0, 5 and 8 respectively. Find the vectors.

22. If $\vec{a} = 5\hat{i} - \hat{i} + 7\hat{k}$, $\hat{b} = \hat{i} - \hat{j} - \lambda\hat{k}$, find λ such that $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are orthogonal.

24. Let $\vec{a}$ and $\vec{b}$ be vectors such that $\left|\vec{a}\right| = \left|\vec{b}\right| = \left|\vec{a} - \vec{b}\right| = 1$, then find $\left|\vec{a} + \vec{b}\right|$.

25. If $\left|\vec{a}\right| = 2$, $\left|\vec{b}\right| = 5$ and $\vec{a} \times \vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$, find the value of $\vec{a} \cdot \vec{b}$.

26. $\vec{a}$, $\vec{b}$, $\vec{c}$ are three vectors such that $\vec{b} \times \vec{c} = \vec{a}$ and

$\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{c}$. Prove that $\overrightarrow{a}, \overrightarrow{b}$ and $\overrightarrow{c}$ are mutually perpendicular to each other and $|\overrightarrow{b}| = 1, |\overrightarrow{c}| = |\overrightarrow{a}|$.

ANSWERS

(1 MARK QUESTIONS)

1. $-\frac{5\sqrt{3}}{2}, \frac{5}{2}$.

2. $a = \frac{1}{3}$

3. $\overrightarrow{x}$ and $\overrightarrow{y}$ are like parallel vectors.

4. $\sqrt{126}$ sq units.

5. $\frac{\pi}{3}$

6. $\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}$

7. (6, 11)

8. $\left(5, \frac{14}{3}, -6\right)$

9. $4\hat{i} + 6\hat{j} + 4\sqrt{3}\hat{k}$.

10. $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$.

11. $\cos^{-1}\left(\frac{3}{7}\right)$.

12. $\frac{3\hat{i} + 4\hat{j} - \hat{k}}{\sqrt{26}}$.

13. 0

14. 4

15. –9

16. 2

17. $\frac{\pi}{2}$.

18. $\sqrt{5}$

19. $\frac{3}{2}$ sq. units.

20. $\sqrt{13}$

21. $\frac{2\pi}{3}$

22. –1

23. $\frac{\pi}{4}$

24. $\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$

25. $-\frac{3}{2}$

26. $\frac{\pi}{3}$

(4 MARKS QUESTIONS)

3. $\lambda = 1$

5. $\overrightarrow{\beta} = \left(\frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}\right) + \left(\frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}\right).$

7. 60°

8. $\frac{1}{4}\hat{i} + \frac{1}{4}\hat{j} + \frac{3}{4}\hat{k}.$

9. $\frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}.$

10. $\frac{-1}{\sqrt{165}}\left(10\hat{i} + 7\hat{j} - 4\hat{k}\right).$

12. $2\left|\overrightarrow{a}\right|^2$

15. $x = -2$

21. $\hat{i} + 2\hat{j} + \hat{k}$

22. $\pm\sqrt{73}$

24. $\sqrt{3}$

25. $\frac{91}{10}$

CHAPTER 11

THREE DIMENSIONAL GEOMETRY

POINTS TO REMEMBER

- Distance between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

- (i) The coordinates of point R which divides line segment PQ where $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $m : n$ internally are

$$\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}, \frac{mz_2 + nz_1}{m + n}\right).$$

  (ii) The co-ordirates of a point which divides join of (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio of $m : n$ externally are

$$\left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}, \frac{mz_2 - nz_1}{m - n}\right).$$

- Direction ratios of a line through (x_1, y_1, z_1) and (x_2, y_2, z_2) are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$.
- Direction cosines of a line whose direction ratios are a, b, c are given by

$$l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \; m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}, \; n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

- (i) Vector equation of a line through point $\overrightarrow{a}$ and parallel to vector $\overrightarrow{b}$ is $\overrightarrow{r} = \overrightarrow{a} + \lambda\overrightarrow{b}$.

  (ii) Cartesian equation of a line through point (x_1, y_1, z_1) and having direction ratios proportional to a, b, c is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

- (i) Vector equation of line through two points

  $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\left(\vec{b} - \vec{a}\right)$.

  (ii) Cartesian equation of a line through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$.

- Angle 'θ' between lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ is given by $\cos\theta = \dfrac{\vec{b_1} \cdot \vec{b_2}}{\left|\vec{b_1}\right|\left|\vec{b_2}\right|}$.

- Angle θ between lines $\dfrac{x - x_1}{a_1} = \dfrac{y + y_1}{b_1} = \dfrac{z - z_1}{c_1}$ and $\dfrac{x - x_2}{a_2} = \dfrac{y + y_2}{b_2} = \dfrac{z - z_2}{c_2}$ is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

- Two lines are perpendicular to each other if

  $\vec{b_1} \cdot \vec{b_2} = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

- Equation of plane :

  (i) At a distance of p unit from origin and perpendicular to $\hat{n}$ is $\vec{r} \cdot \hat{n} = p$ and corresponding cartesian form is $lx + my + nz = p$ when *l, m* and n are *d.c.*s of normal to plane.

  (ii) Passing through $\vec{a}$ and normal to $\vec{n}$ is $\vec{n}\left(\vec{r} - \vec{a}\right).\vec{n} = 0$ and corresponding cartesian form is $a(x - x_1) + b\,(y - y_1) + c(z - z_1) = 0$ where *a, b, c* are *d.r.*'s of normal to plane and (x_1, y_1, z_1) lies on the plane.

  (iii) Passing through three non collinear points is

  $\left(\vec{r} - \vec{a}\right) \cdot \left[\left(\vec{b} - \vec{a}\right) \times \left(\vec{c} - \vec{a}\right)\right] = 0$

$$\text{or} \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

(iv) Having intercepts *a, b* and *c* on co-ordinate axis is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$

(v) Planes passing through the line of intersection of planes $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$ is

$$\left(\vec{r} \cdot \vec{n_1} - d_1\right) + \lambda\left(\vec{r} \cdot \vec{n_2} - d_2\right) = 0.$$

- (i) Angle 'θ' between planes $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$ is given by $\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$.

  (ii) Angle θ between $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

  (iii) Two planes are perpendicular to each other iff $\vec{n_1} . \vec{n_2} = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

  (iv) Two planes are parallel iff $\vec{n_1} = \lambda \vec{n_2}$ for some scaler $\lambda \neq 0$ or $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

- (i) Distance of a point from plane $\vec{r} \cdot \vec{n} = d$ is $\frac{|\vec{a} \cdot \vec{n} - d|}{|\vec{n}|}$.

  (ii) Distance of a point (x_1, y_1, z_1) from plane $ax + by + cz = d$ is $\frac{|ax_1 + by_1 + cz_1 - d|}{|a^2 + b^2 + c^2|}$.

12. (i) Two lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ are coplanar. Iff $\left(\vec{a_2} - \vec{a_1}\right) \cdot \left(\vec{b_1} \times \vec{b_2}\right) = 0$ and equation of plane, containing these lines is $\left(\vec{r} - \vec{a_1}\right) \cdot \left(\vec{b_1} \times \vec{b_2}\right) = 0.$

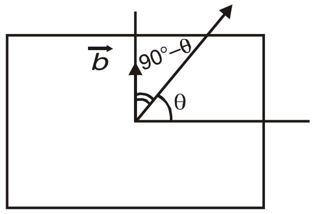
(ii) Two lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ are coplanar Iff

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

and equation of plane containing them is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

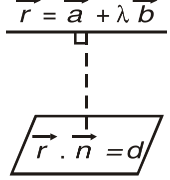
- (i) The angle θ between line $\vec{r} = \vec{a} + \lambda \vec{b}$ and plane $\vec{r} \cdot \vec{n} = d$ is given as $\sin\theta = \frac{\vec{b} \cdot \vec{n}}{|\vec{b}||\vec{n}|}$.



(ii) The angle θ between line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and plane $a_2x + b_2y + c_2 z = d$ is given as

$$\sin\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

(iii) A line $\vec{r} = \vec{a} + \lambda \vec{b}$ is parallel to plane $\vec{r} . \vec{n} = d$ $\Leftrightarrow \vec{b} \cdot \vec{n} = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$.



VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

1. What is the distance of point (a, b, c) from x-axis?
2. What is the angle between the lines $2x = 3y = -z$ and $6x = -y = -4z$?
3. Write the equation of a line passing through (2, –3, 5) and parallel to line $\frac{x-1}{3} = \frac{y-2}{4} = \frac{z+1}{-1}$.
4. In what ratio, the xy plane divides the line segment joining the points (–1, 3, 4) and (2, –5, 6)?
5. Write the equation of a line through (1, 2, 3) and perpendicular to $\vec{r} \cdot (\hat{i} - \hat{j} + 3\hat{k}) = 5$.
6. What is the value of λ for which the lines $\frac{x-1}{2} = \frac{y-3}{5} = \frac{z-1}{\lambda}$ and $\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z}{2}$ are perpendicular to each other.
7. If a line makes angle α, β, and γ with co-ordinate axes, then what is the value of

   $\sin^2 \alpha + \sin^2 \beta + \sin^2\gamma$?
8. Write line $\vec{r} = (\hat{i} - \hat{j}) + \lambda(2\hat{j} - \hat{k})$ into cartesian form.
9. If the direction ratios of a line are 1, –2, 2 then what are the direction cosines of the line?
10. Find the angle between the planes $2x - 3y + 6z = 9$ and xy – plane.
11. Write equation of a line passing through (0, 1, 2) and equally inclined to co-ordinate axes.
12. What is the perpendicular distance of plane $2x - y + 3z = 10$ from origin?
13. What is the y-intercept of the plane $x - 5y + 7z = 10$?
14. What is the distance between the planes $2x + 2y - z + 2 = 0$ and $4x + 4y - 2z + 5 = 0$.
15. What is the equation of the plane which cuts off equal intercepts of unit length on the coordinate axes.
16. Are the planes $x + y - 2z + 4 = 0$ and $3x + 3y - 6z + 5 = 0$ intersecting?

17. What is the equation of the plane through the point (1, 4, – 2) and parallel to the plane $-2x + y - 3z = 7$?

18. Write the vector equation of the plane which is at a distance of 8 units from the origin and is normal to the vector $\left(2\hat{i} + \hat{j} + 2\hat{k}\right)$.

19. What is equation of the plane if the foot of perpendicular from origin to this plane is (2, 3, 4)?

20. Find the angles between the planes $\vec{r} \cdot \left(\hat{i} - 2\hat{j} - 2\hat{k}\right) = 1$ and $\vec{r} \cdot \left(3\hat{i} - 6\hat{j} + 2\hat{k}\right) = 0$.

21. What is the angle between the line $\dfrac{x+1}{3} = \dfrac{2y-1}{4} = \dfrac{2-z}{-4}$ and the plane $2x + y - 2z + 4 = 0$?

22. If O is origin $OP = 3$ with direction ratios proportional to –1, 2, – 2 then what are the coordinates of P?

23. What is the distance between the line $\vec{r} = 2\hat{i} - 2\hat{j} + 3\hat{k} + \lambda\left(\hat{i} + \hat{j} + 4\hat{k}\right)$ from the plane $\vec{r} \,.\left(-\hat{i} + 5\hat{j} - \hat{k}\right) + 5 = 0$.

24. Write the line $2x = 3y = 4z$ in vector form.

SHORT ANSWER TYPE QUESTION (4 MARKS)

1. The line $\dfrac{x-4}{1} = \dfrac{2y-4}{2} = \dfrac{k-z}{-2}$ lies exactly in the plane

   $2x - 4y + z = 7$. Find the value of k.

2. Find the equation of a plane containing the points (0, –1, –1), (–4, 4, 4) and (4, 5, 1). Also show that (3, 9, 4) lies on that plane.

3. Find the equation of the plane which is perpendicular to the plane $\vec{r} \cdot \left(5\hat{i} + 3\hat{j} + 6\hat{k}\right) + 8 = 0$ and which is containing the line of intersection of the planes $\vec{r} \cdot \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) = 4$ and $\vec{r} \cdot \left(2\hat{i} + \hat{j} - \hat{k}\right) + 5 = 0$.

4. If l_1, m_1, n_1, and l_2, m_2, n_2 are direction cosines of two mutually perpendicular

lines, show that the direction cosines of line perpendicular to both of them are

$$m_1n_2 - n_1m_2,\ n_1l_2 - l_1n_2,\ l_1m_2 - m_1l_2.$$

5. Find vector and cartesian equation of a line passing through a point with position vectors $2\hat{i} + \hat{j} + \hat{k}$ and which is parallel to the line joining the points with position vectors $-\hat{i} + 4\hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + 2\hat{k}$.

6. Find the equation of the plane passing through the point (3, 4, 2) and (7, 0, 6) and is perpendicular to the plane $2x - 5y = 15$.

7. Find equation of plane through line of intersection of planes $\vec{r} \cdot \left(2\hat{i} + 6\hat{j}\right) + 12 = 0$ and $\vec{r} \cdot \left(3\hat{i} - \hat{j} + 4\hat{k}\right) = 0$ which is at a unit distance from origin.

8. Find the image of the point (3, –2, 1) in the plane $3x - y + 4z = 2$.

9. Find the equation of a line passing through (2, 0, 5) and which is parallel to line $6x - 2 = 3y + 1 = 2z - 2$.

10. Find image (Reflection) of the point (7, 4, – 3) in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$.

11. Find equations of a plane passing through the points (2, –1, 0) and (3, –4, 5) and parallel to the line $2x = 3y = 4z$.

12. Find distance of the point (– 1, – 5, – 10) from the point of intersection of line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$.

13. Find equation of the plane passing through the point (2, 3, – 4) and (1, –1, 3) and parallel to the x–axis.

14. Find the distance of the point (1, –2, 3) from the plane $x - y + z = 5$, measured parallel to the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$.

15. Find the equation of the plane passing through the intersection of two plane $3x - 4y + 5z = 10$, $2x + 2y - 3z = 4$ and parallel to the line $x = 2y = 3z$.

16. Find the distance between the planes $2x + 3y - 4z + 5 = 0$ and $\vec{r}.\left(4\hat{i} + 6\hat{j} - 8\hat{k}\right) = 11$.

17. Find the equations of the planes parallel to the plane $x - 2y + 2z - 3 = 0$ whose perpendicular distance from the point (1, 2, 3) is 1 unit.

18. Show that the lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and

$\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$ intersect each other. Find the point of intersection.

19. Find the shortest distance between the lines

$\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda\left(2\hat{i} + 3\hat{j} + 4\hat{k}\right)$ and

$\vec{r} = \left(2\hat{i} + 4\hat{j} + 5\hat{k}\right) + \lambda\left(3\hat{i} + 4\hat{j} + 5\hat{k}\right)$.

20. Find the distance of the point (–2, 3, –4) from the line $\frac{x+2}{3} = \frac{2y+3}{4} = \frac{3z+4}{5}$ measured parallel to the plane $4x + 12y - 3z + 1 = 0$.

21. Find the equation of plane passing through the point (–1, –1, 2) and perpendicular to each of the plane

$$\vec{r} \cdot \left(2\hat{i} + 3\hat{j} - 3\hat{k}\right) = 2 \text{ and } \vec{r} \cdot \left(5\hat{i} - 4\hat{j} + \hat{k}\right) = 6.$$

22. Find the equation of a plane passing through (–1, 3, 2) and parallel to each of the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and $\frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5}$.

23. Show that the plane $\vec{r} \cdot \left(\hat{i} - 3\hat{j} + 5\hat{k}\right) = 7$ contains the line $\vec{r} = \left(\hat{i} + 3\hat{j} + 3\hat{k}\right) + \lambda\left(3\hat{i} + \hat{j}\right)$.

LONG ANSWER TYPE QUESTIONS (6 MARKS)

1. Check the coplanarity of lines

$$\vec{r} = \left(-3\hat{i} + \hat{j} + 5\hat{k}\right) + \lambda\left(-3\hat{i} + \hat{j} + 5\hat{k}\right)$$

$$\vec{r} = \left(-\hat{i} + 2\hat{j} + 5\hat{k}\right) + \mu\left(-\hat{i} + 2\hat{j} + 5\hat{k}\right)$$

If they are coplanar, find equation of the plane containing the lines.

2. Find shortest distance between the lines :

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7} \text{ and } \frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}.$$

3. Find shortest distance between the lines :

$$\vec{r} = (1-\lambda)\hat{i} + (\lambda-2)\hat{j} + (3-2\hat{\lambda})\hat{k}$$

$$\vec{r} = (\mu+1)\hat{i} + (2\mu-1)\hat{j} + (2\mu+1)\hat{k}.$$

4. A variable plane is at a constant distance $3p$ from the origin and meet the coordinate axes in A, B, C. Show that the locus of centroid of ΔABC is $x^{-2} + y^{-2} + z^{-2} = p^{-2}$.

5. A vector $\vec{n}$ of magnitude 8 units inclined to x–axis at 45°, y axis at 60° and an acute angle with z-axis. If a plane passes through a point $(\sqrt{2}, -1, 1)$ and is normal to $\vec{n}$, find its equation in vector form.

6. Find the foot of perpendicular from the point $2\hat{i} - \hat{j} + 5\hat{k}$ on the line $\vec{r} = (11\hat{i} - 2\hat{j} - 8\hat{k}) + \lambda(10\hat{i} - 4\hat{j} - 11\hat{k})$. Also find the length of the perpendicular.

7. A line makes angles α, β, λ, δ with the four diagonals of a cube. Prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}.$$

8. Find the equations of the two lines through the origin such that each line is intersecting the line $\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}$ at an angle of $\frac{\pi}{3}$.

9. Find the equation of the plane passing through the intersection of planes $2x + 3y - z = -1$ and $x + y - 2z + 3 = 0$ and perpendicular to the plane $3x - y - 2z = 4$. Also find the inclination of this plane with xy-plane.

ANSWERS

(1 MARK QUESTIONS)

1. $\sqrt{b^2 + c^2}$

2. 90°

3. $\frac{x-2}{3} = \frac{y+3}{4} = \frac{z-5}{-1}$.

4. 2 : 3 externally

5. $\vec{r} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + \lambda\left(\hat{i} - \hat{j} + 3\hat{k}\right)$

6. $\lambda = 2$

7. 2

8. $\frac{x-1}{0} = \frac{y+1}{2} = \frac{z}{-1}$.

9. $\pm\frac{1}{\sqrt{3}}, \mp\frac{2}{\sqrt{3}}, \pm\frac{2}{\sqrt{3}}$

10. $\cos^{-1}$ (6/7).

11. $\frac{x}{a} = \frac{y-1}{a} = \frac{z-2}{a}$, $a \in R - \{0\}$

12. $\frac{10}{\sqrt{14}}$

13. –2

14. $\frac{1}{6}$

15. $x + y + z = 1$

16. No

17. $-2x + y - 3z = 8$

18. $\vec{r} \cdot \left(2\hat{i} + \hat{j} + 2\hat{k}\right) = 24$

19. $2x + 3y + 4z = 29$

20. $\cos^{-1}\left(\frac{11}{21}\right)$

21. 0 (line is parallel to plane)

22. (–1, 2, –2)

23. $\frac{10}{3\sqrt{3}}$

24. $\vec{r} = \vec{o} + \lambda\left(6\hat{i} + 4\hat{j} + 3\hat{k}\right)$

SHORT ANSWER TYPE QUESTION (4 MARKS)

1. $k = 7$

2. $5x - 7y + 11z + 4 = 0$.

3. $\vec{r} \cdot \left(-51\hat{i} - 15\hat{j} + 50\hat{k}\right) = 173$

5. $\vec{r} = \left(2\hat{i} - \hat{j} + \hat{k}\right) + \lambda\left(2\hat{i} - 2\hat{j} + \hat{k}\right)$ and $\frac{x-2}{2} = \frac{y+1}{-2} = \frac{z-1}{1}$.

6. $x - 2y + 3z = 1$

7. $\vec{r} \cdot (8\hat{i} + 4\hat{j} + 8\hat{k}) + 12 = 0$ or $\vec{r} \cdot (-4\hat{i} + 8\hat{j} - 8\hat{k}) + 12 = 0$

8. $(0, -1, -3)$

9. $\dfrac{x-2}{1} = \dfrac{y}{2} = \dfrac{z-5}{3}$.

10. $\left(\dfrac{47}{7}, -\dfrac{18}{7}, \dfrac{43}{7}\right)$

11. $29x - 27y - 22z = 85$

12. 13

13. $7y + 4z = 5$

14. 1

15. $x - 20y + 27z = 14$

16 $\dfrac{21}{2\sqrt{29}}$ units.

17. $x - 2y + 2z = 0$ or $x - 2y + 2z = 6$

18. $\left(\dfrac{1}{2}, -\dfrac{1}{2}, -\dfrac{3}{2}\right)$

19. $\dfrac{1}{\sqrt{6}}$

20. $\dfrac{17}{2}$

21. $\vec{r} \cdot (9\hat{i} + 17\hat{j} + 23\hat{k}) = 20$

22. $2x - 7y + 4z + 15 = 0$

LONG ANSWER TYPE QUESTIONS (6 MARKS)

1. $x - 2y + z = 0$

2. $\dfrac{16}{7}$

3. $\dfrac{8}{\sqrt{29}}$

5. $\vec{r} \cdot (\sqrt{2}\hat{i} + \hat{j} + \hat{k}) = 2$

6. $\hat{i} + 2\hat{j} + 3\hat{k}, \sqrt{14}$

8. $\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{-1}, \dfrac{x}{-1} = \dfrac{y}{1} = \dfrac{z}{-2}$

9. $7x + 13y + 4z = 9, \cos^{-1}\left(\dfrac{4}{\sqrt{234}}\right)$.

CHAPTER 12

LINEAR PROGRAMMING

POINTS TO REMEMBER

- Linear programming is the process used to obtain minimum or maximum value of the linear objective function under known linear constraints.
- **Objective Functions :** Linear function $z = ax + by$ where a and b are constants, which has to be maximized or minimized is called a linear objective function.
- **Constraints :** The linear inequalities or inequations or restrictions on the variables of a linear programming problem.
- **Feasible Region :** It is defined as a set of points which satisfy all the constraints including non-negative constraints $x \geq 0,\ y \geq 0$.
- **To Find Feasible Region :** Draw the graph of all the linear inequations and shade common region determined by all the constraints.
- **Feasible Solutions :** Points within and on the boundary of the feasible region represents feasible solutions of the constraints.
- **Optimal Feasible Solution :** Feasible solution which optimizes the objective function is called optimal feasible solution.

LONG ANSWER TYPE QUESTIONS (6 MARKS)

1. Solve the following L.P.P. graphically

Minimise and maximise $z = 3x + 9y$

Subject to the constraints
$$x + 3y \leq 60$$
$$x + y \geq 10$$
$$x \leq y$$
$$x \geq 0,\ y \geq 0$$

2. Determine graphically the minimum value of the objective function $z = -50x + 20y$, subject to the constraints

$$2x - y \geq -5$$

$$3x + y \geq 3$$

$$2x - 3y \leq 12$$

$$x \geq 0, \; y \geq 0$$

3. Two tailors *A* and *B* earn Rs. 150 and *Rs*. 200 per day respectively. *A* can stitch 6 shirts and 4 pants per day, while *B* can stich 10 shirts and 4 pants per day. Formulate the above L.P.P. mathematically and hence solve it to minimise the labour cost to produce at least 60 shirts and 32 pants.

4. There are two types of fertilisers *A* and *B*. *A* consists of 10% nitrogen and 6% phosphoric acid and *B* consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, a farmer finds that he needs at least 14 kg of nitrogen and 14 kg of phosphoric acid for his crop. If *A* costs Rs. 61 kg and *B* costs Rs. 51 kg, determine how much of each type of fertiliser should be used so that nutrient requirements are met at minimum cost. What is the minimum cost?

5. A man has Rs. 1500 to purchase two types of shares of two different companies S_1 and S_2. Market price of one share of S_1 is Rs 180 and S_2 is Rs. 120. He wishes to purchase a maximum of ten shares only. If one share of type S_1 gives a yield of Rs. 11 and of type S_2 yields Rs. 8 then how much shares of each type must be purchased to get maximum profit? And what will be the maximum profit?

6. A company manufactures two types of lamps say *A* and *B*. Both lamps go through a cutter and then a finisher. Lamp *A* requires 2 hours of the cutter's time and 1 hours of the finisher's time. Lamp *B* requires 1 hour of cutter's and 2 hours of finisher's time. The cutter has 100 hours and finishers has 80 hours of time available each month. Profit on one lamp A is Rs. 7.00 and on one lamp *B* is Rs. 13.00. Assuming that he can sell all that he produces, how many of each type of lamps should be manufactured to obtain maximum profit?

7. A dealer wishes to purchase a number of fans and sewing machines. He has only Rs. 5760 to invest and has space for almost 20 items. A fan and sewing machine cost Rs. 360 and Rs. 240 respectively. He can sell a fan at a profit of Rs. 22 and sewing machine at a profit of Rs. 18. Assuming

that he can sell whatever he buys, how should he invest his money to maximise his profit?

8. If a young man rides his motorcycle at 25 km/h, he has to spend Rs. 2 per km on petrol. If he rides at a faster speed of 40 km/h, the petrol cost increases to Rs. 5 per km. He has Rs. 100 to spend on petrol and wishes to find the maximum distance he can travel within one hour. Express this as L.P.P. and then solve it graphically.

9. A producer has 20 and 10 units of labour and capital respectively which he can use to produce two kinds of goods *X* and *Y*. To produce one unit of *X*, 2 units of capital and 1 unit of labour is required. To produce one unit of *Y*, 3 units of labour and one unit of capital is required. If *X* and *Y* are priced at Rs. 80 and Rs. 100 per unit respectively, how should the producer use his resources to maximise the total revenue?

10. A factory owner purchases two types of machines *A* and *B* for his factory. The requirements and limitations for the machines are as follows:

Machine	***Area Occupied***	***Labour Force***	***Daily Output (In units)***
A	1000 m^2	12 men	60
B	1200 m^2	8 men	40

He has maximum area of 9000 m^2 available and 72 skilled labourers who can operate both the machines. How many machines of each type should he buy to maximise the daily output.

11. A manufacturer makes two types of cups *A* and *B*. Three machines are required to manufacture the cups and the time in minutes required by each in as given below :

Types of Cup	***Machine***		
	I	***II***	***III***
A	12	18	6
B	6	0	9

Each machine is available for a maximum period of 6 hours per day. If the profit on each cup *A* is 75 paise and on *B* is 50 paise, find how many cups of each type should be manufactured to maximise the profit per day.

12. A company produces two types of belts *A* and *B*. Profits on these belts are Rs. 2 and Rs. 1.50 per belt respectively. A belt of type *A* requires twice as

much time as belt of type B. The company can produce almost 1000 belts of type B per day. Material for 800 belts per day is available. Almost 400 buckles for belts of type A and 700 for type B are available per day. How much belts of each type should the company produce so as to maximize the profit?

13. Two Godowns X and Y have a grain storage capacity of 100 quintals and 50 quintals respectively. Their supply goes to three ration shop A, B and C whose requirements are 60, 50 and 40 quintals respectively. The cost of transportation per quintals from the godowns to the shops are given in following table :

To / From	Cost of transportation (in Rs. per quintal)	
	X	***Y***
A	6.00	4.00
B	3.00	2.00
C	2.50	3.00

How should the supplies be transported to ration shops from godowns to minimize the transportation cost?

14. An Aeroplane can carry a maximum of 200 passengers. A profit of Rs. 400 is made on each first class ticket and a profit of Rs. 300 is made on each second class ticket. The airline reserves at least 20 seats for first class. However atleast four times as many passengers prefer to travel by second class than by first class. Determine, how many tickets of each type must be sold to maximize profit for the airline.

15. A diet for a sick person must contain atleast 4000 units of vitamins, 50 units of minerals and 1400 units of calories. Two foods A and B are available at a cost of Rs. 5 and Rs. 4 per unit respectively. One unit of food A contains 200 unit of vitamins, 1 unit of minerals and 40 units of calories whereas one unit of food B contains 100 units of vitamins, 2 units of minerals and 40 units of calories. Find what combination of the food A and B should be used to have least cost but it must satisfy the requirements of the sick person.

ANSWERS

1. Min $z = 60$ at $x = 5$, $y = 5$.

   Max $z = 180$ at the two corner points (0, 20) and (15, 15).

2. No minimum value.
3. Minimum cost = Rs. 1350 at 5 days of *A* and 3 days of *B*.
4. 100 kg. of fertiliser *A* and 80 kg of fertilisers *B*; minimum cost *Rs.* 1000.
5. Maximum Profit = Rs. 95 with 5 shares of each type.
6. Lamps of type *A* = 40, Lamps of type *B* = 20.
7. Fan : 8; Sewing machine : 12, Max. Profit = Rs. 392.
8. At 25 km/h he should travel 50/3 km, At 40 km/h, 40/3 km. Max. distance 30 km in 1 hr.
9. *X* : 2 units; *Y* : 6 units; Maximum revenue Rs. 760.
10. Type *A* : 6; Type *B* : 0
11. Cup *A* : 15; Cup *B* : 30
12. Maximum profit Rs. 1300, No. of belts of type *A* = 200 No. of bells of type *B* = 600.
13. From *X* to *A*, *B* and *C* : 10 quintals, 50 quintals and 40 quintals respectively.

    From *Y* to *A, B, C* : 50 quintals, NIL and NIL respectively.
14. No. of first class tickets = 40, No. of 2nd class tickets = 160.
15. Food *A* : 5 units, Food *B* : 30 units.

CHAPTER 13

PROBABILITY

POINTS TO REMEMBER

- **Conditional Probability :** If A and B are two events associated with any random experiment, then $P(A/B)$ represents the probability of occurence of event-A knowing that event B has already occurred.

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \; P(B) \neq 0$$

  $P(B) \neq 0$ = means that the events should not be impossible.

$$P(A \cap B) = P(A \text{ and } B) = P(B) \times P(A/B)$$

  Similarly $\quad P(A \cap B \cap C) = P(A) \times P(B/A) \times P(C/AB)$

- **Multiplication Theorem on Probability :** If the event A and B are associated with any random experiment and the occurrence of one depends on the other then

$$P(A \cap B) = P(A) \times P(B/A) \text{ where } P(A) \neq 0$$

- When the occurrence of one does not depends on the other then these events are said to be independent events.

  Here $\quad P(A/B) = P(A)$ or $P(B/A) = P(B)$

$$P(A \cap B) = P(A) \times P(B)$$

- Theorem on total probability : If $E_1, E_2, E_3..., E_n$ be a partition of sample space and $E_1, E_2... E_n$ all has non-zero probability. A be any event associated with sample space S, then

$$P(A) = P(E_1) \,.\, P(A/E_1) + P(E_2) \,.\, P(A/E_2) + ... + P(E_n) \,.\, P(A/E_n).$$

- Bayes theorem : Let S be the sample space and $E_1, E_2 ... E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If

A is any event which occurs with E_1, or E_2 or ... E_n, then.

$$p(E_i/A) = \frac{P(E_i)\,P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\,P(A/E_i)}$$

- **Random variable :** It is real valued function whose domain is the sample space of random experiments.
- **Probability distribution :** It is a system of numbers of random variable (x), such that

X:	x_1	x_2	x_3...	x_n
$P(x)$:	$P(x_1)$	$P(x_2)$	$P(x_3)$...	$P(x_n)$

Where $P(x_i) > 0$ and $\sum_{i=1}^{n} P(x_i) = 1$

- Mean or expectation of a random variables (x) denoted by $E(x)$

$$E(x) = \mu = \sum_{i=1}^{n} x_i \; P(x_i)$$

- Variance of X denoted by var(X) or σ_x^2 and

$$\text{var}(x) = \sigma_x^2 = \sum_{i=1}^{n} (x_i - \mu)^2 \, P(x_i)$$

- The non-negative number $\sigma_x = \sqrt{\text{var}(x)}$ is called standard diviation of random variable X.
- **Bernoulli Trials :** Trials of random experiment are called Bernoulitrials if:

(i) No. of trials are finite.

(ii) Trials are independent.

(iii) Each trial has exactly two outcomes either success or failure.

(iv) Probability of success remains same in each trial.

- **Binomial Distribution :**

  $P(X = r) = {}^nc_r\, q^{n-r}\, p^r,$ where $r = 0, 1, 2, \ldots n$

  p = prob. of Success

  q = prob. of Failure

  n = total no. of trails

  r = value of random variable.

VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

1. Find $P(A/B)$ if $P(A) = 0.4$, $P(B) = 0.8$ and $P(B/A) = 0.6$

2. Find $P(A \cap B)$ if A and B are two events such that $P(A) = 0.5$, $P(B) = 0.6$ and $P(A \cup B) = 0.8$

3. A policeman fires three bullets on a dacoit. The probability that the dacoit will be killed by one bullet is 0.7. What is the probability that the dacoit is still alive.

4. What is the probability that a leap year has 53 Sundays?

5. 20 cards are numbered 1 to 20. One card is them drawn at random. What is the probability that the number on the card will be a multiple of 4?

6. Three coins are tossed once. Find the probability of getting at least one head.

7. The probability that a student is not a swimmer is $\frac{1}{5}$. Find the probability that out of 5 students, 4 are swimmers.

8. Find $P(A/B)$, if $P(B) = 0.5$ and $P(A \cap B) = 0.32$

9. A random variable x has the following probability distribution.

X	0	1	2	3	4	5
P(x)	$\frac{1}{15}$	k	$\frac{15k - 2}{15}$	k	$\frac{15k - 1}{15}$	$\frac{1}{15}$

   Find the value of k.

10. A random variable X, taking values 0, 1, 2 has the following probability distribution for some number k.

$$P(X) = \begin{cases} k & \text{if } X = 0 \\ 2k & \text{if } X = 1 \\ 3k & \text{if } X = 2 \end{cases}, \text{ find } k.$$

ANSWERS

VERY SHORT ANSWER TYPE QUESTIONS

1. 0.3
2. $\frac{3}{10}$
3. $(0.3)^3$
4. $\frac{2}{7}$
5. $\frac{1}{4}$
6. $\frac{7}{8}$
7. $\left(\frac{4}{5}\right)^4$
8. $\frac{16}{25}$
9. $k = \frac{1}{5}$
10. $k = \frac{1}{6}$

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

1. A problem in Mathematics is given to 3 students whose chances of solving it are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$. What is the probability that the problem is solved.

2. A die is rolled. If the outcome is an even number. What is the probability that it is a prime?

3. If A and B are two events such that $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{8}$. Find P (not A and not B).

4. In a class of 25 students with roll numbers 1 to 25, a student is picked up at random to answer a question. Find the probability that the roll number of the selected student is either a multiple of 5 or 7.

5. A can hit a target 4 times in 5 shots *B* three times in 4 shots, *C* twice in 3 shots. They fire a volley. What is the probability that two shots at least hit.

6. Two dice are thrown once. Find the probability of getting an even number on the first die or a total of 8.

7. *A* and *B* throw a die alternatively till one of them throws a '6' and wins the game. Find their respective probabilities of winning, if *A* starts first.

8. If *A* and *B* are events such that $P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{5}$ and $P(B) = p$ find p if events

   (i) are mutually exclusive,

   (ii) are independent.

9. A drunkard man takes a step forward with probability 0.4 and backward with probability 0.6. Find the probability that at the end of eleven steps he is one step away from the starting point.

10. Two cards are drawn from a pack of well shuffled 52 cards. Getting an ace or a spade is considered a success. Find the probability distribution for the number of success.

11. In a game, a man wins a rupee for a six and looses a rupee for any other number when a fair die is thrown. The man decided to throw a die thrice but to quit as and when he gets a six. Find the expected value of the amount he wins/looses.

12. Suppose that 10% of men and 5% of women have grey hair. A grey haired person is selected at random. What is the probability of this person being male? Assume that there are 60% males and 40% females?

13. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn. What is the probability that they both are diamonds?

14. Ten eggs are drawn successively with replacement from a lot containing 10% defective eggs. Find the probability that there is at least one defective egg.

15. Find the variance of the number obtained on a throw of an unbiased die.

VERY SHORT ANSWER TYPE QUESTIONS

1. $\frac{3}{4}$
2. $\frac{1}{3}$
3. $\frac{3}{8}$
4. $\frac{8}{25}$
5. $\frac{5}{6}$
6. $\frac{5}{9}$
7. $\frac{6}{11}, \frac{5}{11}$
8. (i) $p = \frac{1}{10}$, (ii) $p = \frac{1}{5}$
9. 0.3678
10.

x	**0**	**1**	**2**
$p(x)$	81/169	72/169	16/169

11. $-\frac{91}{54}$
12. $\frac{3}{4}$
13. $\frac{1}{17}$
14. $1-\left(\frac{9}{10}\right)^{10}$
15. $\text{var}(X) = \frac{35}{12}$.

LONG ANSWER TYPE QUESTIONS (6 MARKS)

1. In a hurdle race, a player has to cross 8 hurdles. The probability that he will clear each hurdle is $\frac{4}{5}$, what is the probability that he will knock down fewer than 2 hurdles.

2. Bag A contain 4 red, 3 white and 2 black balls. Bag B contain 3 red, 2 white and 3 black balls. One ball is transferred from bag A to bag B and then a ball is drawn from bag B. The ball so drawn is found to be red find the probability that the transferred ball is black.

3. If a fair coin is tossed 10 times, find the probability of getting.

   (i) exactly six heads,

   (ii) at least six heads,

   (iii) at most six heads.

4. A doctor is to visit a patient. From the past experience, it is known that the probabilities that he will come by train, bus, scooter by other means of transport are resp. $\frac{3}{13}, \frac{1}{5}, \frac{1}{10}$ and $\frac{2}{5}$. The probabilities that he will be late are $\frac{1}{4}, \frac{1}{3}$, and $\frac{1}{12}$ if he comes by train, bus and scooter resp. but if comes by other means of transport, then he will not be late. When he arrives, he is late. What is the probability that he comes by train?

5. A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is six. Find the probability that it is actually a six.

6. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accidents are 0.01, 0.03 and 0.15 resp. one of the insured persons meets with an accident. What is the probability that he is a scooter driver.

7. Two cards from a pack of 52 cards are lost. One card is drawn from the remaining cards. If drawn card is Heart, find the probability that the lost cards were both hearts.

8. A box *X* contain 2 white and 3 red balls and a bag Y contain 4 white and 5 red balls. One ball is drawn at random from one of the bag and is found to be red. Find the probability that it was drawn from bag *Y*.

9. In answering a question on a multiple choice, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be incorrect with probability $\frac{1}{4}$. What is the probability that the student knows the answer, given that he answered correctly.

10. Suppose a girl throws a die. If she gets 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4 she tosses

a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she throws 1, 2, 3 or 4 with the die.

11. In a bolt factory machines *A, B* and *C* manufacture 60%, 30% and 10% of the total bolts respectively, 2%, 5% and 10% of the bolts produced by them resp. are defective. A bolt is picked up at random from the product and is found to be defective. What is the probability that it has been manufactured by machine *A*.

12. Two urns *A* and *B* contain 6 black and 4 white, 4 black and 6 white balls respectively. Two balls are drawn from one of the urns. If both the balls drawn are white, find the probability that the balls are drawn from urn *B*.

13. Two cards are drawn from a well shuffled pack of 52 cards. Find the mean and variance for the number of face cards obtained.

14. Write the probability distribution for the number of heads obtained when three coins are tossed together. Also, find the mean and variance of the probability distribution.

15. Two groups are competing for the position on the Board of Directors of a corporations. The probabilities that the first and the second groups will win are 0.6 and 0.4 respectively. Further if the first group wins, the probability of introducing a new product is 0.7 and the corresponding probability is 0.3 if the second group wins. Find the probability that the new product introduced was by the second group.

LONG ANSWER TYPE QUESTIONS

1. $\frac{12}{5}\left(\frac{4}{5}\right)^7$.

2. $\frac{2}{33}$

3. (i) $\frac{105}{512}$ (ii) $\frac{193}{512}$ (iii) $\frac{53}{64}$

4. $\frac{1}{2}$

5. $\frac{3}{8}$

6. $\frac{1}{52}$

7. $\frac{22}{425}$

8. $\dfrac{25}{52}$

9. $\dfrac{12}{13}$

10. $\dfrac{8}{11}$

11. $\dfrac{12}{37}$

12. $\dfrac{5}{7}$

13. Mean $= \dfrac{8}{13}$, Variance $= \dfrac{1200}{287}$

14.

X	0	1	2	3
$P(X)$	$\dfrac{1}{8}$	$\dfrac{3}{8}$	$\dfrac{3}{8}$	$\dfrac{1}{8}$

Mean $= \dfrac{3}{2}$

Variance $= \dfrac{3}{4}$

15. $\dfrac{2}{9}$

MODEL PAPER - I

MATHEMATICS

Time allowed : 3 hours ***Maximum marks : 100***

General Instructions

1. All questions are compulsory.
2. The question paper consists of 29 questions divided into three sections A, B and *C.* Section A comprises of 10 questions of one mark each, Section B comprises of 12 questions of four marks each and Section C comprises of 7 questions of six marks each.
3. All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.
4. There is no overall choice. However, internal choice has been provided in 4 questions of four marks each and 2 questions of six marks each. You have to attempt only one of the alternatives in all such questions.
5. Use of calculators is not permitted.

SECTION A

Question number 1 to 10 carry one mark each.

1. Find the value of x, if

$$\begin{pmatrix} 5x + y & -y \\ 2y - x & 3 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ -3 & 3 \end{pmatrix}$$

2. Let * be a binary operation on N given by $a * b$ = HCF (a, b), $a, b \in N.$ Write the value of 6 * 4.

3. Evaluate : $\displaystyle\int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1 - x^2}}\, dx$

4. Evaluate : $\int \frac{\sec^2(\log x)}{x} dx$

5. Write the principal value of $\cos^{-1}\left(\cos \frac{7\pi}{6}\right)$.

6. Write the value of the determinant :

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

7. Find the value of x from the following :

$$\begin{vmatrix} x & 4 \\ 2 & 2x \end{vmatrix} = 0$$

8. Find the value of $(\hat{i} \times \hat{j}) \cdot \hat{k} + (\hat{j} \times \hat{k}) \cdot \hat{i} + (\hat{k} \times \hat{i}) \cdot \hat{j}$

9. Write the direction cosines of the line equally inclined to the three coordinate axes.

10. If $\overrightarrow{p}$ is a unit vector and $(\overrightarrow{x} - \overrightarrow{p}).(\overrightarrow{x} + \overrightarrow{p}) = 80$, then find $|\overrightarrow{x}|$.

SECTION B

Question numbers 11 to 22 carry 4 marks each.

11. The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing the rate of 4 cm/minute. When x = 8 cm and y = 6 cm, find the rate of change of (*a*) the perimeter, (*b*) the area of the rectangle.

OR

Find the intervals in which the function f given by $f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$ is strictly increasing or strictly decreasing.

12. If $\sin y = x(a + y)$, prove that $\frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a}$.

OR

If $(\cos x)^y = (\sin y)^x$, find $\frac{dy}{dx}$.

13. Lef $f : N \to N$ be defined by

$$f(x) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases} \quad \text{for all } n \in N.$$

Find whether the function f is bijective.

14. Evaluate : $\int \frac{dx}{\sqrt{5 - 4x - 2x^2}}$.

OR

Evaluate : $\int x \sin^{-1} x \, dx$.

15. If $y = \frac{\sin^{-1} x}{\sqrt{1 - x^2}}$, show that $\left(1 - x^2\right)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0$.

16. In a multiple choice examination with three possible answers (out of which only one is correct) for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing?

17. Using properties of determinants, prove the following :

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix} = 1.$$

18. Solve the following differential equation :

$$x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right).$$

19. Solve the following differential equation :

$$\cos^2 x \cdot \frac{dy}{dx} + y = \tan x.$$

20. Find the shortest distance between the lines

$$\vec{r} = (1 + \lambda)\hat{i} + (2 - \lambda)\hat{j} + (\lambda + 1)\hat{k}$$

$$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu\left(2\hat{i} + \hat{j} + 2\hat{k}\right).$$

21. Prove the following :

$$\cot^{-1}\left[\frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}\right] = \frac{x}{2},\ x \in \left(0, \frac{\pi}{4}\right).$$

OR

Solve for x :

$$2 \tan^{-1} (\cos x) = \tan^{-1} (2 \operatorname{cosec} x)$$

22. The scalar product of the vector $\hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of λ.

SECTION C

Question number 23 to 29 carry 6 marks each.

23. Find the equation of the plane determined by the points A (3, –1, 2), B (5, 2, 4) and C (–1, –1, 6). Also find the distance of the point P(6, 5, 9) from the plane.

24. Find the area of the region included between the parabola $y^2 = x$ and the line $x + y = 2$.

25. Evaluate : $\displaystyle\int_0^{\pi} \frac{x\,dx}{a^2 \cos^2 x + b^2 \sin^2 x}$.

26. Using matrices, solve the following system of equation :

$$x + y + z = 6$$

$$x + 2z = 7$$

$$3x + y + z = 12$$

OR

Obtain the inverse of the following matrix using elementary operations:

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}.$$

27. Coloured balls are distributed in three bags as shown in the following table :

Bag	*Colour of the Ball*		
	Red	***White***	***Black***
I	1	2	3
II	2	4	1
III	4	5	3

A bag is selected at random and then two balls are randomly drawn from the selected bag. They happen to be black and red. What is the probability that they came from bag I?

28. A dealer wishes to purchase a number of fans and sewing machines. He has only Rs. 5760 to invest and has space for at most 20 items. A fan costs him Rs. 360 and a sewing machine Rs. 240. His expectation is that he can sell a fan at a profit of Rs. 22 and a sewing machine at a profit of Rs. 18. Assuming that be can sell all the items that he can buy, how should he invest his money in order to maximise the profit? Formulate this as a linear programming problem and solve it graphically.

29. If the sum of the lengths of the hypotenuse and a side of a right-angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$.

OR

A tank with rectangular base and rectangular sides open at the top is to be constructed so that it's depts is 2m and volume is $8m^3$. If building of tank cost Rs. 70 per *sq* meter for the base and Rs. 45 per *sq* meter for the sides. What is the cost of least expensive tank.

MODEL PAPER - I

SOLUTION

SECTION A

1. We are given

$$\begin{bmatrix} 5x + y & -y \\ 2y - x & 3 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -5 & 3 \end{bmatrix}$$

$\therefore$ $5x + y = 4$ and $-y = 1$

$\therefore$ $y = -1$ and $5x - 1 = 4$

or $5x = 5$

$\therefore$ $x = 1$

2. 6 * 4 = HCF of 6 and 4 = 2.

3. $$\int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1 - x^2}} dx = \left|\sin^{-1} x\right|_0^{1/\sqrt{2}}$$

$$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1} 0$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

4. Let $$I = \int \frac{\sec^2 (\log x)}{x} dx$$

Let $\log x = t$

then $\frac{1}{x} dx = dt$

or $dx = x\, dt$

$\therefore \qquad I = \int \sec^2 t \; dt$

$= \tan t + c$

$= \tan (\log x) + c$

5. $\cos^{-1}\left(\cos \frac{7\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi - \frac{5\pi}{6}\right)\right]$

$= \cos^{-1}\left[\cos\left(\frac{5\pi}{6}\right)\right]$

$= \frac{5\pi}{6}$

6. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = \begin{vmatrix} a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{vmatrix}$

$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix}$

$= 0$

7. Here $\begin{vmatrix} x & 4 \\ 2 & 2x \end{vmatrix} = 0$

or $\quad 2x^2 - 8 = 0$

or $\quad x^2 - 4 = 0$

$x = \pm 2$

8. $(\hat{i} \times \hat{j}) \cdot \hat{k} + (\hat{j} \times \hat{k}) \cdot \hat{i} + (\hat{k} \times \hat{i}) \cdot \hat{j}$

$= \hat{k} \cdot \hat{k} + \hat{i} \cdot \hat{i} + \hat{j} \cdot \hat{j}$

$= 1 + 1 + 1$

$= 3$

9. The d.c. of a line equally inclined to the coordinate axes are $\left(\frac{\pm 1}{\sqrt{3}}, \frac{\pm 1}{\sqrt{3}}, \frac{\pm 1}{\sqrt{3}}\right)$.

10. $$(\vec{x} - \vec{p}).(\vec{x} + \vec{p}) = 80$$

$\therefore$ $$|\vec{x}|^2 - |\vec{p}|^2 = 80$$

As $\vec{p}$ is a unit vector,

$$|\vec{p}| = 1$$

$\therefore$ $$|\vec{x}|^2 - 1 = 80$$

or $$|\vec{x}|^2 = 81$$

$\therefore$ $$|x| = 9$$

SECTION B

11. Let P be the perimeter and A be the area of the rectangle at any time t, then

$$P = 2(x + y) \text{ and } A = xy$$

It is given that $$\frac{dx}{dt} = -5 \text{ cm/minute}$$

and $$\frac{dy}{dt} = 4 \text{ cm/minute}$$

(i) We have $P = 2(x + y)$

$\therefore$ $$\frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

$$= 2\ (-5 + 4) \text{ cm/minute}$$

$$= -\ 2 \text{ cm/minute}$$

(ii) We have $A = xy$

$$\therefore \quad \frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

$= [8 \times 4 + 6\,(-5)]$ cm^2/minute ......($\because$ $x = 8$ and $y = 6$)

$= (32 - 30)$ cm^2/minute

$= 2$ cm^2/minute

OR

The given function is

$$f(x) = \sin x + \cos x,\ 0 \le x \le 2\pi$$

$$\therefore \quad f'(x) = \cos x - \sin x$$

$$= -\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x\right)$$

$$= -\sqrt{2}\left(\sin x\cos\frac{\pi}{4} - \cos x\sin\frac{\pi}{4}\right)$$

$$= -\sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$$

For decreasing function,

$$f'(x) < 0$$

$$\therefore \quad -\sqrt{2}\sin\left(x - \frac{\pi}{4}\right) < 0$$

or $$\sin\left(x - \frac{\pi}{4}\right) > 0$$

or $$0 < x - \frac{\pi}{4} < \pi$$

or $$\frac{\pi}{4} < x < \pi + \frac{\pi}{4}$$

or $\frac{\pi}{4} < x < \frac{5\pi}{4}$

Thus $f(x)$ is a decreasing function in

$$\left\{x : \frac{\pi}{4} < x < \frac{5\pi}{4}\right\}$$

As sin x and cos x are well defined in $(0, 2\pi)$,

$f(x) = \sin x + \cos x$ is an increasing function in the complement of interval

$\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

i.e., in $\left[0, \frac{\pi}{4}\right] \cup \left[\frac{5\pi}{4}, 2\pi\right]$

12. We are given $\sin y = x \sin (a + y)$

$$\therefore \qquad x = \frac{\sin y}{\sin(a + y)}$$

Differentiating w.r.t. y, we get

$$\frac{dx}{dy} = \frac{\sin(a + y)\cos y - \sin y \cos(a + y)}{\sin^2(a + y)}$$

$$= \frac{\sin(a + y - y)}{\sin^2(a + y)}$$

$$= \frac{\sin a}{\sin^2(a + y)}$$

$$\therefore \qquad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{\sin^2(a + y)}{\sin(a)}$$

OR

12. We are given

$$(\cos x)^y = (\sin y)^x$$

Taking log of both sides, we get

$$y \log \cos x = x \log \sin y$$

Differentiating w.r.t. x, we get

$$y \cdot \frac{1}{\cos x} \cdot (-\sin x) + \log \cos x \cdot \frac{dy}{dx}$$
$$= x \cdot \frac{1}{\sin y} \cdot (\cos y) \frac{dy}{dx} + \log \sin y \cdot 1$$

or $\quad -y \tan x + \log \cos x \frac{dy}{dx} = x \cot y \frac{dy}{dx} + \log \sin y$

$\Rightarrow \quad \frac{dy}{dx}(\log \cos x - x \cot y) = \log \sin y + y \tan x$

$\therefore \quad \frac{dy}{dx} = \frac{\log \sin y + y \tan x}{\log \cos x - x \cot y}$

13. The function f is defined as

$$f(x) = \begin{cases} \frac{n+1}{2}, \text{ if } n \text{ is odd} \\ \frac{n}{2}, \text{ if } n \text{ is even} \end{cases} \quad \text{for all } n \in N$$

Now let $n = 1$, $\quad f(1) = \frac{1+1}{2} = 1$

and when $n = 2$ $f(2) = \frac{2}{2} = 1$

$\therefore$ f is not one-one function.

Hence, f is not bijective.

14. Let $\quad I = \int \frac{dx}{\sqrt{5 - 4x - 2x^2}}$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{5}{2} - 2x - x^2}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{5}{2} - \left(2x + x^2 + 1 - 1\right)}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{7}{2} - (x + 1)^2}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(\frac{7}{2}\right)^2 - (x + 1)^2}}$$

$$= \frac{1}{\sqrt{2}} \sin^{-1} \left(\frac{x + 1}{\frac{\sqrt{7}}{\sqrt{2}}} \right) + c$$

$$= \frac{1}{\sqrt{2}} \sin^{-1} \left(\frac{\sqrt{2}\,(x + 1)}{\sqrt{7}} \right) + c$$

OR

Let $\quad I = \int x \sin^{-1} x \; dx$

$$= \int \sin^{-1} x . x \; dx$$

$$= \sin^{-1} x \cdot \frac{x^2}{2} - \int \frac{1}{\sqrt{1 - x^2}} \cdot \frac{x^2}{2} \, dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \frac{-x^2}{\sqrt{1 - x^2}} \, dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \frac{1 - x^2 - 1}{\sqrt{1 - x^2}} \, dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2}\int \sqrt{1 - x^2}dx - \frac{1}{2}\int \frac{dx}{\sqrt{1 - x^2}}$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \cdot \left[\frac{x}{2}\sqrt{1 - x^2} + \frac{1}{2}\sin^{-1} x\right] - \frac{1}{2}\sin^{-1} x + c$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{x}{4}\sqrt{1 - x^2} + \frac{1}{4}\sin^{-1} x - \frac{1}{2}\sin^{-1} x + c$$

$$= \frac{x^2 \sin^{-1} x}{2} - \frac{1}{4}\sin^{-1} x + \frac{x}{4}\sqrt{1 - x^2} + c$$

15. We have

$$y = \frac{\sin^{-1} x}{\sqrt{1 - x^2}}$$

$$\Rightarrow \quad y\sqrt{1 - x^2} = \sin^{-1} x$$

Differentiating w.r.t. x, we get

$$y.\frac{(-2x)}{2\sqrt{1 - x^2}} + \sqrt{1 - x^2}\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

or $$-xy + \left(1 - x^2\right)\frac{dy}{dx} = 1$$

Differentiating again,

$$-x\frac{dy}{dx} - y + \left(1 - x^2\right)\frac{d^2y}{dx^2} + \frac{dy}{dx}(-2x) = 0$$

or $$\left(1 - x^2\right)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0$$

which is the required result.

16. Here $p = \frac{1}{3}$, $q = 1 - \frac{1}{3} = \frac{2}{3}$, and $n = 5$

Let x denote the number of successes.

Probability of r successes is given by

$$P(X = r) = {}^nC_r p^r \, q^{n-r}, \; r = 1, 2, 3.....$$

$$\therefore \quad P(X = 4 \text{ or } 5) = P(X = 4) + P\,(X = 5)$$

$$= {}^5C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^1 + {}^5C_5 \left(\frac{1}{3}\right)^5$$

$$= 5 \cdot \frac{2}{3^5} + 1 \cdot \frac{1}{3^5}$$

$$= \frac{10}{243} + \frac{1}{243} = \frac{11}{243}.$$

17. Let $\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix}$

Applying $R_2 \to R_2 - 2R_1$, we get

$$\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & -1+p \\ 3 & 6+3p & 1+6p+3q \end{vmatrix}$$

Applying $R_3 \to R_3 - 3R_1$, we get

$$\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & -1+p \\ 0 & 3 & -2+3q \end{vmatrix}$$

$$= 1[(-2 + 3p) - 3\,(-1 + p)]$$

$$= [-2 + 3p + 3 - 3p)]$$

$$= 1$$

Hence the result.

18. The given differential equation is

$$x \frac{dy}{dx} = y - x \tan\left(\frac{y}{x}\right)$$

or $$\frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right) \qquad ...(i)$$

Let $$\frac{y}{x} = z \text{ or } y = zx$$

$\therefore$ $$\frac{dy}{dx} = z + x\frac{dz}{dx}$$

$\therefore$ From (*i*), $$z + x\frac{dz}{dx} = z - \tan z$$

or $$x\frac{dz}{dx} = -\tan z$$

or $$\frac{dz}{\tan z} + \frac{dx}{x} = 0$$

or $$\int \cot z \, dz + \int \frac{dx}{x} = 0$$

$\therefore$ $$\log \sin z + \log x = \log c$$

or $$\log (x \sin z) = \log c$$

or $$x \sin\left(\frac{y}{x}\right) = c$$

which is the required solution.

19. The given differential equation is

$$\cos^2 x \frac{dy}{dx} + y = \tan x$$

or $$\frac{dy}{dx} + \sec^2 x. \ y = \tan x. \sec^2 x$$

It is a linear differential equation

$$\text{Integrating factor} = e^{\int \sec^2 x\, dx} = e^{\tan x}$$

$\therefore$ Solution of the differential equation is

$$y.e^{\tan x} = \int e^{\tan x}.\tan x \sec^2 x\, dx + c \qquad \text{...(i)}$$

Now, we find $\qquad I_1 = \int e^{\tan x} \cdot \tan x \sec^2 x\, dx$

Let $\qquad \tan x = t, \sec^2 x\, dx = dt$

$\therefore \qquad I_1 = \int te^t\, dt$

$$= t.e^t - \int e^t dt$$

$$= t\, .\, e^t - e^t$$

$$= (t - 1)e^t = (\tan x - 1)\, e^{\tan x}$$

$\therefore$ From (*i*), solution is

$$y\, .\, e^{\tan x} = (\tan x - 1)\, e^{\tan x} + c$$

or $\qquad y = (\tan x - 1) + ce^{-\tan x}$

20. Equations of the two lines are :

$$\vec{r} = (1 + \lambda)\hat{i} + (2 - \lambda)\hat{j} + (\lambda + 1)\hat{k}$$

or $\qquad \vec{r} = \left(\hat{i} + 2\hat{j} + \hat{k}\right) + \lambda\left(\hat{i} - \hat{j} + \hat{k}\right) \qquad$ ...(i)

and $\qquad \vec{r} = \left(2\hat{i} - \hat{j} - \hat{k}\right) + \mu\left(2\hat{i} + \hat{j} + 2\hat{k}\right) \qquad$ ...(ii)

Here $\qquad \overrightarrow{a_1} = \hat{i} + 2\hat{j} + \hat{k}$ and $\overrightarrow{a_2} = 2\hat{i} - \hat{j} - \hat{k}$

and $\qquad \overrightarrow{b_1} = \hat{i} - \hat{j} + \hat{k}$ and $\overrightarrow{b_2} = 2\hat{i} + \hat{j} + 2\hat{k}$

$\therefore \quad \overrightarrow{a_2} - \overrightarrow{a_1} = \left(2\hat{i} - \hat{j} - \hat{k}\right) - \left(\hat{i} + 2\hat{j} + \hat{k}\right)$

$= \hat{i} - 3\hat{j} - 2\hat{k}$

and $\quad \overrightarrow{b_1} \times \overrightarrow{b_2} = \begin{vmatrix} \hat{i} & \hat{i} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$

$= \hat{i}\,(-3) - \hat{j}\,(0) + \hat{k}\,(3)$

$= -3\hat{i} + 3\hat{k}$

$\therefore \quad \left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right| = \sqrt{9 + 9} = 3\sqrt{2}$

$\therefore$ S.D. between the lines $= \dfrac{\left|\left(\overrightarrow{a_2} - \overrightarrow{a_1}\right) \cdot \left(\overrightarrow{b_1} \times \overrightarrow{b_2}\right)\right|}{\left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right|}$

$= \dfrac{\left|\left(\hat{i} - 3\hat{j} - 2\hat{k}\right) \cdot \left(-3\hat{i} + 3\hat{k}\right)\right|}{3\sqrt{2}}$

$= \dfrac{|-3 - 6|}{3\sqrt{2}}$

$= \dfrac{9}{3\sqrt{2}} = \dfrac{3}{\sqrt{2}}$ units

21. $\cot^{-1}\left[\dfrac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}\right]$

$$= \cot^{-1}\left[\frac{\sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2} + \sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}}{\sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2} - \sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}}\right]$$

$\ldots\left[\because x \in \left(0, \dfrac{\pi}{4}\right)\right]$

$$= \cot^{-1}\left[\frac{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) + \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) - \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}\right]$$

$$= \cot^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right)$$

$$= \cot^{-1}\left[\cot\left(\frac{x}{2}\right)\right]$$

$$= \frac{x}{2}$$

OR

The given equation is

$$2\tan^{-1}(\cos x) = \tan^{-1}(2\ \text{cosec}\ x)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2\cos x}{1 - \cos^2 x}\right) = \tan^{-1}(2\text{cosec}\ x)$$

$$\Rightarrow \quad \frac{2\cos x}{\sin^2 x} = 2\ \text{cosec}\ x$$

$$\Rightarrow \quad \cos x = \text{cosec}\ x \,.\, \sin^2 x$$

$$\Rightarrow \quad \cos x = \sin x$$

$$\therefore \quad x = \frac{\pi}{4}$$

22. Unit vector along the sum of vectors

$$\overrightarrow{a} = 2\hat{i} + 4\hat{j} - 5\hat{k} \quad \text{and} \quad \overrightarrow{b} = \lambda\hat{i} + 2\hat{j} + 3\hat{k} \text{ is}$$

$$\frac{\overrightarrow{a} + \overrightarrow{b}}{|\overrightarrow{a} + \overrightarrow{b}|} = \frac{(2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2 + \lambda)^2 + 6^2 + (-2)^2}}$$

$$= \frac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{\lambda^2 + 4\lambda + 44}}$$

We are given that dot product of above unit vector with the vector $\hat{i} + \hat{j} + \hat{k}$ is 1.

$$\therefore \quad \frac{(2+\lambda)}{\sqrt{\lambda^2 + 4\lambda + 44}} \cdot 1 + \frac{6}{\sqrt{\lambda^2 + 4\lambda + 44}} - \frac{2}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1$$

or $\quad 2 + \lambda + 6 - 2 = \sqrt{\lambda^2 + 4\lambda + 44}$

or $\quad (\lambda + 6)^2 = \lambda^2 + 4\lambda + 44$

or $\quad \lambda^2 + 12\lambda + 36 = \lambda^2 + 4\lambda + 44$

or $\quad 8\lambda = 8$

or $\quad \lambda = 1$

SECTION C

23. Equation of the plane through the point A (3, – 1, 2) is

$$a\,(x - 3) + b\,(y + 1) + c\,(z - 2) = 0 \qquad ...(i)$$

As (i) Also passes through the points B(5, 2, 4) and C (– 1, – 1, 6), we have

$$a\,(5 - 3) + b\,(2 + 1) + c\,(4 - 2) = 0$$

and $\quad a\,(-1 - 3) + b\,(-1 + 1) + c\,(6 - 2) = 0$

or $\quad 2a + 3b + 2c = 0 \qquad ...(ii)$

and $\quad -4a + 0b + 4c = 0 \qquad ...(iii)$

$\therefore$ Solving (ii) and (iii), $\quad \frac{a}{12} = \frac{b}{-8-8} = \frac{c}{12}$

or $\quad \frac{a}{12} = \frac{b}{-16} = \frac{c}{12}$

or $\quad \frac{a}{3} = \frac{b}{-4} = \frac{c}{3} = k\text{(says)}$

$\therefore \quad a = 3k,\ b = -4k,$ and $c = 3k$

$\therefore$ From (*i*), equation of the required plane is

$$3k(x-3) - 4k(y+1) + 3k(z-2) = 0$$

or $$3x - 4y + 3z - 19 = 0$$

Distance of point P (6, 5, 9) from plane $3x - 4y + 3z - 19 = 0$

$$= \frac{|3 \times 6 - 4 \times 5 + 3 \times 9 - 19|}{\sqrt{9 + 16 + 9}} \text{ units}$$

$$= \frac{|18 - 20 + 27 - 19|}{\sqrt{34}} \text{ units}$$

$$= \frac{6}{\sqrt{34}} \text{ unit}$$

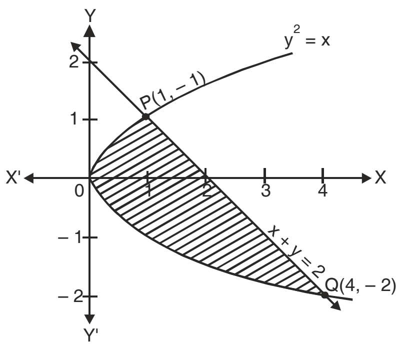
24. The given parabola is $y^2 = x$ ....(i)

It represents a parabola with vertex at O (0, 0)

The given line is

$$x + y = 2$$

or $$x = 2 - y$$ ....(ii)



Solving (*i*) and (*ii*), we get the point of intersection P (1, 1) and Q (4, – 2)

the required area = Area of the shaded region

$$= \int_{-2}^{1} \left[(2-y) - y^2\right] dy$$

$$= \left(2y - \frac{y^2}{2} - \frac{y^3}{3}\right)_{-2}^{1}$$

$$= \left[\left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right)\right] sq \text{ units}$$

$$= \left(2 - \frac{1}{2} - \frac{1}{3} + 4 + 2 - \frac{8}{3}\right) \text{sq units}$$

$$= \frac{12 - 3 - 2 + 24 + 12 - 16}{6} \text{ sq units}$$

$$= \frac{27}{6} \text{ sq unit}$$

$$= \frac{9}{2} \text{ sq unit}$$

25. Let $\quad I = \int_0^{\pi} \frac{x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x}$

or $\quad I = \int_0^{\pi} \frac{(\pi - x)dx}{a^2 \cos^2(\pi - x) + b^2 \sin^2(\pi - x)}$

or $\quad I = \int_0^{\pi} \frac{(\pi - x)dx}{a^2 \cos^2 x + b^2 \sin^2 x}$ ....(ii)

Adding (*i*) and (*ii*), we get

$$2I = \pi\int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} \qquad \text{....(iii)}$$

or $\quad 2I = \pi.2\int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$

or $$I = \pi \int_0^{\pi/2} \frac{\sec^2 x \; dx}{a^2 + b^2 \tan^2 x}$$

Let $\tan x = t$ then $\sec^2 x \; dx = dt$

When $x = 0$, $t = 0$ and when $x \to \frac{\pi}{2}, t \to \infty$

$$\therefore \quad I = \pi \int_0^{\infty} \frac{dt}{a^2 + b^2 t^2}$$

$$= \frac{\pi}{b^2} \int_0^{\infty} \frac{dt}{\left(\frac{a}{b}\right)^2 + t^2}$$

$$= \frac{\pi}{b^2} \cdot \frac{1}{a/b} \left[\tan^{-1} \frac{t}{a/b}\right]_0^{\infty}$$

$$= \frac{\pi}{ab} \left[\tan^{-1} \frac{bt}{a}\right]_0^{\infty}$$

$$= \frac{\pi}{ab} \left[\frac{\pi}{2}\right]$$

$$= \frac{\pi^2}{2ab}$$

26. The given system of equations is

$$x + y + z = 6$$

$$x + 2z = 7$$

$$3x + y = z = 12$$

Rewriting it in matrix from.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

or $\quad AX = B$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

or $\quad X = A^{-1}B \quad$ ...(i)

Now $$|A| = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}$$

$$= 1\ (-2) - 1\ (-5) + 1\ (1)$$

$$= -2 + 5 + 1$$

$$= 4 \neq 0$$

$\therefore \quad A^{-1}$ exists

Now cofactors of elements of Matrix A are :

$A_{11} = (-1)^2\ (-2) = -2$, $\quad A_{12} = (-1)^3\ (-5) = 5$,
$A_{13} = (-1)^4\ (l) = 1$

$A_{21} = (-1)^3\ (0) = 0$, $\quad A_{22} = (1)^4\ (-2) = -2$,
$A_{23} = (-1)^5\ (-2) = 2$

$A_{31} = (-1)^4\ (2) = 2$, $\quad A_{32} = (-1)^5\ (1) = -1$,
$A_{33} = (-1)^6\ (-1) = -1$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|} \Rightarrow A^{-1} = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

$\therefore \quad$ From (i), $X = A^{-1}\ B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -12 + & 0 + & 24 \\ 30 - & 14 - & 12 \\ 6 + & 14 - & 12 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 12 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore$ $x = 3$, $y = 1$ and $z = 2$

OR

26. By using elementary row transformations, we can write

$$A = IA$$

i.e., $$\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_2 - R_2$, we get

$$\begin{bmatrix} 1 & -3 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 2R_1$, we get

$$\begin{bmatrix} 1 & -3 & -1 \\ 0 & 9 & 2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 + R_3$, we get

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 9 & 2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 2R_3$, we get

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \to R_3 - 4R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 8 & 12 & 9 \end{bmatrix} A$$

$$A^{-1} = \begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 8 & -12 & 9 \end{bmatrix}$$

27. Let the events be

E_1 : Bag I is selected

E_2 : Bag II is selected

E_3 : Bag III is selected

and A : A back and a red ball is drawn

$\therefore$ $p(E_1) = p(E_2) = p(E_3) = \frac{1}{3}$

$$p(A/E_1) = \frac{1 \times 3}{{}^6C_2} = \frac{3}{15} = \frac{1}{5}$$

$$p(A/E_2) = \frac{2 \times 1}{{}^7C_2} = \frac{2}{21}$$

$$p(A/E_3) = \frac{4 \times 3}{{}^{12}C_2} = \frac{4 \times 3}{66} = \frac{2}{11}$$

$$\therefore \quad p(E_1/A) = \frac{p(A/E_1).p(E_1)}{p(A/E_1)\ p/(E_1) + p(A/E_2)\ p\left(E_2\right) + p(A/E_3).P(E_3)}$$

$$= \frac{\frac{1}{3} \times \frac{1}{5}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{2}{21} \times \frac{1}{3} \times \frac{2}{11}}$$

$$= \frac{\frac{1}{15}}{\frac{1}{15} + \frac{2}{63} \times \frac{2}{33}}$$

$$= \frac{\frac{1}{15}}{\frac{551}{3465}}$$

$$= \frac{1}{15} \times \frac{3465}{551} = \frac{231}{551}$$

28. Let us suppose that the dealer buys x fans and y sewing machines,

$\therefore$ By the given condition,

$$x + y \leq 20$$

Cost of a fan = Rs 360

and cost of a sewing machine = Rs 240

$\therefore$ By the given condition,

$$360x + 240y \leq 5760$$

Profit on one fan = Rs 22

and Profit on one sewing machine = Rs 18

$\therefore$ Objective function is

$$Z = 22x + 18y$$

Clearly $x \geq 0,\ y \geq 0$

Thus L.P. problem is to

maximise $Z = 22x + 18y$

subject to constraints,

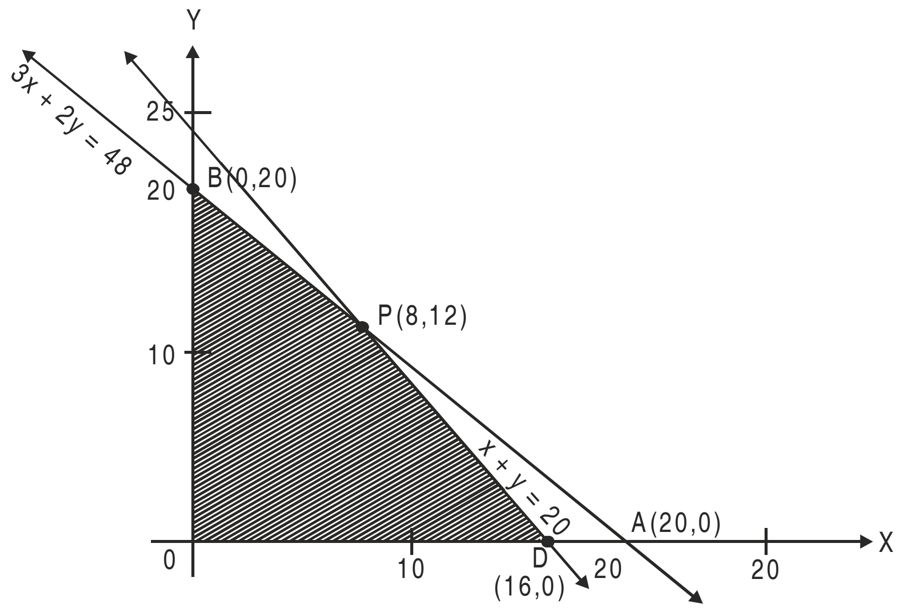
$$x + y \leq 20$$

$$360x + 240y \leq 5760 \text{ or } 3x + 2y \leq 48$$

$$x \geq 0,\ y \geq 0$$

We now draw the graphs of

$$x + y = 20 \text{ and } 3x + 2y = 48$$



The feasible region ODPB of the L.P.P. is the shaded region which has the corners O (0, 0), D (16, 0), P (8, 12) and B (0, 20)

The values of the objective function Z at O, D, P and B are :

	At O,	$Z = 22 \times 0 + 18 \times 0 = 0$
	At D,	$Z = 22 \times 16 + 18 \times 0 = 352$
	At P,	$Z = 22 \times 8 + 18 \times 12 = 392 \rightarrow$ Maximum
and	At B,	$Z = 22 \times 0 + 18 \times 20 = 360$

Thus Z is maximum at $x = 8$ and $y = 12$ and the maximum value of $z =$ Rs 392.

Hence the dealer should purchase 8 fans and 12 sewing machines to obtain maximum profit.

29. Let ABC be a right angled triangle with base BC = x and hypotenuse AB = y

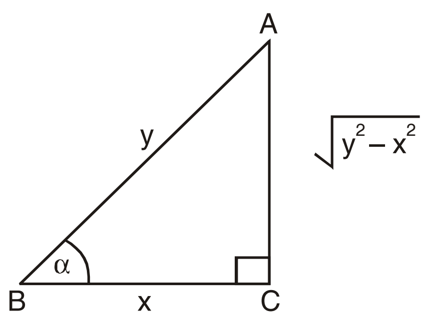
Such that

$$x + y = k \text{ where } k \text{ is a constant}$$

Let α be the angle between the base and the hypotenuse. Then area of the triangle

$$A = \frac{1}{2} BC \times AC$$

$$= \frac{1}{2} x\sqrt{y^2 - x^2}$$



$$\therefore \qquad A^2 = \frac{x^2}{4}(y^2 - x^2)$$

$$= \frac{x^2}{4}\left[(k - x)^2 - x^2\right]$$

$$\text{or} \qquad A^2 = \frac{x^2}{4}\left[k^2 - 2kx\right] = \frac{k^2x^2 - 2kx^3}{4} \qquad \text{..(i)}$$

Differentiating w.r.t. x we get

$$2A\frac{dA}{dx} = \frac{2k^2x - 6kx^2}{4} \qquad \text{....(ii)}$$

$$\text{or} \qquad \frac{dA}{dx} = \frac{k^2x - 3kx^2}{4A}$$

For maximum or minimum,

$$\text{or} \qquad \frac{dA}{dx} = 0$$

$$\Rightarrow \qquad \frac{k^2x - 3kx^2}{4} = 0$$

$$\Rightarrow \qquad x = \frac{k}{3}$$

Differentiating (ii) *w.r.t.x.* we get

$$2\left(\frac{dA}{dx}\right)^2 + 2A\frac{d^2A}{dx^2} = \frac{2k^2 - 12kx}{4}$$

Putting, $\frac{DA}{dx} = 0$ and $x = \frac{k}{3}$, we get

$$\frac{d^2A}{dx^2} = \frac{-k^2}{4A} < 0$$

$\therefore$ A is maximum when $x = \frac{k}{3}$

Now $$x = \frac{k}{3} \Rightarrow y = k - \frac{k}{3} = \frac{2k}{3}$$

$$\therefore \quad \cos\alpha = \frac{x}{y} \Rightarrow \cos\alpha = \frac{k/3}{2k/3} = \frac{1}{2}$$

$$\alpha = \frac{\pi}{3}$$

OR

30. Let the length of the tank be x metres and breadth by y metres

$\therefore$ Depth of the tank = 2m

$\therefore$ Volume = x × y × 2 = 8

$$xy = 4$$

or $$y = \frac{4}{x}$$

Area of base = xy sq m

Area of 4 walls = 2 [2x + 2y] sq m = 4 (x + y) sq m

$\therefore$ Cost C (x ,y) = 70 (xy) + 45 (4x + 4y)

or C (x, y) = 70 × 4 + 180 (x + y)

$$\therefore \quad C(x) = 280 + 180\left(x + \frac{4}{x}\right)$$

Now $$\frac{dC}{dx} = 180\left(1 - \frac{4}{x^2}\right)$$

For maximum or minimum $\frac{dC}{dx} = 0$

$$\therefore \quad 180\left(1 - \frac{4}{x^2}\right) = 0$$

or $x^2 = 4$

or $x = 2$

and $$\frac{d^2C}{dx^2} = 180\left(\frac{8}{x^3}\right) > 0$$

$\therefore$ C is minimum at $x = 2$

Least Cost = Rs [(280 + 180 (2 + 2)]

= Rs [(280 + 720] = Rs 1000

MODEL PAPER - II

MATHEMATICS

Time allowed : 3 hours
Maximum marks : 100

General Instructions

1. All question are compulsory.
2. The question paper consists of 29 questions divided into three sections A, B and *C.* Section A comprises of 10 questions of one mark each, Section B comprises of 12 questions of four marks each and Section C comprises of 7 questions of six marks each.
3. All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.
4. There is no overall choice. However, internal choice has been provided in 4 questions of four marks each and 2 questions of six marks each. You have to attempt only one of the alternatives in all such questions.
5. Use of calculators is not permitted.

SECTION A

Question number 1 to 10 carry one mark each.

1. Evaluate : $\int \frac{1}{x + x \log x} dx$.
2. Evaluate : $\int_0^1 \frac{1}{\sqrt{4x + 1}} dx$.
3. If the binary operation * defined on Q, is defined as $a * b = 2a + b - ab$, for all a, $b \in Q$, find the value of 3 * 4.
4. If $\begin{pmatrix} y + 2x & 5 \\ -x & 3 \end{pmatrix} = \begin{pmatrix} 7 & 5 \\ -2 & 3 \end{pmatrix}$, find the value of y.

5. Find a unit vector in the direction of $2\hat{i} - \hat{j} + 2\hat{k}$.

6. Find the direction cosines of the line passing through the following points :

$(-2, 4, -5), (1, 2, 3)$

7. If $A = \left[a_{ij}\right] = \begin{pmatrix} 2 & 3 & -5 \\ 1 & 4 & 9 \\ 0 & 7 & -2 \end{pmatrix}$ and $B = \left[b_{ij}\right] = \begin{pmatrix} 2 & 1 & -1 \\ -3 & 4 & 4 \\ 1 & 5 & 2 \end{pmatrix}$, then find $a_{22} + b_{21}$.

8. If $\left|\vec{a}\right| = \sqrt{3}, \left|\vec{b}\right| = 2$ and $\vec{a} \cdot \vec{b} = \sqrt{3}$, find the angle between $\vec{a}$ and $\vec{b}$.

9. If $A = \begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix}$, then find the value of k if $|2A| = k\,|A|$.

10. Write the principal value of $\tan^{-1}\left[\tan \frac{3\pi}{4}\right]$.

SECTION B

Question number 11 to 22 carry 4 marks each.

11. Evaluate : $\int \frac{\cos x\, dx}{(2 + \sin x)(3 + 4 \sin x)}$.

OR

Evaluate : $\int x^{2} \cos^{-1} x \; dx$.

12. Show that the relation R in the set of real numbers, defined as $R = \{(a, b) : a \leq b^2\}$ is neither reflexive, symmetric, nor transitive.

13. If $\log (x^2 + y^2) = 2 \tan^{-1}\left(\frac{y}{x}\right)$, then show that $\frac{dy}{dx} = \frac{x + y}{x - y}$.

OR

If $x = a(\cos t + t \sin t)$ and $y = a(\sin t - \cos t)$, then find $\frac{d^2y}{dx^2}$.

14. Find the equation of the tangent to the curve $y = \sqrt{4x - 2}$ which is parallel to the line $4x - 2y + 5 = 0$.

OR

Using differentials, find the approximate value of $f(2.01)$, where $f(x) = 4x^3 + 5x^2 + 2$.

15. Prove the following :

$$\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right) = \frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right).$$

OR

Solve the following for x :

$$\cos^{-1}\left(\frac{x^2 - 1}{x^2 + 1}\right) + \tan^{-1}\left(\frac{2x}{x^2 - 1}\right) = \frac{2\pi}{3}.$$

16. Find the angle between the line $\frac{x + 1}{2} = \frac{3y + 5}{9} = \frac{3 - z}{-6}$ and the plane $10x + 2y - 11z = 3$.

17. Solve the following differential equation :

$$(x^3 + y^3)\, dy - x^2 y\, dx = 0$$

18. Find the particular solution of the differential equation

$\frac{dy}{dx} + y \cot x = \operatorname{cosec} x$, $(x \neq 0)$, given that $y = 1$ when $x = \frac{\pi}{2}$.

19. Using properties of determinants, prove the following :

$$\begin{vmatrix} a^2+1 & ab & ac \\ ba & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2$$

20. The probability that A hits a target is $\frac{1}{3}$ and the probability that B hits it is $\frac{2}{5}$. If each one of A and B shoots at the target, what is the probability that

 (i) the target is hit?

 (ii) exactly one of them hits the target?

21. Find $\frac{dy}{dx}$, if $y^x + x^y = a^b$, where a, b are constants.

22. If $\vec{a}, \vec{b}$ and $\vec{c}$ are vectors such that $\vec{a}.\vec{b} = \vec{a} \cdot \vec{c}$ and $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}, \vec{a} \neq \vec{0}$ then prove that $\vec{b} = \vec{c}$.

SECTION C

Question number 23 to 29 carry 6 marks each.

23. One kind of cake requires 200 g of flour and 25g of fat, and another kind of cake requires 100g of flour and 50g of fat. Find the maximum number of cakes which can be made from 5 kg of flour and 1 kg of fat assuming that there is no shortage of the other ingredients used in making the cakes. Formulate the above as a linear programming problem and solve graphically.

24. Using integration, find the area of the region :

$$\{(x, y) : 9x^2 + y^2 \leq 36 \text{ and } 3x + y \geq 6\}$$

25. Show that the lines $\frac{x+3}{-3} - \frac{y-1}{1} = \frac{z-5}{5}$ and $\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5}$ are coplanar. Also find the equation of the plane containing the lines.

26. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}\cdot$ Also find the maximum volume.

OR

Show that the total surface area of a closed cuboid with square base and given volume, is minimum, when it is a cube.

27. Using matrices, solve the following system of linear equations :

$$3x - 2y + 3z = 8$$

$$2x + y - z = 1$$

$$4x - 3y + 2z = 4$$

28. Evaluate : $\int \frac{x^4 dx}{(x-1)(x^2+1)}$.

OR

Evaluate : $\int_1^4 [|x-1| + |x-2| + |x-4|]\, dx$.

29. Two cards are drawn simultaneously (or successively without replacement) from a well shuffled pack of 52 cards. Find the mean and variance of the number of red cards.