EDUCATIVE COMMENTARY ON
JEE 2014 ADVANCED MATHEMATICS PAPERS

Contents

The year 2013 represented a drastic departure in the history of the JEE. Till 2012, the selection of the entrants to the IITs was entirely left to the IITs and for more than half a century they did this through the JEE which acquired a world wide reputation as one of the most challenging tests for enrty to an engineering programme. Having cleared the JEE was often a passport for many lucrative positions in all walks of life (many of them having little to do with engineering). It is no exaggeration to say that the coveted positions of the IIT's was due largely to the JEE system which was renowned not only for its academic standards, but also its meticulous punctuality and its unimpeachable integrity.

The picture began to change since 2013. The Ministry of Human Resources decided to have a common examination for not only the IITs, but all NIT's and other engineering colleges who would want to come under its umbrella. This common test would be conducted by the CBSE. Serious concerns were raised that this would result in a loss of autonomy of the IITs and eventually of their reputation. Finally a compromise was reached that the common entrance test conducted by the CBSE would be called the JEE (Main) and a certain number of top rankers in this examination would have a chance to appear for another test, to be called JEE (Advanced), which would be conducted solely by the IITs, exactly as they conducted their JEE in the past.

So, in effect, the JEE (Advanced) from 2013 took the role of the JEE in the past except that the candidates appearing for it are selected by a procedure over which the IITs have no control. So, this arrangement is not quite the same as the JEE in two tiers which prevailed for a few years. It was hoped that now that the number of candidates appearing for the JEE (Advanced) is manageable enough to permit evaluation by humans, the classic practice of requiring the candidates to give justifications for their answers would be revived at least from 2014, if not from 2013 (when there might not have been sufficient time to make the switch-over). But this has not happened. The JEE (Advanced) 2014 is completely multiple choice type and its pattern is differs from that of the JEE (Advanced) 2013 only marginally. Single correct answer MCQ's in Paper I have been dropped. Instead, the number of questions with a single correct integer value answer is increased.

Academically (and socially), the JEE (Advanced) has the same status as the JEE in the past. So, from 2013, the Educative Commentary to JEE Mathematics Papers is confined only to JEE (Advanced). The numbering of the questions will be that in Code 5 of the question papers. As in the past, unless otherwise stated, all the references made are to the author's book *Educative JEE (Mathematics)* published by Universities Press, Hyderabad.

PAPER 1

Contents

SECTION I

Single Correct Choice Type

This section contains **ten** multiple choice questions. Each question has 4 choices out of which **ONE OR MORE THAN ONE** is/are correct. A correct answer gets 3 points. No points if the question is not answered or if some but not all the alternatives marked are correct. No negative marking for an incorrect answer.

Q.41 Let $f : (0, \infty) \longrightarrow I\!R$ be given by $f(x) = \displaystyle\int_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}$. Then

(A) $f(x)$ is monotonically increasing on $[1, \infty)$
(B) $f(x)$ is monotonically decreasing on $(0, 1)$
(C) $f(x) + f(1/x) = 0$, for all $x \in (0, \infty)$
(D) $f(2^x)$ is an odd function of x on $I\!R$

Answer and Comments: (A, C, D). The given function is of the type

$$f(x) = \int_{u(x)}^{v(x)} g(t)dt \qquad (1)$$

where $g(t)$ is some continuous function of t and $u(x), v(x)$ are differentiable functions of x. From the second fundamental theorem of calculus (or rather, its slight extension given on p. 631), we have

$$f'(x) = \frac{e^{-\left(x+\frac{1}{x}\right)}}{x} + xe^{-\left(\frac{1}{x}+x\right)}\frac{1}{x^2} = \frac{2e^{-\left(x+\frac{1}{x}\right)}}{x} \qquad (2)$$

which is positive for all $x \in (0, \infty)$. Hence (A) is true and (B) false. For (C), note that $f(1/x)$ is obtained from $f(x)$ by interchanging the upper and the lower limits of integration in (1). As this results into changing the sign of the integral, we see that $f(1/x) = -f(x)$, whence (C) is also true. For (D), we have to compare $f(2^x)$ and $f(2^{-x})$. But since 2^x and 2^{-x} are reciprocals of each other, by (C), $f(2^x) = -f(2^{-x})$. Hence $f(2^x)$ is an odd function of x. So (D) is also true.

This question tests various (and unrelated) properties of the integrals. A candidate who quickly observes that (D) follows from (C) is rewarded for his alertness in terms of the time saved.

Q.42 Let $a \in I\!R$ and $f : I\!R \longrightarrow I\!R$ be given by $f(x) = x^5 - 5x + a$. Then,

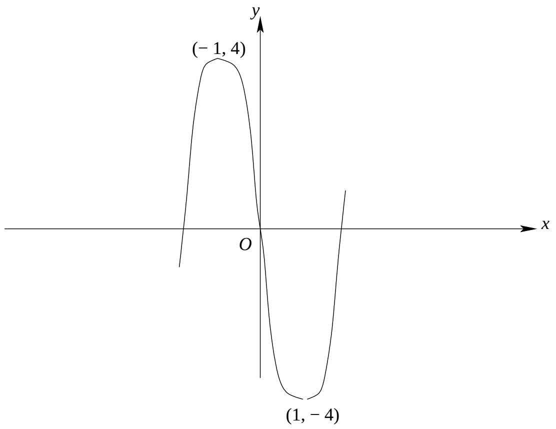
(A) $f(x)$ has three real roots if $a > 4$

(B) $f(x)$ has only one real root if $a > 4$

(C) $f(x)$ has three real roots if $a < -4$

(D) $f(x)$ has three real roots if $-4 < a < 4$

Answer and Comments: (B, D). There is no formula for the roots of a fifth degree polynomial. Nor can they be identified by inspection in the present case. At any rate, we are not asked to identify the real roots but only to test some statements about the numbers of real roots (as opposed to the roots themselves). So methods from calculus have to be used and often the answers are obtained by merely looking at a (well-drawn) graph. Note that $f(x)$ is a different function of x for different values of a. But their graphs are all qualitatively same and can be obtained from each other by suitable vertical shifts. All the four alternatives given can be paraphrased in terms of this function. Hence it suffices to draw only the graph of the function $y = f(x) = x^5 - 5x$. $f'(x) = 5(x^4 - 1) = 5(x^2 + 1)(x^2 - 1)$ has two distinct, simple real roots, viz. -1 and 1. So, f' is positive on $(-\infty, -1)$, negative on $(-1, 1)$ and positive on $(1, \infty)$. Since f is an odd function and $f(-1) = 4$ and $f(1) = -4$, the graph looks like this:



Now, coming to the various alternatives, for any $a \in I\!R$, the roots of the equation $x^5 - 5x + a = 0$ correspond precisely the points of intersection of the graph above with the horizontal line $y = -a$. From the graph it is clear that a horizontal line $y = b$ intersects the graph in three distinct points if $b \in (-4, 4)$ and only in one point if $b < -4$ ot $b > 4$. This shows that (B) and (D) are correct while (A) and (C) are false.

A simple, straightforward question, testing elementary curve sketching.

Q.43 For every pair of continuous functions, $f, g : [0,1] \longrightarrow I\!R$, such that $\max\{f(x) : x \in [0,1]\} = \max\{g(x) : x \in [0,1]\}$, the correct statement(s) is/are

(A) $(f(c))^2 + 3f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0,1]$

(B) $(f(c))^2 + f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0,1]$

(C) $(f(c))^2 + 3f(c) = (g(c))^2 + g(c)$ for some $c \in [0,1]$

(D) $(f(c))^2 = (g(c))^2$ for some $c \in [0,1]$

Answer and Comments: (A, D). The expressions in the alternatives (A) and (D), when brought on one side of the equality have $f(c) - g(c)$ has a factor. So, evidently, the question deals with the vanishing of the difference function $f(x) - g(x)$. By the Intermediate Value Property, if this difference never vanishes on $[0,1]$, then either it is throughout positive or throughout negative. In other words, either $f(x) > g(x)$ for all $x \in [0,1]$ or $f(x) < g(x)$ for all $x \in [0,1]$. Without loss of generality, assume that the first possibility holds. Let x^* be a point in $[0,1]$ where g attains its maximum. Then

$$\max\{f(x) : x \in [0,1]\} \geq f(x^*) > g(x^*) = \max\{g(x) : x \in [0,1]\}$$

which contradicts that the first and the last terms are given to be equal. Hence $f(x) - g(x)$ must vanish at some point. Therefore (A) and (D) are true. This does not quite prove that these are the only correct alternatives. That has to be done only by counter-examples for (B) and (C). An easy counterexample which will work for both is to take both f and g to be the constant functions k for some $k \neq 0$.

A good problem on the use of Intermediate Value Property. Unfortunately, instead of asking that f and g are equal somewhere, the alternatives are given in a twisted form. The fake alternatives (B) and (C) are added, apparently, only to make up the total equal 4. A scrupulous student who tries to disprove them by counterexamples will be punished in terms of time.

Q.44 A circle S passes through the point $(0,1)$ and is orthogonal to the circles $(x-1)^2 + y^2 = 16$ and $x^2 + y^2 = 1$. Then

(A) radius of S is 8 (B) radius of S is 7
(C) centre of S is at $(-7,1)$ (D) centre of S is at $(-8,1)$

Answer and Comments: (B, C). A circle in the plane is uniquely determined by three independent conditions. This is consistent with the fact that the general equation of a circle has three independent constants. In the present problem, we are given that the circle S passes through the point $(0,1)$. The other two conditions are its orthogonality with each of the other two given circles. A mechanical way to do the problem would be to start by taking the equation of S in the form

$$x^2 + y^2 + 2gx + 2fy + c = 0 \tag{1}$$

and write a system of three equations for the three unknowns g, f and c. One equation comes from the fact that (1) is satisfied by $(0,1)$. And hence

$$2f + c + 1 = 0 \tag{2}$$

The other two equations can be written down from the formulas that express the orthogonality conditions for two circles in terms of their equations. But instead of following this mechanical approach, let us take an advantage of the fact that the point $(0, 1)$ happens to lie on the circle, say C_1 whose equation is $x^2+y^2 = 1$. Moreover the tangent to C_1 at the point $(0, 1)$ is the line $y = 1$. Since S also passes through $(0, 1)$ and cuts C_1 orthogonally at $(0, 1)$, the tangent to S at $(0, 1)$ must be the y-axis. So the centre of S must be of the form $(a, 1)$ and its radius must be $|a|$ for some real number a. So, instead of three unknowns, we now have to determine only one unknown, viz. a. For this we need one equation in a. It is provided by the orthogonality of S with the other circle, say C_2 whose equation is $(x - 1)^2 + y^2 = 16$. Hence the centre of C_2 is $(1, 0)$ while its radius is 4. By orthogonality, the square of the distance between the centres of S and C_2 equals the sum of the squares of their radii. This gives

$$(a - 1)^2 + 1 = a^2 + 16 \qquad (3)$$

which reduces to $-2a = 14$. Hence $a = -7$. Therefore the centre of S is at $(-7, 1)$ and its radius is 7.

The problem is simple and rewards a candidate who uses the specific features of the data judiciously to simplify the solution rather than apply readymade formulas mechanically.

Q.45 Let $\vec{x}, \vec{y}$ and $\vec{z}$ be three vectors each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\frac{\pi}{3}$. If $\vec{a}$ is a non-zero vector perpendicular to $\vec{x}$ and $\vec{y} \times \vec{z}$ and $\vec{b}$ is a non-zero vector perpendicular to $\vec{y}$ and $\vec{z} \times \vec{x}$, then

(A) $\vec{b} = (\vec{b} \cdot \vec{z})(\vec{z} - \vec{x})$ (B) $\vec{a} = (\vec{a} \cdot \vec{y})(\vec{y} - \vec{z})$
(C) $\vec{a} \cdot \vec{b} = -(\vec{a} \cdot \vec{y})(\vec{b} \cdot \vec{z})$ (D) $\vec{a} = (\vec{a} \cdot \vec{y})(\vec{z} - \vec{y})$

Answer and Comments: (A, B, C). Since $\vec{b}$ is perpendicular both to $\vec{y}$ and $\vec{z} \times \vec{x}$, it is parallel to their cross product, viz. $\vec{y} \times (\vec{z} \times \vec{x})$ which equals $(\vec{y} \cdot \vec{x})\vec{z} - (\vec{y} \cdot \vec{z})\vec{x}$. From the data the dot product of any two distinct vectors out of $\vec{x}, \vec{y}$ and $\vec{z}$ is $\sqrt{2}\sqrt{2} \cos \frac{\pi}{3}$ which simply equals 1. Therefore we can write

$$\vec{b} = \lambda(\vec{z} - \vec{x}) \qquad (1)$$

for some non-zero scalar λ. Taking dot product with $\vec{z}$, we have

$$\begin{aligned} \vec{b} \cdot \vec{z} &= \lambda(\vec{z} \cdot \vec{z} - \vec{x} \cdot \vec{z}) \\ &= \lambda(\sqrt{2}\sqrt{2} - 1) = \lambda \end{aligned} \tag{2}$$

From (1) and (2) we get (A). In an analogous manner we have

$$\begin{aligned} \vec{a} &= \mu(\vec{x} \times (\vec{y} \times \vec{z})) \\ &= \mu[(\vec{x} \cdot \vec{z})\vec{y} - (\vec{x} \cdot \vec{y})\vec{z}] \\ &= \mu(\vec{y} - \vec{z}) \end{aligned} \tag{3}$$

for some non-zero scalar μ. Taking dot product with $\vec{y}$ we get

$$\begin{aligned} \vec{a} \cdot \vec{y} &= \mu(\vec{y} \cdot \vec{y} - \vec{z} \cdot \vec{y}) \\ &= \mu(2 - 1) = \mu \end{aligned} \tag{4}$$

From (3) and (4), we get (B). It also rules out (D) as otherwise we would have $\vec{a} = -\vec{a}$ contradicting that $\vec{a}$ is a non-zero vector.

Finally, to see if (C) holds, from (3) and (1), we have

$$\begin{aligned} \vec{a} \cdot \vec{b} &= \mu\lambda[(\vec{y} - \vec{z}) \cdot (\vec{z} - \vec{x})] \\ &= \mu\lambda[\vec{y} \cdot \vec{z} - \vec{z} \cdot \vec{z} - \vec{y} \cdot \vec{x} + \vec{z} \cdot \vec{x}] \\ &= \mu\lambda(1 - 2 - 1 + 1) = -\mu\lambda \end{aligned} \tag{5}$$

Substituting from (4) and (2) gives (C).

The problem is a simple application of elementary results and identities about the dot and the cross products of vectors. But the work involved is repetitious. Nothing new is examined once a candidate gets either (A) or (B). Either one of these two should have been the sole question suitable for three minutes.

Q.46 From a point $P(\lambda, \lambda, \lambda)$, perpendiculars PQ and PR are drawn respectively on the lines $y = x, z = 1$ and $y = -x, z = -1$. If P is such that $\angle QPR$ is a right angle, then the possible value(s) of λ is(are)

(A) $\sqrt{2}$ (B) 1 (C) -1 (D) $-\sqrt{2}$

Answer and Comments: (C). Call the two lines as L_1 and L_2 respectively. Then the point Q is of the form $(s, s, 1)$ for some $s \in I\!R$.

Moreover, the vector $\overrightarrow{PQ}$ is perpendicular to $\vec{i}+\vec{j}$ which is a vector parallel to L_1. This gives

$$\begin{aligned} 0 &= \overrightarrow{PQ} \cdot(\vec{i}+\vec{j}) && (1) \\ &= (s-\lambda)+(s-\lambda) && (2) \end{aligned}$$

which gives $s = \lambda$ and hence Q as $(\lambda, \lambda, 1)$.

Similarly, R is of the form $(t, -t, -1)$ for some $t \in I\!\!R$. Perpendicularity of $\overrightarrow{PR}$ and $\vec{i}-\vec{j}$ (which is a vector parallel to L_2 gives

$$\begin{aligned} 0 &= \overrightarrow{PL} \cdot(\vec{i}-\vec{j}) \\ &= (t-\lambda)-(-t-\lambda) && (3) \end{aligned}$$

which gives $t = 0$ and hence R as $(0, 0, -1)$. We are further given that $\angle QPR$ is a right angle, i.e. that the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are perpendicular to each other. This means

$$\begin{aligned} 0 &= \overrightarrow{PQ} \cdot \overrightarrow{PR} \\ &= (1-\lambda)\vec{k} \cdot (-\lambda\vec{i} - \lambda j + (-1-\lambda)\vec{k}) \\ &= \lambda^2 - 1 && (4) \end{aligned}$$

solving which we get $\lambda = \pm 1$. So both (B) and (C) are correct. But if $\lambda = 1$, then the points P and Q coincide and the angle QPR is not well-defined. So we rule out $\lambda = 1$. That leaves (C) as the only correct answer.

The problem is very simple and involves little beyond parametric equations of straight lines and characterisation of orthogonality of vectors in terms of their dot product. At least one of the lines L_1 and L_2 should have been given as an intersection of two planes. Then the candidate would have to first obtain its parametric equations. Even then the problem would hardly merit a place in the advanced JEE.

As the calculations in the problem are utterly simple, perhaps the intention of the paper-setters was more to see if the candidate correctly rules out the possibility $\lambda = 1$ which leads to a degeneracy. Such an intention is indeed commendable, provided it takes some fine, mathematical thinking to weed out some of the crude answers. An excellent example of this occurred in the last year's JEE where the problem was to find all value(s) of $a \neq 1$ for which

$$\lim_{n \to \infty} \frac{(1^a + 2^a + \ldots + n^a)}{(n+1)^{a-1}[(na+1) + (na+2) + \ldots + (na+n)]} = \frac{1}{60}$$

holds true. A hasty calculation leads to a quadratic in a which has 7 and $-17/2$ as its roots. The latter has to be discarded because it can be shown that the limit in the question does not exist for $a < -1$.

In the present problem, however, the exclusion of (B) as a correct answer is somewhat controversial. If the statement '$\angle QPR$ is a right angle' is interpreted as ' $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are at right angles', then it is difficult to rule out the degenerate case where $P = Q$ because when we say that two vectors are orthogonal we do allow the possibility that one (or even both) of them vanish. It would indeed be unfair if a candidate suffers because of something that is more linguistic than mathematical. The numerical data could have been changed to avoid such degeneracy.

Q.47 Let M be a 2×2 symmetric matrix with integer entries. Then M is invertible if

(A) the first column of M is the transpose of the second row of M

(B) the second row of M is the transpose of the first column of M

(C) M is a diagonal matrix with non-zero entries in the main diagonal

(D) the product of the entries in the main diagonal of M is not the square of an integer

Answer and Comments: (C, D). The first two alternatives are really the same because the transpose of the transpose of *any* matrix (which may, in particular, be a row or a column matrix) is the same as the original matrix. Anyway, in the present problem, both of them are false as can be seen by taking M to be a matrix all whose entries are 0 (or any other integer for that matter).

(C) and (D) are the only statements that deserve to be checked. For this, we use the characterisation of the invertibility of a matrix in terms of its determinant. Assume

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{1}$$

where a, b, c, d are integers. Then

$$\det(M) = ad - bc \tag{2}$$

In (C) we have $b = c = 0$ and a, d are non-zero. In that case, $\det(M) = ad \neq 0$. So M is invertible. As for (D), note that $b = c$ as M is given to be symmetric. So, $\det(M) = ad - b^2$. If M were not invertible, then we would have $\det(M) = 0$, i.e. $ad = b^2$. But we are given that ad is not the square of any integer. Therefore, M is invertible. Hence (D) is also true.

This problem, too, is too simple to deserve a place in the advanced JEE.

Q.48 Let M and N be two 3×3 matrices such that $MN = NM$. Further, if $M \neq N^2$ and $M^2 = N^4$, then

(A) determinant of $(M^2 + MN^2)$ is 0

(B) there is a 3×3 non-zero matrix U such that $(M^2 + MN^2)U$ is the zero matrix

(C) determinant of $(M^2 + MN^2) \geq 1$

(D) for a 3×3 matrix U, if $(M^2 + MN^2)U$ equals the zero matrix then U is the zero matrix.

Answer and Comments: (A, B). All the four alternatives are about the matrix $M^2 + MN^2$ which factors as $M(M + N^2)$. Since the matrices M and N commute with each other, any polynomial expressions involving them are amenable to the same laws as for any two real numbers. In particular, we can factorise $M^2 - N^4$ and write

$$M^2 - N^4 = (M + N^2)(M - N^2) \tag{1}$$

Such a factorisation would be invalid if M and N did not commute with each other. We are given that the L.H.S. is the zero matrix. So, multiplying both the sides of (1) by M we get

$$0 = M(M^2 - N^4) = (M^2 + MN^2)(M - N^2) \tag{2}$$

We are given that $M - N^2 \neq 0$. So, if we take U as $M - N^2$, we get that (B) is correct. Since (D) is the logical negation of (B), it follows that (D) is false.

Let us now check (A). If it were not true, then the matrix $M^2 + MN^2$ would be invertible. But in that case multiplying both the sides of (2) by the inverse of $(M^2 + MN^2)$ we would get

$$\begin{aligned} 0 &= (M^2+MN^2)^{-1}0 = (M^2+MN^2)^{-1}(M^2+MN^2)(M-N^2) \\ &= I_3(M-N^2) = M - N^2 \end{aligned} \tag{3}$$

where I_3 is the 3×3 identity matrix. This is a contradiction. Therefore (A) is true, and automatically, (C) is false.

There is some duplication in the last problem and the present one since both are based on the characterisation of invertible matrices in terms of their determinants. But the present problem is not as trivial as the last one. Its solution is based on the factorisation (1), which requires M and N to commute with each other. Chances are that an unscrupulous candidate will write down (1) without bothering about this justification. A greater danger is that he might even cancel $M - N^2$ which is given to be non-zero and conclude, falsely, that $M + N^2 = 0$. The tragedy is that the statements (A) and (B) also follow quite efficiently from this false assertion. This problem is therefore a glaring example of how a multiple choice question rewards unscrupulous candidates.

Q.49 Let $f : [a, b] \longrightarrow [1, \infty)$ be a continuous function and let $g : I\!R \longrightarrow I\!R$ be defined by

$$g(x) = \begin{cases} 0 & \text{if} \quad x < a \\ \int_a^x f(t)dt & \text{if} \quad a \leq x \leq b \\ \int_a^b f(t)dt & \text{if} \quad b < x \end{cases}$$

Then,

(A) $g(x)$ is continuous but not differentiable at a

(B) $g(x)$ is differentiable on $I\!R$

(C) $g(x)$ is continuous but not differentiable at b

(D) $g(x)$ is continuous and differentiable at either a or b but not both

Answer and Comments: (A,C). Yet another question based on the second form of the fundamental theorem of calculus. What is crucially needed is that because of the continuity of the function $f(t)$, the function defined by $\int_a^x f(t)dt$ for $a \leq x \leq b$ is differentiable at all $x \in (a, b)$ with derivative $f(x)$. However, at the end points a and b all we can say is that it is right differentiable at a with $f(a)$ as the right handed derivative and, similarly, it is left differentiable at b with $f(b)$ as its left handed derivative.

In the present problem, g is identically 0 on $(-\infty, a)$ and also identically constant with value $\int_a^b f(t)dt$ on (b, ∞). As these values coincide with $g(a)$ and $g(b)$ respectively, we get that g is continuous everywhere. Only its differentiability at the points a and b needs to be checked. Note that because of constancy of g on $(-\infty, a]$, g is left differentiable at a with $g'_-(a) = 0$. But as pointed out above, $g'_+(a) = f(a)$. Since f takes values only in $[1, \infty)$, $f(a) \neq 0$. Hence g is not differentiable at a. By a similar reasoning, $g'_+(b) = 0$ while $g'_-(b) = f(b) \neq 0$. So g is not differentiable at b either.

A very simple problem. There was no point in asking it because the second form of the fundamental theorem of calculus was tested already in Q. 41.

Q.50 Let $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow I\!R$ be given by $f(x) = (\log(\sec x + \tan x))^3$. Then

(A) $f(x)$ is an odd function
(B) $f(x)$ is a one-to-one function
(C) $f(x)$ is an onto function
(D) $f(x)$ is an even function

Answer and Comments: (A, B, C). The given function $f(x)$ equals $(g(x))^3$ where

$$g(x) = \log(\sec x + \tan x) \tag{1}$$

for $x \in (-\pi/2, \pi/2)$. Another way of looking at the relationship between $f(x)$ and $g(x)$ is that $f = h \circ g$ where $h : I\!R \longrightarrow I\!R$ is the function $h(y) = y^3$. Note that h is a bijection. Hence f is one-to-one or onto according as g is so. Moreover h itself is an odd function. So, f is odd if and only if g is odd. These elementary observations reduce the problem of checking the various properties of f to that of the checking the corresponding properties of the function g. The latter is simpler and more direct. So we tackle $g(x)$ rather than $f(x)$.

By a direct calculation based on standard trigonometric identities, we have

$$\begin{aligned} g(-x) &= \log(\sec(-x) + \tan(-x)) \\ &= \log(\sec x - \tan x) \\ &= \log\left(\frac{\sec^2 x - \tan^2 x}{\sec x + \tan x}\right) \\ &= \log\left(\frac{1}{\sec x + \tan x}\right) \\ &= -\log(\sec x + \tan x) \\ &= -g(x) \end{aligned} \tag{2}$$

which proves that g and hence f is an odd function. It also follows that f cannot be an even function, since the only function which is both odd and even is the zero function and f is not the zero function. Thus we see that (A) holds and (D) fails.

Now, to check if $g(x)$ (and hence $f(x)$) is one-to-one, if we go by straight definition, we would have to take two points $x_1, x_2 \in (-\pi/2, \pi/2)$ for which $g(x_1) = g(x_2)$ and then show that this can happen only when $x_1 = x_2$. Again, since the logarithm function is one-to-one, to say $g(x_1) = g(x_2)$ is equivalent to saying that $\sec x_1 + \tan x_1 = \sec x_2 + \tan x_2$, or equivalently,

$$\frac{1 + \sin x_1}{\cos x_1} = \frac{1 + \sin x_2}{\cos x_2} \tag{3}$$

and hence to

$$\sin x_1 \cos x_2 - \cos x_1 \sin x_2 = \cos x_1 - \cos x_2 \tag{4}$$

Using trigonometric identities, this further becomes

$$2\sin(\frac{x_1 - x_2}{2})\cos(\frac{x_1 - x_2}{2}) = 2\sin(\frac{x_1 + x_2}{2})\sin(\frac{x_2 - x_1}{2}) \tag{5}$$

One possibility is that $\sin(\frac{x_1 - x_2}{2}) = 0$ which would imply $x_1 = x_2$. But ruling out the other possibility viz. $\sin(\frac{x_1 + x_2}{2}) = \cos(\frac{x_1 - x_2}{2})$ would require more work. So proving that $g(x)$ is one-to-one by this direct method is cumbersome. Proving that it is onto will be even more difficult, because that would amount to solving an equation of the form $\sec x + \tan x = \lambda$ for every positive λ. Such equations cannot be solved easily.

So we abandon this approach and instead turn to calculus. The function $g(x) = \log(\sec x + \tan x)$ is from $(-\pi/2, \pi/2)$ to $(-\infty, \infty)$. By a direct calculation,

$$g'(x) = \sec x \tag{6}$$

which is positive for all $x \in (-\pi/2, \pi/2)$. So, g is strictly increasing and hence one-to-one on $(-\pi/2, \pi/2)$. To see if it is onto, we find its range. Note that as $x \to \pi/2$ from the left, both $\sec x$ and $\tan x$ approach ∞. So $g(x) \to \infty$ as $x \to \pi/2$ from the left. We can similarly find $\lim g(x)$ as x tends to $-\pi/2$ from the right. But that is not necessary. As we have already proved that g is an odd function, the fact that $g(x) \to \infty$ as $x \to \pi/2$ from the left, implies that $g(x) \to -\infty$ as $x \to -\pi/2$ from the right. Thus we have shown that the range of g is $(-\infty, \infty)$, which means g is onto as well.

Summing up, we have shown that $g(x)$ is odd, one-to-one and onto. As already noted, $f(x)$ also has the corresponding properties.

The question is a good one but a bit lengthy for 3 points. Moreover, in a question like this, the ability to give the correct reasoning is quite crucial. So, this question would be ideal as a full length question in an examination.

SECTION - 2

Integer Value Correct Type

This section contains **ten** questions. The answer to each question is a single digit integer ranging from 0 to 9 (both inclusive). There are 4 points for a correct answer, no points if no answer and -1 point in all other cases.

Q.51 Let $n_1 < n_2 < n_3 < n_4 < n_5$ be positive integers such that $n_1 + n_2 + n_3 + n_4 + n_5 = 20$. Then the number of such distinct arrangements $(n_1, n_2, n_3, n_4, n_5)$ is ....

Answer and Comment: 7. The problem asks us to find the number of elements in the set, say, S of all ordered 5-tuples $(n_1, n_2, n_3, n_4, n_5)$ of positive integers in which the entries are strictly increasing and add to 20. One such 5-tuple is $(1, 2, 3, 4, 10)$. Another is $(1, 3, 4, 5, 7)$. But listing all such 5-tuples by hand is prone to omission. So we look for a better way.

There is a standard way to convert a strictly increasing sequence of positive integers to a monotonically increasing sequence of positive integers. We leave the first term as it is, subtract 1 from the second term, 2 from the third, 3 from the fourth and so on. Specifically, we convert a 5-tuple $(n_1, n_2, n_3, n_4, n_5)$ of the desired type to $(m_1, m_2, m_3, m_4, m_5)$ where

$$m_i = n_i - (i - 1) \tag{1}$$

for $i = 1, 2, 3, 4, 5$. Note that $m_1 + m_2 + m_3 + m_4 + m_5 = 20 - (1 + 2 + 3 + 4) = 10$. Conversely, every 5-tuple of the form $(m_1, m_2, m_3, m_4, m_5)$ where the entries are positive integers in an ascending (not necessarily strictly ascending) order and add up to 10, can be converted to a 5-tuple $(n_1, n_2, n_3, n_4, n_5)$ we are looking for by setting

$$n_i = m_i + (i - 1) \tag{2}$$

for $i = 1, 2, 3, 4, 5$.

This correspondence is clearly bijective. Thus we have reduced the original problem to the problem of counting all increasing 5-tuples $(m_1, m_2, m_3, m_4, m_5)$ of positive integers which add up to 10. This is more manageable because 10 is smaller than 20. We can even go further. Let us substract 1 from each of the m_i's to get an ordered 5-tuple $(u_1, u_2, u_3, u_4, u_5)$ where

$$u_i = m_i - 1 \tag{3}$$

for $i = 1, 2, 3, 4, 5$. Note that u_1, u_2, u_3, u_4, u_5 are non-negative integers in an ascending order which add up only to 5. Once again the correspondence between $(m_1, m_2, m_3, m_4, m_5)$ and $(u_1, u_2, u_3, u_4, u_5)$ is reversible.

Thus ultimately, the problem is reduced to finding all possible 5-tuples $(u_1, u_2, u_3, u_4, u_5)$ of non-negative integers in which $u_1 \leq u_2 \leq u_3 \leq$

$u_4 \leq u_5$ and $u_1 + u_2 + u_3 + u_4 + u_5 = 5$. Since 5 is a very small number, the problem can now be done by trial. The largest entry, u_5 can be at most 5 and when it is so all other entries are 0. When $u_5 = 4$, the only possibility is $u_4 = 1$ and $u_1 = u_2 = u_3 = 0$. Reasoning like this, we can list all possible 5-tuples $(u_1, u_2, u_3, u_4, u_5)$. In fact, we need to list only the positive entries. Thus arranging the 5-tuples staring with $u_5 = 5$ and then taking the decreasing values of u_5, the possibilities are : $(5), (1, 4), (2, 3), (1, 1, 3), (1, 2, 2), (1, 1, 1, 2)$ and finally $(1, 1, 1, 1, 1)$. Thus there are 7 possible 5-tuples of the type we are looking for.

The problem is an excellent example of how a counting problem can be reduced by a suitable bijective transformation to another, simpler counting problem. If instead of listing the u_i's in an ascending order, we had taken the descending order, then (ignoring the zero entries), the problem amounts to writing 5 as a sum of positive integers in a descending order. Every such expression is called a **partition** of 5. More generally, for any positive integer n, a partition of n means writing n as a sum of positive integers in descending order. The number of all possible partitions of n is generally denoted by $p(n)$. Thus the present problem really amounts to finding $p(5)$ and the answer is 7.

The partition function $p(n)$ has been extensively studied because the number of partitions of an integer n props up in many counting problems in many different branches of mathematics. There is no formula for expressing $p(n)$ directly in terms of n. Sometimes, instead of considering all partitions of an integer n, one considers partitions with some restrictions, for example, only those partitions in which the part sizes are all odd. Many interesting identities are known which relate the number of partitions of various types. For example one can show that the number of partitions of a positive integer n in exactly m parts is the same as the number of partitions of n in which the largest part size is m. A few non-trivial identities about partitions are due to the great Indian mathematician **Ramanujan**.

Q.52 Let $n > 2$ be an integer. Take n distinct points on a circle and join each pair of points by a line segment. Colour the line segment joining every pair of adjacent points by blue and the rest by red. If the red and blue line segments are equal, then the value of n is ..... .

Answer and Comments: 5. The paper-setters have tried to make the problem look colourful. But that was quite unnecessary. In essence, all that the problem involves is to count the number of adjacent pairs and the number of non-adjacent pairs of vertices of an n-gon inscribed in a circle. Obviously the former is n. By complementary counting, the latter is $\binom{n}{2} - n$ because the total number of all pairs (adjacent as well as non-adjacent) of points is clearly $\binom{n}{2}$. Hence the problem amounts to solving the equation

$$n = \binom{n}{2} - n \tag{1}$$

which reduces to

$$4n = n(n-1) \tag{2}$$

As $n \neq 0$ the only solution is $n = 5$.

It is shocking that such an utterly trivial problem is asked in JEE Advanced. There are indeed some good problems which involve the colouring of the edges and diagonals of an n-gon by red and blue colours. See, for example, the problem on p. 218 which is the forerunner of Ramsey theory. The present problem comes nowhere close. In fact, as already stated, the colouring of the segments is quite superficial to the problem.

Q.53 Let $f : I\!\!R \longrightarrow I\!\!R$ and $g : I\!\!R \rightarrow I\!\!R$ be respectively given by $f(x) = |x|+1$ and $g(x) = x^2 + 1$. Define $h : I\!\!R \longrightarrow I\!\!R$ by

$$h(x) = \begin{cases} \max\{f(x), g(x)\} & \text{if} \quad x \leq 0 \\ \min\{f(x), g(x)\} & \text{if} \quad x > 0 \end{cases}$$

The number of points at which $h(x)$ is not differentiable is ..... .

Answer and Comments: 3. As both f and g are continuous, and the maximum as well as the minimum of two continuous functions is continuous, there is no difficulty about the continuity of $h(x)$ anywhere. Note that f is differentiable everywhere except at $x = 0$ while $g(x)$ is differentiable everywhere. To test the differentiability of $h(x)$, we need to split its domain $I\!\!R$ into subintervals over each one of which $h(x)$

equals either $f(x)$ or $g(x)$. The end-points of these subintervals, along with 0 are the only possible points of non-differentiability.

For $x \leq 0$, $f(x) = -x + 1$ and $g(x) = x^2 + 1$. The two are equal only when $x = 0$ or -1. For $x > 0, f(x) = x + 1$ and $g(x) = x^2 + 1$ are equal only at $x = 1$. So, we partition $I\!R$ at these three nodal points $-1, 0$ and 1 and study the behaviour of h on each of the subintervals $(-\infty, -1], [-1, 0], [0, 1]$ and $[1, \infty)$. Note that $f(x) \leq g(x)$ on the first and the last of these four subintervals while $f(x) \geq g(x)$ on the two middle ones.

Keeping this in mind, we get the following description of $h(x)$.

$$h(x) = \begin{cases} g(x) = x^2 + 1, & x \in (-\infty, -1] \\ f(x) = -x + 1, & x \in [-1, 0] \\ g(x) = x^2 + 1, & x \in [0, 1] \\ f(x) = x + 1, & x \in [1, \infty) \end{cases} \tag{1}$$

we now calculate the right and left handed derivatives of h at the nodal points $-1, 0$ and 1.

$$h'_-(-1) = g'_-(-1) = -2 \tag{2}$$

$$h'_+(-1) = f'_+(-1) = -1 \tag{3}$$

$$h'_-(0) = f'_-(0) = -1 \tag{4}$$

$$h'_+(0) = g'_+(0) = 0 \tag{5}$$

$$h'_-(1) = g'_-(1) = 2 \tag{6}$$

$$\text{and, finally} \quad h'_+(1) = f'_+(1) = 1 \tag{7}$$

As the left and right handed derivatives do not match with each other at any of the three points $-1, 0$ and 1, we get that h is not differentiable at any of these three points. All other points are interior points of these subintervals and over them h coincides with either f or g both of which are differentiable. So, h is differentiable everywhere else.

A simple, straightforward problem. A careless candidate often thinks that when a function has a different formula on the two sides of a point, then it is non-differentiable at that point. In the present problem such a careless student will get the correct answer without bothering to calculate the right and the left handed derivatives at these points. To penalise such a candidate, it would have been better if the data was so designed that h would be differentiable at some of the nodal points.

Q.54 Let a, b, c be positive integers such that $\frac{b}{a}$ is an integer. If a, b, c are in geometric progression and the arithmetic mean of a, b, c ia $b + 2$, then the value of $\frac{a^2 + a - 14}{a + 1}$ is ..... .

Answer and Comments: 4. Let $b = ar$. Then $c = ar^2$ since a, b, c are in G.P. As we are given that their A.M. is $b + 2$ we get

$$a\frac{1 + r + r^2}{3} = ar + 2 \tag{1}$$

Here a and r are unknowns and a single equation in them is insufficient in general to determine them uniquely. But here we are given further that a and r are positive integers and this considerably restricts the number of solutions. Indeed, we can recast (1) as

$$a(r - 1)^2 = 6 \tag{2}$$

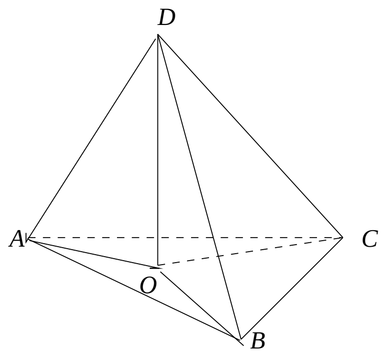
The only perfect square which divides 6 is 1. So $r - 1 = \pm 1$ which gives $r = 2$ since $r = 0$ is untenable. If $r = 2$ then $a = 6$. By direct substitution, the value of $\frac{a^2 + a - 14}{a + 1}$ is 28/7, i.e. 4.

A simple, good problem. Although the number theory part is very elementary, it brings a refreshing breeze to the otherwise drab world of progressions.

Q.55 Let $\vec{a}, \vec{b}, \vec{c}$ be three non-coplanar unit vectors such that the angle between every pair of them is $\frac{\pi}{3}$. If $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} = p\vec{a} + q\vec{b} + r\vec{c}$, where p, q, r are scalars, then the value of $\frac{p^2 + 2q^2 + r^2}{q^2}$ is ..... .

Answer and Comments: 4. The information that $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar is redundant because in a plane it is simply impossible to accommodate three unit vectors every two of which are inclined to each other at 60°. We can, however, have three unit vectors, say the position vectors of the complex numbers $1, \omega$ and ω^2 every two of which are inclined at an angle $2\pi/3$. Denote their terminal points by A, B, C and their common initial point by O, the origin. Now suppose we raise the initial point O vertically to a point D at a height h above O. Then the terminal points A, B, C will move closer to the origin and will lie on a circle of radius $\sqrt{1-h^2}$ centred at O. The angle, say θ, between any two of these unit vectors $\overrightarrow{DA}, \overrightarrow{DB}$ and $\overrightarrow{DC}$ will decrease as h increases. If this angle is $\pi/3$, then the triangles DAB, DBC and DCA are all equilateral ones with each side 1.

So, we can take the vectors $\vec{a}, \vec{b}, \vec{c}$ to be respectively $\overrightarrow{DA}, \overrightarrow{DB}$ and $\overrightarrow{DC}$ where $DABC$ is a regular tetrahedron with each side 1 as shown in the diagram below.



We are given that

$$\vec{a} \times \vec{b} + \vec{b} \times \vec{c} = p\vec{a} + q\vec{b} + r\vec{c} \tag{1}$$

If we could determine p, q, r explicitly, then we would be in a position to find the value of the expression $\dfrac{p^2 + 2q^2 + r^2}{q^2}$ by direct substitution. Unfortunately, since the vectors $\vec{a}, \vec{b}, \vec{c}$ do not form an orthonormal system, resolving along them will not be so easy. Doing it geometrically from the diagram above will also involve considerable trigonometric calculations. So we abandon this approach and instead work directly in terms of the unknown coefficients p, q, r by getting a system of equations in them. We note that even though $\vec{a}, \vec{b}, \vec{c}$ do not form an orthonormal system, they are all unit vectors and because of the data, the dot product of any two (distinct) vectors out of these three vectors is $\cos(\pi/3)$, i.e. $1/2$. This suggests that we can take the dot product of both the sides of (1) with each of the three vectors $\vec{a}, \vec{b}$ and $\vec{c}$, to get a system of three equations in the three unknowns p, q, r.

Taking the dot product with $\vec{a}$, we get

$$[\vec{a}\ \vec{a}\ \vec{b}] + [\vec{a}\ \vec{b}\ \vec{c}] = p(\vec{a}\cdot\vec{a}) + q(\vec{a}\cdot\vec{b}) + r(\vec{a}\cdot\vec{c}) \tag{2}$$

where the bracketed expressions on the L.H.S. are the scalar triple products. The first scalar triple product vanishes since two of its vectors are identical. The second one equals the volume of the box or the parallelepiped with sides DA, DB, DC. We can calculate it right now from the figure above using trigonometry. But since the answer to the problem depends only on the relative proportions of p, q, r, rather than their actual values, we may not need the value of the box product $[\vec{a}\ \vec{b}\ \vec{c}]$. So we leave it as it is for the moment. We shall calculate it later if the need arises.

The R.H.S. of (2) poses no problems since we know the values of all the dot products occurring in it. So, we get

$$p + \frac{q}{2} + \frac{r}{2} = [\vec{a}\ \vec{b}\ \vec{c}] \tag{3}$$

as our first equation in p, q and r.

The other two equations are obtained similarly by taking the dot products of both the sides of (1) with $\vec{b}$ and $\vec{c}$ respectively. As the justifications are similar, we write them down without much elaboration. They are

$$\frac{p}{2} + q + \frac{r}{2} = [\vec{b}\ \vec{a}\ \vec{b}] + [\vec{b}\ \vec{b}\ \vec{c}] = 0 \tag{4}$$

$$\text{and}\quad \frac{p}{2} + \frac{q}{2} + r = [\vec{c}\ \vec{a}\ \vec{b}] + [\vec{c}\ \vec{b}\ \vec{c}] = [\vec{a}\ \vec{b}\ \vec{c}] \tag{5}$$

Because of (4), we have

$$q = -\frac{1}{2}(p + r) \tag{6}$$

while if we subtract (5) from (3), we get

$$p = r \tag{7}$$

without calculating the value of $[\vec{a}\,\vec{b}\,\vec{c}]$ which simply cancels out. So the gamble of putting off its calculation has paid. The last two equations give

$$p = r = -q \tag{8}$$

and this is sufficient to calculate $\dfrac{p^2 + 2q^2 + r^2}{q^2}$ as 4.

This is an excellent problem which tests the ability of realising how *not* to do a problem. The geometric approach is tempting but a candidate who pursues it will pay a heavy price in terms of time. Also, the calculation of the box product $[\vec{a}\,\vec{b}\,\vec{c}]$ is not needed and realising this early enough saves precious time. Incidentally, we mention that if we do want the value of the box product $[\vec{a}\,\vec{b}\,\vec{c}]$, it is six times the volume of the regular tetrahedron pictured above. As all the sides are equal to 1, the area, say Δ, of the base ABC is $\frac{1}{2}\sin 60^\circ$, i.e. $\dfrac{\sqrt{3}}{4}$ while its height, say h, is $DO = \sqrt{(DA)^2 - (OA)^2}$ which comes out to be $\dfrac{\sqrt{2}}{\sqrt{3}}$. Therefore the volume of the tetrahedron is $\dfrac{1}{3}\Delta h = \dfrac{1}{3} \times \dfrac{\sqrt{3}}{4} \times \dfrac{\sqrt{2}}{\sqrt{3}} = \dfrac{1}{6\sqrt{2}}$. Hence, finally, the box product $[\vec{a}\,\vec{b}\,\vec{c}]$ equals $\dfrac{1}{\sqrt{2}}$.

Q. 56 The slope of the tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point $(1, 3)$ is ....

Answer and Comments: 8. This is obviously a problem on implicit differentiation. Differentiating both the sides w.r.t. x gives

$$2(y - x^5)(\frac{dy}{dx} - 5x^4) = (1 + x^2)^2 + 4x^2(1 + x^2) \tag{1}$$

solving which we get

$$\frac{dy}{dx} = 5x^4 + \frac{(1 + x^2)(1 + 5x^2)}{2(y - x^5)} \tag{2}$$

At the point $(1, 3)$, we have

$$\frac{dy}{dx} \;=\; 5 + \frac{12}{2(3 - 1)} = 5 + 3 = 8 \tag{3}$$

There can hardly be a more straightforward problem. Just compare this one with the last one. There is simply no comparison, either conceptually or in terms of the calculations involved. While it is obviously impossible to design all questions of equal levels of difficulty, one certainly does not expect such a glaring difference in the levels of difficulty. To make the problem suitable for an advanced selection, the data could have been changed in such a way that the equation resulting after differentiation cannot be solved so easily as in the present problem.

Q.57 The value of $\int_0^1 4x^3 \left\{ \frac{d^2}{dx^2}(1-x^2)^5 \right\}$ is ....

Answer and Comments: 2. It is tempting to evaluate the expression in the curly brackets. That would, indeed, be the most straightforward way of starting the solution. But as differentiation is to be done twice the resulting expression would be fairly complicated. So we try another approach. The integrand is given to be the product of two functions, the second one of which is the derivative (actually the second derivative) of something. But the second derivative is simply the first derivative of the first derivative. This is exactly the setting we need to apply the rule of integration by parts. So, if we call the given integral as I, then as a starter we get,

$$I = \left[4x^3 \frac{d}{dx}(1-x^2)^5\right]\Big|_0^1 - \int_0^1 12x^2 \frac{d}{dx}(1-x^2)^5 dx \qquad (1)$$

Since $\frac{d}{dx}(1-x^2)^5$ is divisible by $(1-x^2)$ (actually, even by $(1-x^2)^4$), it vanishes at both the end points 0 and 1. So the first term on the R.H.S. iz 0. Call the integral in the second term as J. Another (and a more straightforward) application of integration by parts gives

$$\begin{aligned} J &= \int_0^1 12x^2 \frac{d}{dx}(1-x^2)^5 dx \\ &= \left[12x^2(1-x^2)^5\right]\Big|_0^1 - \int_0^1 24x(1-x^2)^5 dx \qquad (2) \end{aligned}$$

Once again, the first term vanishes. The last integral can be evaluated using the substitution $u = 1-x^2$. Thus, from (1) and (2)

$$I = \int_0^1 24x(1-x^2)^5 dx = \int_0^1 12u^5 du = 2u^6\Big|_0^1 = 2 \qquad (3)$$

Q.58 The largest value of the non-negative integer a for which $\lim_{x\to 1}\left(\frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right)^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4}$ is ..... .

Answer and Comments: 0. The function, say $f(x)$, whose limit is considered is an exponential function of the form

$$f(x) = u(x)^{v(x)} \tag{1}$$

where

$$u(x) = \frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1} \tag{2}$$

$$\text{and} \quad v(x) = \frac{1-x}{1-\sqrt{x}} \tag{3}$$

In a small deleted neighbourhood of 1, the exponent $v(x)$ can be rewritten as $1+\sqrt{x}$ and hence tends to the limit 2 as $x \to 1$. So, if L is the limit of the base $u(x)$ as $x \to 1$, then $\lim_{x\to 1} f(x)$ would be L^2. So, let us first evaluate $L = \lim_{x\to 1} \frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}$. If we make the substitution $t = x-1$, then L takes a somewhat simpler form, viz.

$$L = \lim_{t\to 0} \frac{\sin t - at}{\sin t + t} \tag{4}$$

To evaluate this limit we divide both the numerator and the denominator by u and use the fact that $\lim_{t\to 0} \frac{\sin t}{t} = 1$. Thus,

$$\begin{aligned} L &= \lim_{t\to 0} \frac{\sin t - at}{\sin t + t} \\ &= \lim_{t\to 0} \frac{\frac{\sin t}{t} - a}{\frac{\sin t}{t} + 1} \\ &= \frac{\lim_{t\to 0} \frac{\sin t}{t} - a}{\lim_{t\to 0} \frac{\sin t}{t} + 1} \\ &= \frac{1-a}{2} \end{aligned} \tag{5}$$

Hence we finally have

$$\lim_{x\to 1} f(x) = L^2 = \left(\frac{1-a}{2}\right)^2 \tag{6}$$

If this is to equal 1/4, then we must have $\frac{1-a}{2} = \pm\frac{1}{2}$. So a is either 0 or 2. As we want the largest value of a, the answer would be 2. But there is a subtle catch. The expression whose limit is taken is a power. The exponent, viz. $v(x) = \frac{1-x}{1-\sqrt{x}}$ is *not* an integer for values of x near 1. In such a case, the power would make sense only if the base $u(x)$ is positive. For $a = 2$, the expression $\frac{\sin t - 2t}{\sin t + t}$ is negative for small positive values of t. (This follows from the inequality that for such values of t, $\sin t < t$.) Hence the possibility $a = 2$ has to be discarded. So the correct answer is 0.

Unlike in Q.46, where the elimination of one of the possibilities was somewhat controversial, in the present problem it requires a careful consideration of the conditions under which powers make sense. This problem is therefore a very good test of fine thinking. In our solution, when we calculated the limit of $u(x)^{v(x)}$ as $(\lim u(x))^{\lim v(x)}$, we implicitly used the joint continuity of the function x^y as a function of two variables. But those who feel uncomfortable because functions of several variables are not in the JEE syllabus, may instead consider $g(x) = \log(f(x)) = v(x)\log u(x)$ and then calculate $\lim\limits_{x\to 1} \log u(x)$. The work involved is exactly the same and so is the catch too because for the log to make sense the expression must be positive. There is actually hardly any difference between the two approaches, because when the exponent b is not an integer, the very definition of the power u^b is $e^{b \log u}$. So, whichever approach is taken, logarithms are inevitable.

Q.59 Let $f : [0, 4\pi] \longrightarrow [0, \pi]$ be defined by $f(x) = \cos^{-1}(\cos x)$. The number of points $x \in [0, 4\pi]$ satisfying the equation $f(x) = \frac{10-x}{10}$ is .... .

Answer and Comments: 3. Call $\cos^{-1}(\cos x)$ as y. Then we have

$$\cos y = \cos x \tag{1}$$

which means

$$x = 2n\pi \pm y \tag{2}$$

for some integer n. Note that y always lies in $[0, \pi]$. But x varies over $[0, 4\pi]$. So there is a unique value of x in each of the four intervals

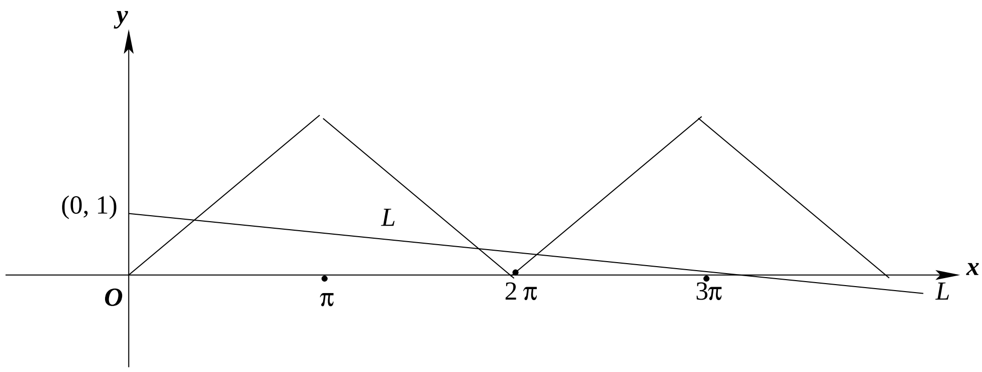
$[0, \pi], [\pi, 2\pi], [2\pi, 3\pi]$ and $[3\pi, 4\pi]$ which satisfies (1). This gives the following expression for $f(x)$.

$$f(x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi \\ 2\pi + x, & 2\pi \le x \le 3\pi \\ 4\pi - x, & 3\pi \le x \le 4\pi \end{cases} \tag{3}$$

We now have to see if the equation

$$f(x) = \frac{10 - x}{10} \tag{4}$$

has solutions in each of these four intervals. Note that both the sides of the equation are linear over each of the four intervals. So their graphs are straight line segments. So, if at all they intersect, they will intersect only in one point. Our interest is not so much in these points of intersection *per se* but merely in how many of them there are. So it is preferable to do the problem graphically. The graph of $y = f(x)$ consists of straight line segments while that of $y = \dfrac{10 - x}{10}$ consists of a single straight line, say L. Both are shown in the diagram below.



It is abundantly clear from the graph that the line L meets the first three segments, but not the fourth one. So there are three points of intersection and hence three solutions to the given equation.

A simple problem on elementary properties of inverse trigonometric functions. What makes it really nice is the simplification brought about by working in terms of the graphs rather than analytically.

Q.60 For a point P in the plane, let $d_1(P)$ and $d_2(P)$ be the distances of P from the lines $x - y = 0$ and $x + y = 0$ respectively. The area of the region R consisting of all points P lying in the first quadrant of the plane and satisfying $2 \leq d_1(P) + d_2(P) \leq 4$, is ....

Answer and Comments: 6. The first task is obviously to sketch the region R. Suppose $P = (a, b)$ where a, b are both non-negative. Then we have

$$d_1(P) = \frac{|a-b|}{\sqrt{2}} \tag{1}$$

$$\text{and} \quad d_2(P) = \frac{|a+b|}{\sqrt{2}} \tag{2}$$

As a, b are both non-negative we have

$$d_2(P) = \frac{a+b}{\sqrt{2}} \tag{3}$$

for all P. However, for $d_1(P)$ we have to make a distinction depending upon whether P lies below the line $L : y = x$ or above it. In the former case, $a \geq b$ while in the latter $a \leq b$. Let us first assume that P lies below L. Then

$$d_1(P) = \frac{a-b}{\sqrt{2}} \tag{4}$$

and so the inequality

$$2 \leq d_1(P) + d_2(P) \leq 4 \tag{5}$$

reduces to

$$2 \leq \frac{2a}{\sqrt{2}} \leq 4 \tag{6}$$

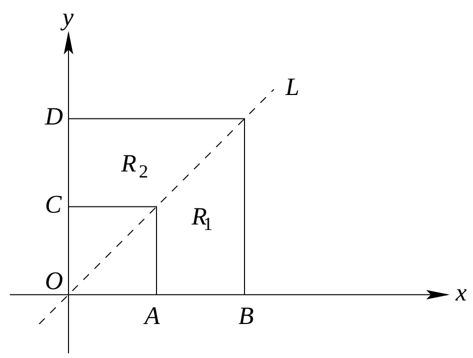
or equivalently, to

$$\sqrt{2} \leq a \leq 2\sqrt{2} \tag{7}$$

By a similar reasoning (or by symmetry), for a point P lying above the line L, the inequality (5) becomes

$$\sqrt{2} \leq b \leq 2\sqrt{2} \tag{8}$$

It is now easy to sketch R. The portion, say R_1, of R below the line L is a vertical trapezium between the lines $x = \sqrt{2}$ and $x = 2\sqrt{2}$, bounded above by the line L and below by the x-axis. Similarly, the portion, say R_2 of R lying above L is a horizontal trapezium between the lines $y = \sqrt{2}$ and $y = 2\sqrt{2}$, bounded on the left by the y-axis and on the right by he line L. It is shown in the figure below where the points A, B, C, D are $(\sqrt{2}, 0), (2\sqrt{2}, 0), (0, \sqrt{2})$ and $(0, 2\sqrt{2})$ respectively.



To find the area of R, we could find the areas of the two trapezia R_1 and R_2 and add. But a much better way is to notice that R is simply the difference between two squares, one with sides OB and OD and the other with sides OA and OC. Therefore the area of R is $(2\sqrt{2})^2 - (\sqrt{2})^2 = 8 - 2 = 6$ square units.

A very good problem involving only elementary ideas from coordinate geometry.

PAPER - 2

Contents

SECTION -1

Only One Options Correct Type

This section contains **ten** multiple choice questions. Each question has 4 choices out of which **ONLY ONE** is correct. A correct answer gets 3 points. No points if the question is not answered. -1 point in all other cases.

Q.41 Three boys and two girls stand in a queue. The probability that the number of boys ahead of every girl is at least one more than the number of girls ahead of her is

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{3}{4}$

Answer and Comments: (A). Here the sample space, say S, consists of all sequences of the form $(x_1, x_2, x_3, x_4, x_5)$ where each x_i is either B (for a boy) or G (for a girl) and exactly three of the terms are B and the remaining 2 are G. Clearly,

$$|S| = \binom{5}{2} = 10 \tag{1}$$

Now let F be the set of all favourable sequences, i.e. sequences in which every G is preceded by at least one more B's than G's. Then the desired probability, say p is given by

$$p = \frac{|F|}{|S|} = \frac{|F|}{10} \tag{2}$$

A clever student can now bypass the calculation of $|F|$. Out of the four given alternatives, the first one is the only one where the denominator is a divisor of 10. So it is the correct one! The examiners should have given at least one fake alternative (e.g. $\frac{2}{5}$ or $\frac{3}{10}$) to preclude this kind of a sneaky answer.

But if we do not want to take an unfair advantage of the carelessness on the part of the paper-setters, then here is a honest calculation of $|F|$. Clearly the first place x_1 must be a B. The remaining four places are to be filled with 2 B's and two G's in such a way that for every G, the number of B's before it (not counting the B at x_1) is at least the number of G's before it. As 4 is a small number, this is best done by hand. There are $\binom{4}{2}$ i.e. 6 ways to arrange two B's and two G's in a row and out of these the only forbidden arrangement is (G, G, B, B). All the remaining five arrangements are favourable. So $|F| = 5$ and hence the desired probability is $\frac{|F|}{|S|} = \frac{5}{10} = \frac{1}{2}$.

A very straightforward and simple problem. It could have been made a little more challenging by giving 4 boys and 3 girls, and very challenging by giving n boys and $n-1$ girls for some positive integer n. (In the present problem, $n = 3$.) In this general form the problem can be looked slightly differently. The sample space S now has $\binom{2n-1}{n}$ elements. But calculating $|F|$ is tricky. To do it, add one more girl so that now there are n boys and n girls. The requirement that every girl must be preceded by more boys than girls remains the same. Now think of a boy as a left parenthesis and a girl as a right parenthesis. Then a favourable arrangement is precisely a balanced arrangement of n left and n right parentheses. It can be shown (see p. 18) that the number of such balanced arrangements is $\frac{1}{n+1}\binom{2n}{n-1}$, i.e. $\frac{(2n)!}{n!(n+1)!}$. Hence when there are n boys and $n-1$ girls, the desired probability, say p_n is given by

$$p_n = \frac{|F|}{|S|} = \frac{\frac{(2n)!}{n!(n+1)!}}{\frac{(2n-1)!}{n!(n-1)!}} \tag{3}$$

which comes out to be $\frac{2}{n+1}$ upon simplification. In our problem,

$n = 3$ and so $p_3 = \frac{2}{4} = \frac{1}{2}$ which tallies with the answer we got earlier.

There is a surprisingly large number of combinatorial problems which can be reduced to the problem of counting the number of balanced arrangements of n left and n right parentheses. There is, therefore, a name given to their number. It is called the n-th **Catalan number**. As mentioned above, its value is $\frac{(2n)!}{n!(n+1)!}$. The proof is a remarkable combination of the techniques of transformations and complementary counting.

Q.42 In a triangle the sum of two sides is x and the product of the same two sides is y. If $x^2 - c^2 = y$, where c is the third side of the triangle, then the ratio of the in-radius to the circum-radius of the triangle is

(A) $\frac{3y}{2x(x+c)}$ (B) $\frac{3y}{2c(x+c)}$ (C) $\frac{3y}{4x(x+c)}$ (D) $\frac{3y}{4c(x+c)}$

Answer and Comments: (B). We denote the triangle by ABC and follow the standard notations for its sides a, b, c, in-radius r and circum-radius R. We are asked to find the ratio $\frac{r}{R}$. There are numerous formulas for both r and R and the choice of the right formulas is often the key to the solution. In the present case, a clue is provided by the fact that the ratio $\frac{y}{(x+c)}$ is present in all alternatives. In terms of the sides of the triangle, this ratio is $\frac{ab}{(a+b+c)}$ Both the numerator and the denominator can be related to the area, say Δ, of the triangle by the relations

$$\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}y\sin C \tag{1}$$

and

$$\Delta = \frac{1}{2}r(a+b+c) = \frac{1}{2}r(x+c) \tag{2}$$

From (1) and (2), we have

$$r = \frac{y\sin C}{(x+c)} \tag{3}$$

and hence

$$\frac{r}{c} = \frac{y \sin C}{c(x+c)} \tag{4}$$

Yet another identity relates the side c to the circum-radius R by

$$c = 2R \sin C \tag{5}$$

Substituting (5) into (4), we get

$$\frac{r}{R} = \frac{2y \sin^2 C}{c(x+c)} \tag{6}$$

The derivation so far is valid for any triangle. We have not yet used the special condition given to us, viz. $x^2 - c^2 = y$. In terms of the sides of the triangle, this translates as

$$(a+b)^2 - c^2 = ab \tag{7}$$

which can be rewritten as

$$\frac{a^2 + b^2 - c^2}{2ab} = -\frac{1}{2} \tag{8}$$

The L.H.S. is simply $\cos C$. So we get $\cos C = -\frac{1}{2}$ Therefore

$$\sin^2 C = 1 - \cos^2 C = 1 - \frac{1}{4} = \frac{3}{4} \tag{9}$$

If we substitute this into (6), we get

$$\frac{r}{R} = \frac{3y}{2c(x+c)} \tag{10}$$

This is a very well-designed problem about solution of triangles. Although there are numerous formulas for r and R and hence also for their ratio, (6) is not a standard formula. By making the ratio $\frac{y}{(x+c)}$ appear as a part of all alternate answers, the paper-setters have hinted at this formula implicitly. Actually the ratio $\frac{y}{x+c}$ is simply $x - c$. A good candidate will be led to think that there must be some purpose in giving this ratio instead of simply $x-c$ which is more logical. And once he sees this purpose, the solution unfolds itself. But the proportionate time of only 3 minutes is far too inadequate for the problem.

Q.43 Six cards and six envelopes are numbered 1, 2, 3, 4, 5, 6 and the cards are to be placed into the envelopes so that each envelope contains exactly one card and no card is placed in the envelope bearing the same number and moreover, the card numbered 1 is always placed in the envelope numbered 2. Then the number of ways it can be done is

(A) 264 (B) 265 (C) 53 (D) 67

Answer and Comments: (C). A very similar problem where we had to count the number of ways to put four balls of different colours into four boxes of the same colours so that no box is empty and no ball goes to the box of its own colour, was asked in JEE 1992. See Exercise (1.46). The answer was D_4, the number of derangements of 4 symbols, where a derangement means a permutation without any fixed points. Exercise (1.45) gives a formula for D_n, viz.

$$D_n = \frac{n!}{2!} - \frac{n!}{3!} + \frac{n!}{4!} - \frac{n!}{5!} + \ldots + \frac{(-1)^n n!}{n!} \tag{1}$$

so that $D_4 = 12 - 4 + 1 = 9$. The reasoning for (1) is based on what is called the **principle of inclusion and exclusion**, given in Exercise (1.43). The essential idea is that if $A_1, A_2, \ldots, A_n$ are subsets of a finite set S, and A_i' denotes the complement of A_i in S, then

$$\left|\bigcap_{i=1}^{n} A_i'\right| = s_0 - s_1 + s_2 - s_3 + \ldots + (-1)^n s_n = \sum_{r=0}^{n} (-1)^r s_r \tag{2}$$

where

$$\begin{aligned}
s_0 &= |S| \\
s_1 &= |A_1| + |A_2| + \ldots + |A_n| = \sum_{1 \leq i \leq n} |A_i| \\
s_2 &= \sum_{1 \leq i < j \leq n} |A_i \cap A_j| \\
s_3 &= \sum_{1 \leq i_1 < i_2 < i_3 \leq n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| \\
&\vdots \\
s_n &= |A_1 \cap A_2 \cap \ldots \cap A_n|
\end{aligned}$$

This can be proved by induction on n. To derive (1) from (2), take S to be the set of all permutations of n symbols and A_i to be the set of those permutations which have i as a fixed point (and possibly others too) for $i = 1, 2, 3, \ldots, n$.

In the present problem, if the restriction that the card numbered 1 goes to the envelope numbered 2 is removed (without any other change), then the answer would be

$$D_6 = 360 - 120 + 30 - 6 + 1 = 265 \tag{3}$$

which would be obtained by applying (2) to the set of all permutations of the set $\{1, 2, 3, 4, 5, 6\}$. Apparently, the alternate answer (B) is given with this in mind.

But this is not the correct answer for us because in our problem the set S is not the set of all permutations of $\{1, 2, 3, 4, 5, 6\}$ (which has $6! = 720$ elements) but only the subset of it consisting of those permutations $f : \{1, 2, 3, 4, 5, 6\} \longrightarrow \{1, 2, 3, 4, 5, 6\}$ for which $f(1) = 2$. (Here we are thinking of a permutation of a set as a bijection from that set to itself.) In symbols,

$$S = \{f : \{1, 2, 3, 4, 5, 6\} \longrightarrow \{1, 2, 3, 4, 5, 6\} | f \text{ a bijection}, f(1) = 2\} \tag{4}$$

Evidently, we have

$$|S| = 5! = 120 \tag{5}$$

because every such restricted permutation is uniquely determined by a bijection from the set $\{2, 3, 4, 5, 6\}$ to the set $\{1, 3, 4, 5, 6\}$. Our interest is in finding $|T|$ where

$$T = \{f \in S : f(i) \neq i, \text{ for } i = 2, 3, 4, 5, 6\} \tag{6}$$

Note that the requirement $f(2) \neq 2$ is automatically satisfied because $f(1) = 2$ forces $f(2) =\neq 2$ since f is one-to-one. So we apply the principle of inclusion and exclusion with only four subsets, A_3, A_4, A_5 and A_6 where $A_i = \{f \in S : f(i) = i\}$ for $i = 3, 4, 5, 6$.

Clearly $|A_3| = 4! = 24$ because a permutation in A_3 takes 1 to 2, 3 to 3 and maps the remaining symbols 2, 4, 5 and 6 bijectively to the symbols 1, 4, 5 and 6. Similarly $|A_4| = |A_5| = |A_6| = 24$. Thus, with the notations above

$$s_1 = |A_3| + |A_4| + |A_5| + |A_6| = 96 \tag{7}$$

Next, we consider intersections of the A's taken two at a time. For $3 \leq i < j \leq 6$, we have $|A_i \cap A_j| = 3! = 6$ since every f which takes 1 to 2, i to i and j to j maps the remaining three symbols to the remaining 3 symbols. As this holds for every pair $\{i, j\}$, we have

$$s_2 = \binom{4}{2} 3! = 6 \times 6 = 36 \tag{8}$$

Similarly, $|A_{i_1} \cap A_{i_2} \cap A_{i_3}| = 2! = 2$ for every $3 \leq i_1 < i_2 < i_3 \leq 6$. Hence

$$s_3 = \binom{4}{3} 2! = 8 \tag{9}$$

Finally,

$$s_4 = |A_3 \cap A_4 \cap A_5 \cap A_6| = 1 \tag{10}$$

since the only element in this intersection is the function which interchanges 1 and 2 and maps every other symbol to itself. Hence, by the principle of inclusion and exclusion.

$$\begin{aligned} |T| = |\bigcap_{i=3}^{6} A_i'| &= s_0 - s_1 + s_2 - s_3 + s_4 \\ &= 120 - 96 + 36 - 8 + 1 \\ &= 53 \end{aligned} \tag{11}$$

There is a more elegant way of finding $|T|$. We classify the functions in T into types depending upon whether $f(2) = 1$ or $f(2) \neq 1$. So, let

$$T_1 = \{f \in T : f(2) = 1\} \tag{12}$$

$$\text{and} \quad T_2 = \{f \in T : f(2) \neq 1\} \tag{13}$$

A function in T_1 interchanges 1 and 2 and maps the remaining symbols 3, 4, 5, 6 to themselves bijectively, without any fixed points. So it is like a derangement of these four symbols. Hence

$$|T_1| = D_4 = 9 \tag{14}$$

as calculated above. Now consider a function f in T_2. This corresponds to a bijection from the set $\{2, 3, 4, 5, 6\}$ to the set $\{1, 3, 4, 5, 6\}$ in which $f(2) \neq 1$ and $f(i) \neq i$ for $i = 3, 4, 5, 6$. If we relabel the element 1 in the codomain as 2, then f is nothing but a derangement of the five symbols $2, 3, 4, 5$ and 6. Therefore

$$|T_2| = D_5 = 60 - 20 + 5 - 1 = 44 \tag{15}$$

Adding (14) and (15) we get

$$|T| = |T_1| + |T_2| = 9 + 44 = 53 \tag{16}$$

which is the same answer as before.

In fact, once we know $D_6 = 265$, T can be determined almost instaneously. T is given to be the set of those derangements of $\{1, 2, 3, 4, 5, 6\}$ which take 1 to 2. If instead of 2, we take 3, 4, 5, or 6, we shall get similar subsets and obviously they will all have the same cardinality as T. Since these five sets are mutually disjoint, we get $5|T| = D_6 = 265$. So $|T| = 265/5 = 53$.

Even though these alternate solutions (especially the second one) look shorter, they use the ready-made formula for D_n which was obtained from the principle of inclusion and exclusion in the first place. So, no matter which approach is followed, the use of the principle of inclusion and exclusion is inevitable. When the number of subsets is small, say 2 or 3, this principle can be bypassed by working 'with hands' so to speak. But it is too much to expect this when the number of subsets is 4 or more. Candidates who know this principle will undoubtedly have an easier time over those who don't. The latter are more prone to commit mistakes of omission or double counting. To ensure level ground for all candidates, it is high time that the principle of inclusion and exclusion (which, by itself, is quite elementary) be mentioned explicitly in the JEE syllabus. Those who know this principle are also likely to know the formula for D_n because it is probably the most standard application of this principle. So to ensure uniform justice, the JEE sylabus should also mention derangements explicitly.

Q.44 The common tangents to the circle $x^2+y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the points P, Q and the parabola at the points R, S. Then the area of the quadrilateral $PQRS$ is

(A) 3 (B) 6 (C) 9 (D) 15

Answer and Comments: (D). As both the given circle and the parabola are symmetric about the x-axis, so must be the the two common tangents. So, assume that

$$y = mx + c \tag{1}$$

is a common tangent with $m > 0$. Then its condition for tangency with the parabola $y^2 = 8x$ gives

$$c = \frac{2}{m} \tag{2}$$

and the point of contact R as

$$R = (\frac{2}{m^2}, \frac{4}{m}) \tag{3}$$

while its condition for tangency with the circle $x^2 + y^2 = 2$ gives

$$\frac{c}{\sqrt{1+m^2}} = \sqrt{2} \tag{4}$$

and the point of contact P as

$$P = (-\frac{\sqrt{2}m}{\sqrt{1+m^2}}, \frac{\sqrt{2}}{\sqrt{1+m^2}}) \tag{5}$$

Eliminating c from (2) and (4) gives

$$m^2(1+m^2) = 2 \tag{6}$$

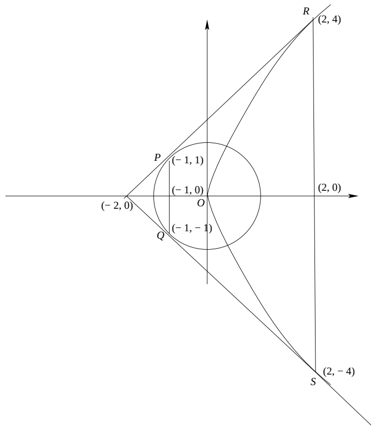
which can be solved to give $m^2 = 1$ or $m^2 = -2$. The latter is discarded as m is real. Further, as $m > 0$, we take $m = 1$. Substituting this into (3) and (4), we get

$$P = (-1, 1), \quad R = (2, 4) \tag{7}$$

As noted before the points Q and S are the reflections of these into the y-axis. Hence

$$Q = (-1, -1), \quad S = (2, -4) \tag{8}$$

It is now quite clear, even without drawing a diagram, that the quadrilateral $PQSR$ is a trapezium with PQ and SR as parallel sides of lengths 2 and 8 respectively and the horizontal distance between them is $2 - (-1) = 3$. So the area of $PQRS$ is $\frac{1}{2}(8+2)3 = 15$ sq. units.



A straightforward problem based on the condition for tangency. The calculations are simple. But once again, the proportionate time of 3 minutes is inadequate, especially so in view of having to find the area of the trapezium, which is essentially a useless addendum not relevant to the main theme and only increases the time required and the chances of numerical mistakes. A student who has cleared JEE (Main) can be presumed to know how to find the area of a trapezium. There is no need to test it in JEE (Advanced). If this presumption is wrong, then correction is due in JEE (Main) rather than in JEE (Advanced).

Q.45 The quadratic equation $p(x) = 0$ with real coefficients has purely imaginary roots. Then the equation $p(p(x)) = 0$ has

(A) only purely imaginary roots
(B) all real roots
(C) two real and two purely imaginary roots
(D) neither real nor purely imaginary roots

Answer and Comments: (D). Let us first identify $p(x)$. As it has real coefficients its complex roots must occur as conjugate pairs. We are given that they are purely imaginary. So they must be of the form ci and $-ci$ for some non-zero real c. Hence

$$p(x) = \lambda(x - ci)(x + ci) = \lambda(x^2 + c^2) \tag{1}$$

where λ is some non-zero real constant.

Now let us call $p(x)$ as y. Then

$$p(p(x)) = p(y) = \lambda(y^2 + c^2) \tag{2}$$

If $p(p(x)) = 0$, then we must have $p(y) = 0$ and hence $y = \pm ci$. Therefore the roots of $p(p(x)) = 0$ are precisely the roots of $p(x) = ci$ and those of $p(x) = -ci$

Since $p(x) = \lambda(x^2+c^2)$, the roots of the equation $p(x) = ci$ are precisely the roots of the equation

$$\lambda(x^2 + c^2) = ci \tag{3}$$

These roots are the two square roots of the complex number $-c^2 + \dfrac{c}{\lambda}i$. Since c and λ are non-zero, this complex number does not lie on the x-axis and so its square roots are neither real nor purely imaginary. A similar reasoning holds for the roots of $p(x) = -ci$. Hence the roots of the equation $p(p(x)) = 0$ are neither real nor purely imaginary.

A simple but unusual problem about complex numbers. Note the role of the number λ. In considering the roots of polynomials, it is often customary to take them as monic polynomials because this does not affect their roots since the roots are independent of the leading coefficient. But in the present case, we are considering the polynomial $p(p(x))$ and its roots could depend on λ. A candidate who misses λ (i.e. takes it as 1) and thereby takes $p(x)$ as $x^2 + c^2$ will also get the same answer. But his work will be unduly simplified. If the candidate were to show his reasoning, then this lapse would come to surface. But otherwise it only serves to reward him by saving his time. So, this is yet another question where the multiple choice format rewards an unscrupulous candidate. Still, the novelty of the problem deserves some appreciation.

Q.46 The integral $\displaystyle\int_{\pi/4}^{\pi/2} (2 \text{ cosec } x)^{17} dx$ is equal to

(A) $\int_0^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{16} du$ (B) $\int_0^{\log(1+\sqrt{2})} (e^u + e^{-u})^{17} du$

(C) $\int_0^{\log(1+\sqrt{2})} (e^u - e^{-u})^{17} du$ (D) $\int_0^{\log(1+\sqrt{2})} 2(e^u - e^{-u})^{16} du$

Answer and Comments: (A). Call the given integral as I. Note that we are *not* asked to evaluate this integral. Our problem is merely to transform it to another definite integral using a suitable substitution. We are also spared the task of drawing this substitution from our own pocket. The various alternatives given suggest that the desired substitution is given by either

$$\begin{aligned} e^u + e^{-u} &= 2 \text{ cosec } x & (1) \\ \text{or} \quad e^u - e^{-u} &= 2 \text{ cosec } x & (2) \end{aligned}$$

(2) seems like a better choice because cosec x is an odd function of x, but the L.H.S. of (1) is an even function of u while that of (2) is an odd function of u. This point would indeed be very critical if the domain of integration were an interval which contained 0 in its interior because in that case a substitution by an even function would not work since an even function cannot assume both positive and negative values. In our case, this is not a problem because x varies only from $\pi/4$ to $\pi/2$ and cosec x is positive throughout this interval. We note further that as x increases from $\pi/4$ to $\pi/2$ cosec x decreases strictly from $\sqrt{2}$ to 1. Therefore the substitution should be such that the upper and lower limits for the values of u should correspond to these end points.

Let us see which of the two substitutions meets this criterion. If we take (1), then cosec $(\pi/4) = \sqrt{2}$ and the equation

$$e^u + e^{-u} = 2\sqrt{2} \tag{3}$$

does have a solution. In fact, if we rewrite it as

$$e^{2u} - 2\sqrt{2}e^u + 1 = 0 \tag{4}$$

we can treat it as a quadratic and get

$$e^u = \sqrt{2} \pm 1 \tag{5}$$

So, there are two possible points, viz. $u = \log(1+\sqrt{2})$ and $u = \log(\sqrt{2}-1)$ which correspond to $x = \pi/4$. (These two values of u are negatives of each other, which is consistent with the fact that $e^u + e^{-u}$ is an even function of u.) And the first value is indeed one of the end-points of the transformed integral in all the four alternatives.

However, if we try the substitution (2) above then we would have to look for a value of u which satisfies

$$e^u - e^{-u} = 2\sqrt{2} \tag{6}$$

This would amount to solving the quadratic in e^u, viz.

$$e^{2u} - 2\sqrt{2}e^u - 1 = 0 \tag{7}$$

and would give $e^u = \sqrt{2} \pm \sqrt{3}$. The negative sign is ruled out since the exponential function does not take negative values. So we must have $u = \log(\sqrt{2}+\sqrt{3})$. But this is not an end-point of the intervals of integration in any of the four alternatives. So (2) is certainly not the right substitution. If at all the choice is between (1) and (2), it has to be (1).

Let us now see what the substitution (1) does to the other end-point, viz. $x = \pi/2$. Here, 2 cosec $(\pi/2) = 2$ and $e^u + e^{-u} = 2$ has $u = 0$ as a (multiple) solution.

Thus we see that the substitution (1) is the right one. We mention, however, that we could have as well chosen $u = \log(\sqrt{2}-1)$ (which is the other root of the equation (4)) as the end-point corresponding to $x = \pi/4$. In that case the interval of integration w.r.t. u would have been $[\log(\sqrt{2}-1), 0]$ and we could have transformed I to an integral over this interval. But as this is not the case in any of the four given choices, we abandon this possibility.

Now that we have found the right substitution, transforming the integral is a routine matter. Differentiating (1), we have

$$(e^u - e^{-u})du = -2\cot x \text{ cosec } x\, dx \tag{8}$$

We already have expressed cosec x in terms of u. As for $\cot x$ we use $\cot x = \sqrt{\text{cosec}^2\, x - 1}$ and take the positive square root because $\cot x$

is posive for all $x \in [\pi/4, \pi/2)$ and 0 at the end point $\pi/2$. Thus

$$\begin{aligned}
\cot x &= \sqrt{(\text{cosec } x)^2 - 1} \\
&= \sqrt{\left(\frac{e^u + e^{-u}}{2}\right)^2 - 1} \\
&= \pm\frac{e^u - e^{-u}}{2} \\
&= \frac{e^u - e^{-u}}{2} \qquad (9)
\end{aligned}$$

where the choice of sign in the last step is dictated by the fact that u is non-negative in the interval $[0, \log(1 + \sqrt{2})]$.

Substitution into (8) gives

$$\text{cosec } x \; dx = -du \qquad (10)$$

So, finally,

$$I = -\int_{\log(1+\sqrt{2})}^{0} 2(e^u + e^{-u})^{16} \; du = \int_{0}^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{16} \; du \qquad (11)$$

After getting the correct substitution, viz. $e^u + e^{-u} = 2\text{cosec } x$, the conversion of the integral could have been done in a shorter (although mathematically hardly different) manner with the introduction of what are called **hyperbolic trigononometric functions**. These are defined in terms of the exponential function. The two basic ones are

$$\begin{aligned}
\sinh x &= \frac{e^x - e^{-x}}{2} \qquad (12) \\
\text{and} \quad \cosh x &= \frac{e^x + e^{-x}}{2} \qquad (13)
\end{aligned}$$

The other four hyperbolic trigonometric functions can be defined in terms of these basic ones. Their notations and definitions are predictable by analogy with the trigonometric functions. For example, $\tanh x$ is defined as $\dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$. These functions satisfy certain identities which are remarkably similar to the trigonometric functions. For example, from (12) and (13) we have

$$\cosh^2 x - \sinh^2 x = 1 \tag{14}$$

$$\frac{d}{dx}\cosh x = \sinh x \tag{15}$$

$$\text{and} \quad \frac{d}{dx}\sinh x = \cosh x \tag{16}$$

for all $x \in I\!R$. Note also that $\cosh x$, like $\cos x$, is an even function of x while $\sinh x$, like $\sin x$, is an odd function of x. Because of the identity (14), we can parametrise the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ as $x = a\cosh u, y = b\sinh u, -\infty < u < \infty$ just as we can parametrise the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as $x = a\cos t, y = b\sin t$. (This explains why these functions are called 'hyperbolic'.)

The similarity between the properties of the ordinary trigonometric functions and the hyperbolic functions is not a matter of coincidence. The two can, in fact, be related with each other using complex numbers. But we shall not go into it. What matters for our purpose here is that the substitution (1) which did the trick in the present problem could have been written as

$$\cosh u = \text{cosec } x \tag{17}$$

and then the conversion of the integral could have been done very efficiently using (15) and (14). We first note that

$$\cot x = \sqrt{(\text{cosec}^2 x - 1} = \sqrt{\cosh^2 u - 1} = \sinh u \tag{18}$$

Now, differentiating (17),

$$\sinh u \, du = -\cot x \text{ cosec } x \, dx \tag{19}$$

(18) and (19) together imply (10). In fact, even if the substitution (17) were not given, it could have been *predicted* at least as a possible try in view of the highly similar identities satisfied by the functions on the two sides, viz.

$$\cosh^2 u - \sinh^2 = 1 = (\text{cosec})^2 x - \cot^2 x \tag{20}$$

As a result, the present problem would have been a good full length question where a candidate has to find the substitution on his own. Even as the problem stands, it would have been more logical to give the alternatives in terms of the hyperbolic functions. The paper-setters have, apparently, not done this for one of the two reasons. They might have thought that the problem would then be too simple. Or they might have thought that not all candidates have studied the hyperbolic functions. The latter is a fact, because these functions are not explicitly mentioned in the JEE syllabus. Still, many good texts introduce them in the chapter on exponential functions and prove their standard identities. Candidates who have studied from such books will definitely have an easier time with this question. To avoid such unintended inequities, it is best to mention hyperbolic functions explicitly in the syllabus. Then nobody would have a cause to complain.

Q.47 The function $y = f(x)$ is the solution of the differential equation $\dfrac{dy}{dx} + \dfrac{xy}{x^2-1} = \dfrac{x^2+2x}{\sqrt{1-x^2}}$ in $(-1,1)$ satisfying $f(0) = 0$. Then $\displaystyle\int_{-\sqrt{3}/2}^{\sqrt{3}/2} f(x)\,dx$ is

(A) $\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{2}$ (B) $\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{4}$ (C) $\dfrac{\pi}{6} - \dfrac{\sqrt{3}}{4}$ (D) $\dfrac{\pi}{8} - \dfrac{\sqrt{3}}{2}$

Answer and Comments: (B). Clearly, the work falls into two parts. First, to identify the function $f(x)$, and the second, to integrate it.

For the first task, we notice that the given differential equation is linear and we can solve it using the readymade formula for the solution of linear, first order equations. But if we look at the nature of the various functions appearing as coefficients and especially the expression $\sqrt{1-x^2}$ and the fact that x varies over $(-1,1)$, the substitution

$$x = \sin\theta \tag{1}$$

suggests itself. This substitution will also help us later in evaluating the integral $\displaystyle\int_{-\sqrt{3}/2}^{\sqrt{3}/2} f(x)\,dx$ because the upper and the lower limits of this integral correspond to $\theta = \pi/3$ and $\theta = -\pi/3$ respectively.

So, let us go ahead with the substitution (1). We first have to rewrite the d.e. in terms of y and θ. We have $dx = \cos\theta d\theta$ and so the given d.e. transforms into

$$\frac{dy}{\cos\theta\ d\theta} - \frac{\sin\theta}{\cos^2\theta} y = \frac{\sin^2\theta + 2\sin\theta}{\cos\theta} \tag{2}$$

which simplifies to

$$\frac{dy}{d\theta} - \tan\theta\ y = \sin^2\theta + 2\sin\theta \tag{3}$$

As expected, this d.e. is also linear. But it is much easier to solve. Indeed the integrating factor is $e^{\int -\tan\theta\ d\theta} = e^{-\log\sec\theta} = e^{\log\cos\theta} = \cos\theta$. Multiplying by it, (3) changes to

$$\cos\theta\frac{dy}{d\theta} - \sin\theta\ y = \sin^2\theta\cos\theta + 2\sin\theta\cos\theta \tag{4}$$

The L.H.S. is nothing but $\frac{d}{d\theta}(y\cos\theta)$. So, integrating we get

$$y\cos\theta = \frac{\sin^3\theta}{3} + \sin^2\theta + c \tag{5}$$

for some arbitrary constant c. To find it we note that that the original initial condition, viz. $y = 0$ when $x = 0$, translates into $y = 0$ when $\theta = 0$. Putting this into (5), we get $c = 0$. Hence the solution is

$$y = \frac{\sin^3\theta}{3\cos\theta} + \frac{\sin^2\theta}{\cos\theta} \tag{6}$$

If we want, we can now express y as a function of x. But that is hardly necessary. Our interest is in evaluating the integral $\int_{-\sqrt{3}/2}^{\sqrt{3}/2} f(x)\ dx$. If we are going to use the substitution $x = \sin\theta$ to evaluate this integral, then it is foolish to write y as a function of x and then convert since we already have it as a function of θ. As pointed out earlier the upper and the lower limits of integration are $\theta = \pi/3$ and $\theta = -\pi/3$ respectively. So the given integral, say I equals

$$\begin{aligned} I &= \int_{-\pi/3}^{\pi/3} \frac{\sin^3\theta}{3\cos\theta} + \frac{\sin^2\theta}{\cos\theta}\ \cos\theta d\theta \\ &= \int_{-\pi/3}^{\pi/3} \frac{\sin^3\theta}{3} + \sin^2\theta\ d\theta \end{aligned} \tag{7}$$

We need not bother about the integral of the first term. The function $\sin^3\theta$ is an odd function of θ and the interval of integration is symmetric about the origin. So the integral of the first term would be 0. The second function $\sin^2\theta$ is an even function. Therefore again by the symmetry of the interval about the origin, we get

$$\begin{aligned} I &= 2\int_0^{\pi/3} \sin^2\theta \, d\theta \\ &= \int_0^{\pi/3} (1-\cos 2\theta)\, d\theta \qquad (8) \\ &= \theta - \sin\theta\cos\theta\Big|_0^{\pi/3} \\ &= \frac{\pi}{3} - \frac{\sqrt{3}}{4} \qquad (9) \end{aligned}$$

The good part of the problem is the realisation that the substitution $x = \sin\theta$ will simplify the work. Unfortunately, even after this simplification, the work that remains can hardly be finished in three minutes.

Q.48 Let $f : [0,2] \longrightarrow I\!R$ be a function which is continuous on $[0,2]$ and differentiable on $(0,2)$ wirh $f(0) = 1$. Let $F(x) = \int_0^{x^2} f(\sqrt{t})\, dt$ for $x \in [0,2]$. If $F'(x) = f'(x)$ for all $x \in (0,2)$, then $F(2)$ equals

(A) $e^2 - 1$ (B) $e^4 - 1$ (C) $e - 1$ (D) e^4

Answer and Comments: (B). Already Paper 1 had two problems based on the second fundamental theorem of calculus, This is one more. But this time there is a slight twist. The integrand is not $f(t)$ but $f(\sqrt{t})$. To tackle it we introduce an intermediate variable u defined by

$$u = \sqrt{t} \qquad (1)$$

Making this substitution into the definition of $F(x)$ we get

$$F(x) = \int_0^{x^2} f(\sqrt{t})\, dt = \int_0^x f(u) 2u\, du \qquad (2)$$

We can *now* apply the second fundamental theorem of calculus and get

$$F'(x) = 2xf(x) \qquad (3)$$

for all $x \in [0, 2]$. As we are also given that $F'(x) = f'(x)$ for all $x \in (0, 2)$, we now have

$$f'(x) = 2xf(x) \tag{4}$$

for all $x \in (0, 2)$. This is a differential equation for $f(x)$ which can be solved by inspection to get $\ln f(x) = x^2 + c$ and hence

$$f(x) = ke^{x^2} \tag{5}$$

for some constant k. The condition $f(0) = 1$ determines k as 1. Hence

$$f(x) = e^{x^2} \tag{6}$$

We now go back to $F(x)$. This time we have the advantage that we know the integrand. So,

$$\begin{aligned} F(x) &= \int_0^{x^2} f(\sqrt{t})\ dt \\ &= \int_0^{x^2} e^t\ dt \\ &= e^t\Big|_0^{x^2} \\ &= e^{x^2} - 1 \end{aligned} \tag{7}$$

In particular $F(2) = e^4 - 1$.

A good problem which makes a candidate think before applying the FTC. Once he does the correct thinking, the calculations needed are reasonable.

Q.49 The coefficient of x^{11} in the expansion of $(1+x^2)^4(1+x^3)^7(1+x^4)^{12}$ is

(A) 1051 (B) 1106 (C) 1113 (D) 1120

Answer and Comments: (C). Every occurrence of x^{11} in the given product will be of the form $x^{n_1}x^{n_2}x^{n_3}$ where n_1, n_2, n_3 are non-negative integers adding up to 11. Since x^{n_1} is a term in the expansion of $(1 + x^2)^2$, n_1 must be even. Similarly, n_2, n_3 will have to be divisible by 3 and 4 respectively. Writing $n_1 = 2m_1, n_2 = 3m_2$ and $n_3 = 4m_3$, we have an equation

$$2m_1 + 3m_2 + 4m_3 = 11 \tag{1}$$

where m_1, m_2, m_3 are non-negative integers. We first have to find all possible solutions of (1) and this is best done by hand. Note that m_3 can take only three possible values, 0, 1, and 2. For each of these values, we find the possible solutions of (1).

If $m_3 = 0$, then $2m_1 + 3m_2 = 11$. Note that $3m_2$ and hence m_2 must be odd. So the only possible values of m_2 are $m_2 = 1$ and $m_2 = 3$. Correspondingly, $m_1 = 4$ and 2. So the possible triples (m_1, m_2, m_3) of this type are $(4, 1, 0)$ and $(1, 3, 0)$.

If $m_3 = 1$, then $2m_1 + 3m_2 = 7$. Again, m_2 is odd and so must be 1. Hence $(2, 1, 1)$ is the only triple of this type.

Finally, if $m_3 = 2$, then $2m_1 + 3m_2 = 3$. Again m_2 must be 1. So $(0, 1, 2)$ is the only triple of this type.

Summing up, we have got four triples (m_1, m_2, m_3) which are solutions of (1), viz. $(4, 1, 0), (1, 3, 0), (2, 1, 1)$ and $(0, 1, 2)$. For each one of these, we have to see how many times $x^{2m_1}x^{3m_2}x^{4m_3}$ occurs in the product $(1 + x^2)^4(1 + x^3)^7(1 + x^4)^{12}$.

Let us begin with the triple $(4, 1, 0)$. Call x^2 as y. Then the coefficient of x^8 in $(1 + x^2)^4$ is the same as that of y^4 in $(1 + y)^4$. This coefficient is $\binom{4}{4}$ i.e. 1. Similarly calling x^3 as z we get the coefficient of x^3 in $(1+x^3)^7$ as equal to the coefficient of z in $(1+z)^7$ which equals $\binom{7}{1}$ i.e. 7. Finally, since the last entry in the triple is 0, we take only the constant term in the expansion of $(1 + w)^{12}$ (where $w = x^2$). This is 1. So the coefficient of x^{11} coming from the triple $(4, 1, 0)$ is

$$\binom{4}{4}\binom{7}{1}\binom{12}{0} = 1 \times 7 \times 1 = 7 \tag{2}$$

By an exactly similar reasoning, the contribution from the triple $(1, 3, 0)$ is

$$\binom{4}{1}\binom{7}{3}\binom{12}{0} = 4 \times 35 \times 1 = 140 \tag{3}$$

that from the triple $(2, 1, 1)$ is

$$\binom{4}{2}\binom{7}{1}\binom{12}{1} = 6 \times 7 \times 12 = 504 \tag{4}$$

while that from $(0, 1, 2)$ is

$$\binom{4}{0}\binom{7}{1}\binom{12}{2} = 1 \times 7 \times 66 = 462 \tag{5}$$

Adding (2), (3), (4) and (5), the coefficient of x^{11} in the given product is 1113.

There is really no elegant way to do a problem like this. One has to resolve it into various possibilities and then add the contributions from each one. As a result, such problems tend to be repetitious. Fortunately, although the reasoning looks lengthy when expressed in words, it does not take much time to conceive it. And once the candidate gets the correct reasoning, the computations involved are simple. Moreover, an answer like 1113 can never come by a fluke. A candidate who gets it has undoubtedly done the correct reasoning. Moreover, in a problem like this, if a candidate begins with a correct reasoning but makes a numerical slip later on, it is highly unlikely that he will get one of the remaining answers. Most likely his answer will not match any of the four alternatives and this will alert him to go back and check his calculations. So, such questions are ideal for multiple choice format.

The multiple choice format has also made interesting binomial identities a thing of the past. Questions like this serve to pay at least a lip service to the binomial theorem.

Q.50 For $x \in (0, \pi)$, the equation $\sin x + 2\sin 2x - \sin 3x = 3$ has

(A) infinitely many solutions (B) three solutions
(C) one solution (D) no solution

Answer and Comments: (D). Yet another problem which asks not so much to solve a trigonometric equation as merely to find how many solutions there are. As the expression involves the sines of three angles, it is unlikely that we can solve it exactly. The only simplification possible is that $\sin x$ is a common factor in all the three terms. Hence the equation becomes

$$\sin x(1 + 4\cos x - 3 + 4\sin^2 x) = 3 \tag{1}$$

The second factor can be written as a quadratic in $\cos x$. Doing so, the equation is

$$\sin x(-4\cos^2 x + 4\cos x + 2) = 3 \tag{2}$$

If the R.H.S. were 0 (instead of 3), then we can solve this equation by equating the factors of the L.H.S. with 0, one at a time. This is of little help here. We note however, that the first factor $\sin x$ lies in $[0, 1]$ for all $x \in [0, \pi]$. So, if (2) is to hold then the second factor must be positive and at least 3. Therefore if x satisfies (2), we must have

$$-4\cos^2 x + 4\cos x \geq 1 \tag{3}$$

or equivalently,

$$\cos x(1 - \cos x) \geq \frac{1}{4} \tag{4}$$

The second factor of the L.H.S. is always non-negative. So, if this inequality is to hold, then the first factor, viz. $\cos x$ must be positive. But in that case, by the A.M.-G.M. inequality, we have

$$\cos x(1 - \cos x) \leq \left(\frac{\cos x + (1 - \cos x)}{2}\right)^2 = \frac{1}{4} \tag{5}$$

with equality holding only when $\cos x = 1 - \cos x$, i.e. when $\cos x = \pi/3$. So this is the only value of x for which the second factor of the L.H.S. of (2) is 3 and it can never be bigger than 3. However, for this value of x, the first factor, viz. $\sin(\pi/3)$ is less than 1. Hence (2) can never hold and therefore (1) and hence the given equation has no solution.

A simple, but tricky problem. Instead of applying the A.M. inequality one can also calculate the maximum of $\cos x(1 - \cos x)$ using calculus and see that it is $\frac{1}{4}$ occurring at $x = \pi/3$.

SECTION - 2

Comprehension Type (Only One Option Correct)

This section contains **3 paragraphs** each describing theory, experiment, data etc. **Six questions** relate to three paragraphs with two questions on each paragraph. Each question of a paragraph has **only one correct answer** among the four choices (A), (B), (C) and (D). There are **3 points** for marking only the correct answer, no points if no answer is marked and -1 point in all other cases.

Paragraph for Questions 51 and 52

Box 1 contains three cards bearing numbers 1,2,3; box 2 contains five cards bearing numbers 1,2,3,4,5 and box 3 contains seven cards bearing numbers 1,2,3,4,5,6,7. A card is drawn from each one of them. Let x_i ne the number n the card drawn from the i-th box, $i = 1, 2, 3$.

Q.51 The probability that $x_1 + x_2 + x_3$ is odd is

(A) $\frac{29}{105}$ (B) $\frac{53}{105}$ (C) $\frac{67}{105}$ (D) $\frac{1}{2}$

Q.52 The probability that x_1, x_2, x_3 are in an arithmetic progression is

(A) $\frac{9}{105}$ (B) $\frac{10}{105}$ (C) $\frac{11}{105}$ (D) $\frac{7}{105}$

Answers and Comments: (B) and (C). In both the problems, the sample space consists of all ordered triples (x_1, x_2, x_3). Since there are respectively, 3, 5 and 7 possible choices for x_1, x_2, x_3 and they are independent of each other, the sample space, say S, has $3 \times 5 \times 7 = 105$ elements. This is also the denominator in all but one alternatives in both the questions. And this assures a candidate that at least his start is correct. (One wonders if this is what is meant by 'comprehension'! If so, then every mathematics problem can be posed as a comprehension problem.)

Now, in the first question, the sum $x_1 + x_2 + x_3$ is odd if and only if either one or all of the summands are odd. So the count of the number of favourable cases falls into four mutually exclusive possibilities: (i) only x_1 odd (and the other even), (ii) only x_2 odd, (iii) only x_3 is odd and (iv) all three are odd. Clearly the numbers of favourable cases in (i) is $2 \times 2 \times 3 = 12$, in (ii) it is $1 \times 3 \times 3 = 9$, in (iii) it is $1 \times 2 \times 4 = 8$ and in (iv) it is $2 \times 3 \times 4 = 24$.

Adding, the total number of favourable cases is $18+9+8+24 = 53$. So the desired probability in Q.51 is $\frac{53}{105}$.

The calculations in Q.52 are similar, and, in a way, simpler. To say that x_1, x_2, x_3 are in an A.P. means $x_1 + x_3 = 2x_2$. So, $x_1 + x_3$ is even. For this to happen either both x_1 and x_3 should be even or both should be odd. The former happens in $1 \times 3 = 3$ ways while the latter happens in 2×4 ways. Hence there are 11 possibilities. It is important to note that for each of these eleven possibilities, the A.M. of x_1 and x_3 indeed lies in the second second set, viz. $\{1, 2, 3, 4, 5\}$. This happens because the minimum and the maximum values of $x_1 + x_3$ are 2 and 10 respectively. So the number of favourable cases is 11 and the desired probability is $\frac{11}{105}$.

The reasoning needed in both the questions is similar, in fact, somewhat repetitious. The second question could have been made a little more interesting by changing the numerical data in such a way that for some x_1 and x_3, their A.M. was not a label on any card in the second box. In that case, a candidate would have to count and subtract the number of such cases.

Although in highly elementary problems like these, little is to be gained by paraphrasing the ideas in terms of sets, it is a good habit one should develop because it pays in more complicated problems. Thus, for $i = 1, 2, 3$, we let S_i be the set of labels on the cards in the i-th box. Then our sample space, say, S is simply the cartesian product of these three sets, i.e.

$$S = S_1 \times S_2 \times S_3 \tag{1}$$

whence

$$|S| = |S_1| \times |S_2| \times |S_3| = 3 \times 5 \times 7 = 105 \tag{2}$$

Now, for each $i = 1, 2, 3$, let A_i be the set of all even labels in S_i and let B_i be the set of all odd labels in S_i. Clearly, A_i and B_i are complements of each other. With these notations the set, say F_1 of all favourable cases is given by

$$F_1 = (B_1 \times A_2 \times A_3) \cup (A_1 \times B_2 \times A_3) \cup (A_1 \times A_2 \times B_3) \cup (B_1 \times B_2 \times B_3) \tag{3}$$

Since all the four cartesian products appearing in this union are mutually disjoint, we have

$$|F_1| = |B_1 \times A_2 \times A_3| + |A_1 \times B_2 \times A_3| + |A_1 \times A_2 \times B_3| + |B_1 \times B_2 \times B_3|$$

$$\begin{aligned}
&= |B_1| \times |A_2| \times |A_3| + |A_1| \times |B_2| \times |A_3| \\
&\quad +|A_1| \times |A_2| \times |B_3| + |B_1| \times |B_2| \times |B_3| \\
&= 2 \times 2 \times 3 + 1 \times 3 \times 3 + 1 \times 2 \times 4 + 2 \times 3 \times 4 \\
&= 12 + 9 + 8 + 24 = 53 \qquad (4)
\end{aligned}$$

From (4) and (1), the desired probability for Q.51 is $\frac{53}{105}$.

For Q.52, we follow the same notations and let F_2 be the favourable set. Then

$$F_2 = (A_1 \times A_3) \cup (B_1 \times B_3) \qquad (5)$$

and as the two sets whose union is taken are mutually disjoint,

$$\begin{aligned}
|F_2| &= |A_1| \times |A_3| + |B_1| \times |B_3| \\
&= 1 \times 3 + 2 \times 4 = 3 + 8 = 11 \qquad (6)
\end{aligned}$$

So the desired probability for Q.52 is $\frac{|F_2|}{|S|} = \frac{11}{105}$.

In fact a true test of comprehension could have been designed by asking the problem the other way. The candidates could be given that S_1, S_2 and S_3 are some finite sets of positive integers. Then the definitions of $A_1, B_1, A_2, B_2, A_3, B_3$ could be given. Then the formulas (1), (3) and (5) should be given as definitions of the sets S, F_1 and F_2 respectively. And then the candidates should be asked to give a verbal description of the events whose probabilities are the ratios $\frac{|F_1|}{|S|}$ and $\frac{|F_2|}{|S|}$. That would be similar to giving them a passage and asking what it is all about. That is really comprehension. The paper-setters have not given a thought to this. The problems they have asked as 'comprehension' are hardly different from any other probability problems where the desired probability is found simply by counting. One really fails to see how they are a test of the ability to comprehend.

Paragraph for Questions 53 and 54

Let a, r, s, t be non-zero real numbers. Let $P(at^2, 2at), Q, R(ar^2, 2ar)$ and $S(as^2, 2as)$ be distinct points on the parabola $y^2 = 4ax$. Suppose PQ is the focal chord and lines QR and PK are parallel where K is the point $(2a, 0)$.

Q.53 The value of r is

$$\text{(A)} \ -\frac{1}{t} \quad \text{(B)} \ \frac{t^2+1}{t} \quad \text{(C)} \ \frac{1}{t} \quad \text{(D)} \ \frac{t^2-1}{t}$$

Q.54 If $st = 1$, then the tangent at P and the normal at S to the parabola meet at a point whose ordinate is

$$\text{(A)} \ \frac{(t^2+1)^2}{2t^3} \quad \text{(B)} \ \frac{a(t^2+1)^2}{2t^3} \quad \text{(C)} \ \frac{a(t^2+1)^2}{t^3} \quad \text{(D)} \frac{a(t^2+2)^2}{t^3}$$

Answers and Comments: (D) and (B). The only point whose coordinates are not given explicitly is the point Q. This is given to be a point on the parabola. So, we can take it as $(au^2, 2au)$ for some $u \in \mathbb{R}$. Here u is to be determined from the fact that PQ is a focal chord, i.e. a chord which passes through the focus, say $F(a, 0)$ of the parabola. The condition for collinearity of P, Q, F gives

$$\frac{2at - 0}{at^2 - a} = \frac{2au - 0}{au^2 - a} \tag{1}$$

which simplifies to $t(u^2 - 1) = u(t^2 - 1)$ and hence to $t - u = -ut(t - u)$. Since the points P and Q are distinct, $t \neq u$ and so we must have

$$u = -1/t \tag{2}$$

(This is a fairly well-known fact and candidates who know it will save some time.)

We are further given that QR and PK are parallel. So equating their slopes

$$\frac{-2a/t - 2ar}{\frac{a}{t^2} - ar^2} = \frac{2at - 0}{at^2 - 2a} \tag{3}$$

which simplifies to

$$\frac{-\frac{1}{t} - r}{\frac{1}{t^2} - r^2} = \frac{t}{t^2 - 2} \tag{4}$$

The denominator of the L.H.S. can be factored as $(1/t + r)(1/t - r)$. The first factor is non-zero since Q and R are distinct points. So the equation above simplifies to

$$(1/t - r)t = -(t^2 - 2) \tag{5}$$

which implies $r = \dfrac{t^2-1}{t}$.

Q.54 is even more straightforward. The equation of the tangent to the parabola at $P(at^2, 2at)$ is

$$y - 2at = \frac{1}{t}(x - at^2) \tag{6}$$

while that of the normal to the parabola at $S(as^2, 2as)$ is

$$y - 2as = -s(x - as^2) \tag{7}$$

which becomes

$$y - \frac{2a}{t} = -\frac{1}{t}(x - \frac{a}{t^2}) \tag{8}$$

since we are given that $st = 1$.

Since we want only the ordinate (i.e. the y-coordinate) of the point of intersection of these two lines, instead of solving (6) and (8) simultaneously, we merely eliminate x between them. This is done most easily by simply adding them to get

$$2y - 2at - \frac{2a}{t} = -at + \frac{a}{t^3} \tag{9}$$

Hence

$$\begin{aligned} y &= \frac{at + \frac{a}{t^3}}{2} + \frac{2a}{t} \\ &= \frac{a(t^4 + 1 + 2t^2)}{2t^3} \\ &= \frac{a(t^2+1)^2}{2t^3} \end{aligned} \tag{10}$$

Apart from the knowledge of the most basic properties of a parabola whose points are taken in a parametric form, the only thing involved in both the questions is algebraic simplification. It is shocking that such questions are asked in an advanced selection.

Paragraph for Questions 55 and 56

Given that for each $a \in (0,1)$, $\displaystyle\lim_{h \to 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1}dt$ exists, let this limit be $g(a)$. In addition it is given that the function $g(a)$ is differentiable on $(0,1)$.

Q.55 The value of $g(\frac{1}{2})$ is

$$\text{(A) } \pi \quad \text{(B) } 2\pi \quad \text{(C) } \frac{\pi}{2} \quad \text{(D) } \frac{\pi}{4}$$

Q.56 The value of $g'(\frac{1}{2})$ is

$$\text{(A) } \frac{\pi}{2} \quad \text{(B) } \pi \quad \text{(C) } -\frac{\pi}{2} \quad \text{(D) } 0$$

Answers and Comments: (A) and (D). The definition of the function $g(a)$ may appear unnecessarily clumsy. Most candidates may wonder why $g(a)$ is not defined simply as the definite integral

$$g(a) = \int_0^1 t^{-a}(1-t)^{a-1}dt \tag{1}$$

The answer is that the usual definition of a definite integral over an interval, as the limit of Riemann sums of the integrand does not apply to the integral on the R.H.S. When we define $\int_a^b f(t)dt$ as the limit of the Riemann sums, it is implicit that the function $f(t)$ is bounded on the interval $[a, b]$ as otherwise the Riemann sums cannot be defined, as the function fails to have finite bounds over some subintervals. If we take the function

$$f(t) = t^{-a}(1-t)^{a-1} = \frac{1}{t^a}\frac{1}{(1-t)^{1-a}} \tag{2}$$

then $f(t) \to \infty$ as $t \to 0^+$ and also as $t \to 1^-$ because in both the cases one of the factors tends to 1 and the other to ∞. Therefore the integral $\int_0^1 f(t)dt$ makes no sense if we follow the ordinary definition. However, for every small positive h, the function $f(t)$ is bounded and in fact continuous on the interval $[h, 1-h]$. So the integral $\int_h^{1-h} f(t)\, dt$ makes sense. Its value depends on h and we can denote it by $I(h)$. It happens sometimes that even though $f(t)$ is unbounded at a and/or at b, the integral $\int_{a+h}^{b-h} f(t)\, dt$ tends to a finite limit as $h \to 0^+$. In such a case the limit is denoted by the (strictly speaking misleading) symbol $\int_a^b f(t)dt$ and is called an **improper integral.** (See p. 686.)

The paper-setters have taken care to avoid the use of improper integrals since they are not a part of the JEE syllabus. That is why they have defined $g(a)$ as the limit of a certain integral. But this care is likely to be lost on most candidates for two reasons. First they may not appreciate the need for it. Secondly, many of the basic results about ordinary definite integrals continue to hold even for improper integrals and sometimes lead to some interesting evaluations. (See, for example, Exercise (18.16).)

So, no harm will arise if we treat (1) like an ordinary integral and subject it to the same laws that we use for ordinary integrals. In a rigorous approach we should apply any such laws only for integrals of the type

$$\int_h^{1-h} t^{-a}(1-t)^{a-1}dt \tag{3}$$

and then take limits as $h \to 0^+$. But we shall not do so the logical perfection gained by doing so is of little value at the JEE level (and is going to be invisible in a multiple choice question anyway).

With this preamble, we now turn to the first question of the paragraph. Without bothering about the impropriety of the integral, we have

$$\begin{aligned} g(\frac{1}{2}) &= \int_0^1 t^{-1/2}(1-t)^{-1/2}dt \\ &= \int_0^1 \frac{1}{\sqrt{t-t^2}}dt \\ &= \int_0^1 \frac{1}{\sqrt{\frac{1}{4}-(t-\frac{1}{2})^2}}dt \qquad (4) \end{aligned}$$

Instead of using a readymade formula for an antiderivative of the integrand here, let us transform it to another integral using the substitution

$$t - \frac{1}{2} = \frac{1}{2}\sin\theta \tag{5}$$

This transforms the interval $[0, 1]$ to $[-\pi/2, \pi/2]$ and the integral to

$$\begin{aligned} g(\frac{1}{2}) &= \int_{-\pi/2}^{\pi/2} \frac{1}{\cos\theta}\cos\theta \; d\theta \qquad (6) \\ &= \int_{-\pi/2}^{\pi/2} 1 d\theta = \pi \qquad (7) \end{aligned}$$

Note that the integral in (6) is *not* improper. So we transformed an improper integral into a proper one. How did this happen? As it stands, the substitution (5) converts the integrand $\frac{1}{\sqrt{t-t^2}}$ to $2\sec\theta$ which is indeed an unbounded function on the the interval $[-\pi/2, \pi/2]$. But the coefficient $\cos\theta$ of $d\theta$ serves to cancel this impropriety. For the puritans, we can apply the substitution (5) only for the 'proper' integral $\int_h^{1-h} \frac{1}{\sqrt{t-t^2}} dt$ to convert it to an integral of the form $\int_\alpha^\beta \frac{1}{\cos\theta}\cos\theta\, d\theta$ where the real numbers α and β are defined by

$$\alpha = \sin^{-1}(2h-1) \quad \text{and} \quad \beta = \sin^{-1}(1-2h) \tag{8}$$

We in fact have, that $\alpha = -\beta$. But that is not so important. What matters is that as $h \to 0$, $\alpha \to -\pi/2$ and $\beta \to \pi/2$ and so

$$g(\frac{1}{2}) \;=\; \lim_{h\to 0^+} \int_h^{1-h} \frac{1}{\sqrt{t-t^2}} dt = \lim_{h\to 0^+} (\beta - \alpha) = \pi/2 - (-\pi/2) = \pi \tag{9}$$

which is the same answer as before but obtained in a 'clean' way. But as pointed out earlier this kind of over-scrupulousness is not going to pay off when you don't have to show the reasoning and it is only the answer that matters.

Let us now turn to Q.56. We are given $g(a)$ as an (improper) integral in which a is a parameter taking values in the interval $(0, 1)$. It can be proved that under certain conditions, a function like this can be 'differentiated under the integral sign'. That is, the derivative of $g(a)$ w.r.t. a is given by the integral (with the same limits) of the derivative (w.r.t. a) of the integrand of $g(a)$. That is,

$$\begin{aligned} g'(a) &= \int_0^1 \frac{d}{da}(t^{-a}(1-t)^{a-1})dt \\ &= \int_0^1 -(\log t)t^{-a}(1-t)^{a-1} + \log(1-t)t^{-a}(1-t)^{a-1}dt \end{aligned} \tag{10}$$

In particular

$$g'(\frac{1}{2}) = \int_0^1 \log(\frac{1-t}{t})\frac{1}{\sqrt{t(1-t)}}dt \tag{11}$$

which, again, is an improper integral. We assume that such improper integrals obey the same laws that are obeyed by the ordinary definite integrals and in particular, the law of reflection stated as Equation (37) in Comment No. 14 of Chapter 18. As a reult,

$$\begin{aligned} g'(\frac{1}{2}) &= \int_0^1 \log\frac{(1-(1-t))}{1-t}\frac{1}{\sqrt{(1-t)t}}dt \\ &= \int_0^1 -\log(\frac{1-t}{t})\frac{1}{\sqrt{t(1-t)}}dt = -g'(\frac{1}{2}) \end{aligned} \tag{12}$$

But this can happen only when $g'(\frac{1}{2}) = 0$.

This proof was based on the technique of differentiation under the integral sign. As this technique is beyond the scope of the JEE syllabus, we give another, and a much simpler, albeit trickier proof. The essence of the derivation (12) was that the integrand of (11) assumed equal but opposite values at the points t and $1-t$. These two points are symmetrically located about the point $\frac{1}{2}$, which is the mid-point of the interval $(0,1)$. In the definition of $g(a)$, we are given another pair of numbers which are symmetric w.r.t. $\frac{1}{2}$, viz. a and $1-a$. So, let us compare $g(1-a)$ with $g(a)$. By straight substitution

$$g(1-a) = \int_0^1 t^{-(1-a)}(1-t)^{(1-a)-1}dt = \int_0^1 t^{a-1}(1-t)^{-a}dt \tag{13}$$

Let us now apply the law of reflection to the second integral. Then we get

$$g(1-a) = \int_0^1 (1-t)^{1-a}t^{a}dt \tag{14}$$

But this is nothing but $g(a)$. Thus we see that the graph of the function $g(a)$ is symmetric about the line $a = \frac{1}{2}$. Therefore the tangent to it at the point $a = \frac{1}{2}$ (which exists since $g'(a)$ is given to exist for all $a \in (0,1)$) must be horizontal. Therefore $g'(\frac{1}{2})$ must vanish. (The argument here is similar to that used in showing that the derivative of an even function, in case it exists, must vanish at 0. Actually, if we make the substitution $b = a - \frac{1}{2}$, then as a varies over $(0,1)$, b varies over $(-1/2, 1/2)$. And if we let $g(a) = g(b+\frac{1}{2}) = h(b)$, then the function h so defined is an even function of b since $h(-b) = g(-b+\frac{1}{2}) = g(1-(b+1/2)) = g(b+1/2) = h(b)$. And $h'(0) = g'(\frac{1}{2})$.)

The technique of differentiation under an integral sign allows us to evaluate many interesting integrals which would be difficult otherwise. (For example, one can show that for $a > 0$, the improper integral $\int_0^\infty \frac{e^{-ax}\sin x}{x}dx$ equals $\cot^{-1} a$ because if we call this integral as $g(a)$, then $g'(a)$ is simply $\int_0^\infty -e^{-ax}\sin x\, dx$ which comes out to be $-\frac{1}{a^2+1}$.) Although this technique is far too advanced for the JEE, the essential idea is not foreign. We do know (and routinely use) the fact that the derivative of a (finite) sum of functions is the sum of their derivatives. In other words, here we differentiate under a summation. Since integration is a refined form of summation, it is not basically surprising that we can also differentiate under the integral. One, of course, has to be wary that we do not do so recklessly as otherwise it leads to absurdities just as extending the laws of finite summations unscrupulously to infinite sums leads to absurdities (a classic example being that if we put $x = 2$ in the identity

$$1 + x + x^2 + x^3 + \ldots + x^n + \ldots = \frac{1}{1-x}$$

then the L.H.S. which is a sum of positive terms, becomes equal to -1).

SECTION - 3

Matching list Type

This section contains **4 multiple choice questions.** Each question has two lists List I and List II. The entries in List I are marked P, Q, R and S while those in List II are numbered from 1 to 4. Each entry in List I matches with **one** entry in List II. There are **three points** if all the matchings are correctly made, **zero point** if no matchings are made and **minus one** point in all other cases.

Q.57

	List I	List II	
(P)	The number of polynomials $f(x)$ with non-negative integer coefficients of degree ≤ 2 satisfying $f(0) = 0$ and $\int_0^1 f(x)dx = 1$ is	(1)	8
(Q)	The number of points in the interval $[-\sqrt{13}, \sqrt{13}]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value, is	(2)	2
(R)	$\int_{-2}^{2} \frac{3x^2}{(1+e^x)} dx$ equals	(3)	4
(S)	$\dfrac{\int\limits_{-1/2}^{1/2} \cos x . \log\left(\frac{1+x}{1-x}\right) dx}{\int\limits_{0}^{1/2} \cos 2x . \log\left(\frac{1+x}{1-x}\right) dx}$ equals	(4)	0

Answer and Comments: (P,2), (Q,3), (R,1), (S,4).

All the parts are completely unrelated and so each has to be tackled separately.

For (P), let $f(x) = ax^2 + bx + c$. $f(0) = 0$ gives $c = 0$. Hence $\int_0^1 f(x)dx = \int_0^1 ax^2 + bxdx = \frac{a}{3} + \frac{b}{2} = 1$ which means $2a + 3b = 6$. As a, b are non-negative integers, the only possibilities are $a = 0, b = 2$ or $a = 2, b = 0$.

For (Q), Put $u = x^2$. Then $u \in [0, 13]$. $f(x) = \sin u + \cos u = \sqrt{2}\sin(u + \frac{\pi}{4})$ which is maximum when $u = 2n\pi + \frac{\pi}{2} - \frac{\pi}{4} = (2n\pi + \frac{\pi}{4})$ for $n = 0, 1, 2, 3, 4, ...$ $u = \pi/4, 9\pi/4$ lie in $[0, 13]$. The next value $17\pi/4$ exceeds 13 because $17\pi \geq 17 \times 3.1 > 52$. So u has 2 possible values and x has 4 possible values.

Denote the integral in (R) by I. Then I also equals $\int_{-2}^{2} \frac{3(-x)^2}{1+e^{-x}} dx$ $= \int_2^2 \frac{3x^2 e^x}{1+e^x} dx$. Adding, $2I = \int_{-2}^{2} 3x^2 dx = 16$. So $I = 8$.

In (S), we note that $\log(\frac{1+x}{1-x})$ is an odd function of x. Since $\cos x$ is an even function, their product is an odd function of x. Therefore the integral in the numerator is 0.

All parts are straightforward, except possibly the slight trick in (R), based on the reflection formula. But this trick is needed so many problems that it is no longer a trick. Already we used it in the solution to Q.56. This duplication could have been avoided.

Q.58 Match the entries in List I with those in List II.

	List I		**List II**
(P)	Let $y(x) = \cos(3\cos^{-1} x)$, $x \in [-1, 1], x \neq \pm\frac{\sqrt{3}}{2}$. Then $\frac{1}{y(x)}\left\{(x^2-1)\frac{d^2y(x)}{dx^2} + x\frac{dy(x)}{dx}\right\}$ equals	(1)	1
(Q)	Let $A_1, A_2, \ldots, A_n$ $(n > 2)$ be the vertices of a regular polygon of n sides with its centre at the origin. Let $\vec{a_k}$ be the position vector of the point A_k, $k = 1, 2, \ldots, n$. If $\left\lvert\sum_{k=1}^{n-1}(\vec{a_k} \times \vec{a_{k+1}})\right\rvert = \left\lvert\sum_{k=1}^{n-1}(\vec{a_k} \cdot \vec{a_{k+1}})\right\rvert$, then the minimum value of n is	(2)	2
(R)	If the normal from the point $P(h, 1)$ on the ellipse $\frac{x^2}{6} + \frac{y^2}{3} = 1$ is perpendicular to the line $x + y = 8$, then the value of h is	(3)	8
(S)	Number of positive solutions satisfying the equation $\tan^{-1}\left(\frac{1}{2x+1}\right) + \tan^{-1}\left(\frac{1}{4x+1}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$ is	(4)	9

Answer and Comments: (P,4), (Q,3), (R,2), (S,1). As in the last question, in the present problem, each item is independent of the others. So each one has to be tackled separately.

In (P), a direct calculation of the derivative, especially the second derivative, will be complicated. So we introduce an auxiliary variable u by

$$u = \cos^{-1} x \tag{1}$$

We then have

$$\begin{aligned} x &= \cos u & (2) \\ \text{and} \quad y &= \cos(3u) & (3) \end{aligned}$$

Therefore

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/du}{dx/du} = \frac{-3\sin(3u)}{-\sin u} \\ &= \frac{9\sin u - 12\sin^3 u}{\sin u} \\ &= 9 - 12\sin^2 u = 12\cos^2 u - 3 \\ &= 12x^2 - 3 & (4) \end{aligned}$$

where we have used the identity $\sin 3u = 3\sin u - 4\sin^3 u$. Hence,

$$\frac{d^2y}{dx^2} = 24x \tag{5}$$

Call the given expression as E. Then

$$\begin{aligned} E &= \frac{1}{y(x)}\left\{(x^2-1)\frac{d^2y(x)}{dx^2} + x\frac{dy(x)}{dx}\right\} \\ &= \frac{24(x^2-1)x + x(12x^2-3)}{y} \\ &= \frac{36x^3 - 27x}{y} \\ &= \frac{9(4\cos^3 u - 3\cos u)}{\cos 3u} \\ &= 9 & (6) \end{aligned}$$

because of the identity $\cos 3u = 4\cos^3 u - 3\cos u$.

This problem is a good application of the identities for $\sin 3u$ and $\cos 3u$. But it would take a super-human brain to do it in 45 seconds!

The question would have been more appropriate as an integer answer type question in Section -1 of Paper -1.

The other three parts are not so demanding. (Of course, it is again doubtful if anybody can do them in 45 seconds each!) In (Q), assume that r is the circum-radius of the polygon. Then for every $k = 1, 2, \ldots, n-1$, the angle between $\vec{a_k}$ and $\vec{a_{k+1}}$ is $2\pi/n$ and so

$$\begin{aligned} \vec{a_k} \times \vec{a_{k+1}} &= r^2 \sin(2\pi/n)\vec{k} \\ \text{and} \quad \vec{a_k} \cdot \vec{a_{k+1}} &= r^2 \cos(2\pi/n) \end{aligned} \tag{7}$$

where $\vec{k}$ is a unit vector perpendicular to the plane of the polygon. As a result, the equality

$$\left|\sum_{k=1}^{n-1}(\vec{a_k} \times \vec{a_{k+1}})\right| = \left|\sum_{k=1}^{n-1}(\vec{a_k} \cdot \vec{a_{k+1}})\right| \tag{8}$$

in the statement of the problem becomes

$$(n-1)\sin(2\pi/n) = (n-1)\cos(2\pi/n) \tag{9}$$

or equivalently,

$$\tan(2\pi/n) = 1 \tag{10}$$

We are given $n > 2$. The values $n = 3, 4$ clearly do not satisfy this equation since $\tan(2\pi/3)$ is negative while $\tan(\pi/2)$ is undefined. For $n > 4$, $2\pi/n$ is an acute angle. But there is only one acute angle, viz. $\pi/4$, whose tangent is 1. So, we get

$$\frac{2\pi}{n} = \frac{\pi}{4} \tag{11}$$

which gives $n = 8$ as the *only* value of n. The words 'minimum value of n' in the statement are put either carelessly or intentionally to confuse the candidate and mar the cute beauty of the problem, which is a good combination of elementary vectors and trigonometry. Again, this question would be more appropriate for an integer answer type.

Careless of the paper-setters is also evident in (R). As the point $P(h, 1)$ lies on the ellipse

$$\frac{x^2}{6} + \frac{y^2}{3} = 1 \tag{12}$$

we have

$$h^2 = 6(1 - \frac{1}{3}) = 4 \tag{13}$$

whence $h = \pm 2$. As -2 does not appear in List - II, the correct answer has to be 2 without doing any further work. However, for an honest solution, the data is a twisted way of saying that the tangent to the ellipse is parallel to the line $x + y = 1$ and hence has slope -1. Since the equation of the tangent at the point $(h, 1)$ is

$$\frac{xh}{6} + \frac{y}{3} = 6 \tag{14}$$

and hence its slope is $-dsfrac3h6 = -\frac{h}{2}$. Equating this with -1 gives $h = 2$. Actually, the second coordinate of the point P need not have been given. If P were (h, k), then instead of (14), the equation of the tangent at P would ahve been

$$\frac{xh}{6} + \frac{yk}{3} = 1 \tag{15}$$

i.e. $h + 2k = 3$. Solving this simulateneously with

$$\frac{h^2}{6} + \frac{k^2}{3} = 6 \tag{16}$$

would give P as either $(2, 1)$ or $(-2, -1)$. This is consistent with the fact that for any line, there are two points on the ellipse at which the tangents are parallel to the line. Apparently, by giving $k = 1$, the paper-setters have tried to make life simple for the candidate. But they have made it so simple that he does not even have to know the equation of the tangent!

In the last part (S), the L.H.S., say E of the given equation is

$$\begin{aligned} E &= \tan^{-1}\left(\frac{1}{2x+1}\right) + \tan^{-1}\left(\frac{1}{4x+1}\right) \\ &= \tan^{-1}\left(\frac{\frac{1}{2x+1} + \frac{1}{4x+1}}{1 - \frac{1}{(2x+1)(4x+1)}}\right) \\ &= \tan^{-1}\left(\frac{6x+2}{8x^2+6x}\right) \\ &= \tan^{-1}\left(\frac{3x+1}{4x^2+3x}\right) \qquad (17) \end{aligned}$$

Hence the equation to be solved reduces to

$$x^2(3x+1) = 2(4x^2+3x) \tag{18}$$

i.e.

$$x(3x^2-7x-6) = 0 \tag{19}$$

whose solutions are $x = 0$, $x = 3$ and $x = -\frac{2}{3}$. Only one of the solutions is 1. Actually, since we are not interested in the solutions *per se*, we need not even solve the quadratic. The product of its roots is $-6/7$ which is negative. So exactly one of its roots is positive and the other negative. Since the actual solutions are not to found, the problem could have been made more interesting by changing the numerical data so that the resulting equation in x would be a cubic whose roots are not easy to find and so their signs have to be determined using calculus.

Q.59 Let $f_1 : I\!\!R \longrightarrow I\!\!R$, $f_2 : [0,\infty) \longrightarrow I\!\!R$, $f_3 : I\!\!R \longrightarrow I\!\!R$ and $f_4 : I\!\!R \longrightarrow [0,\infty)$ be defined by $f_1(x) = \begin{cases} |x| & \text{if } x < 0 \\ e^x & \text{if } x \geq 0 \end{cases}$; $f_2(x) = x^2$;
$f_3(x) = \begin{cases} \sin x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$ and $f_4(x) = \begin{cases} f_2(f_1(x)) & \text{if } x < 0 \\ f_2(f_1(x)) - 1 & \text{if } x \geq 0 \end{cases}$.

List I		**List II**	
(P)	f_4 is	(1)	onto but not one-one
(Q)	f_3 is	(2)	neither continuous nor one-one
(R)	$f_2 \circ f_1$ is	(3)	differentiable but not one-one
(S)	f_2 is	(4)	continuous and one-one

Answer and Comments: (P,1), (Q,3), (R,2), (S,4).

The definitions of f_1, f_2, f_3 are directly in terms of x. But f_4 is defined in terms of f_1 and f_2. So, let us first express it directly in terms of x. It comes out to be

$$f_4(x) = \begin{cases} x^2 & \text{if } x < 0 \\ e^{2x} - 1 & \text{if } x \geq 0 \end{cases} \tag{1}$$

It is gratifying to note that f_4 takes only non-negative values as it should, because to begin with f_4 is given to be from $I\!R$ to $[0, \infty)$. Clearly f_4 maps both $(-\infty, 0]$ and $[0, \infty)$ to $[0, \infty)$. So it is onto but not one-one. This shows that (P) matches with 1. Since the composite $f_2 \circ f_1$ also appears in (R), let us tackle it before (Q). We modify the last equation and get

$$(f_2 \circ f_1)(x) = \begin{cases} x^2 & \text{if } x < 0 \\ e^{2x} & \text{if } x \geq 0 \end{cases} \tag{2}$$

The graph of $f_2 \circ f_1$ is obtained from that of f_4 by pulling its right half (i.e. its portion over $[0, \infty)$ by 1 unit. This creates a discontinuity at 0. The right half now has range $[-1, \infty)$. Since this includes $[0, \infty)$ which is the range of the left half, we see that $f_2 \circ f_1$ is not one-one. So, (R) matches with (2).

The remaining two parts are more straightforward since they deal with single functions that are defined directly in terms of x. The sine function is not one-one. But it is differentiable on $(-\infty, 0]$ and the left handed derivative at 0 is the same as that of $\sin x$ and hence equals $\cos 0$, i.e. 1. The function x is differentiable on $[0, \infty)$ and its right handed derivative 0 (in fact, at every point) is 1. This makes f_3 differentiable everywhere but not one-one. So (Q) matches with (3).

Finally f_2 is the squaring function which is continuous everywhere. Also on its domain, viz. on the set $[0, \infty)$, it is one-one. So (S) matches with (4).

A simple, but highly arbitrary question. There is no common theme to the various parts. When too many functions with subscripted notations appear in a problem, even a good candidate is likely to mistake one of them in a hurry. And because of the severe marking scheme, he pays a heavy price of 4 points even though his reasoning is correct otherwise.

Q.60 Let $z_k = \cos\left(\frac{2\pi}{10}\right) + i \sin\left(\frac{2k\pi}{10}\right), \; ; k \in \{1, 2, , \ldots, 9\}$.

Match the entries in List I with those in List II.

List I	List II
(P) For each z_k there exists z_j such that $z_k.z_j = 1$.	(1) True
(Q) There exists a $k \in \{1, 2, \ldots, 9\}$ such that $z_1 \cdot z = z_k$ has no solution z in the set of complex numbers.	(2) False
(R) $\dfrac{\lvert 1-z_1\rvert\lvert 1-z_2\rvert \ldots \lvert 1-z_9\rvert}{10}$ equals	(3) 1
(S) $1 - \sum_{k=1}^{9} \cos\left(\frac{2k\pi}{10}\right)$ equals	(4) 2

Answer and Comments: (P,1), (Q,2), (R,3), (S,4).

This question differs from the last three ones in that there is a common theme to all the four parts. The numbers z_k given in the preamble are what are called the complex 10-th roots of unity, because their tenth powers all equal 1. Numerous identities are known about them (see the first seven comments of Chapter 7). And those who are familiar with these identities will undoubtedly have an easier time in this problem. But we shall give a solution which is self-sufficient.

The first two parts actually do not require these identities. They merely require that if we extend the definition of z_k to all integers k (not just those between 1 and 9), then for every integer k we have

$$z_k = e^{2\pi k/10} \tag{1}$$

Note that $z_{10} = e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$. The laws of indices (or De'Moivre's laws) then imply tht for any two integers k and j,

$$z_k \cdot z_j = z_{k+j} \tag{2}$$

So, if $k\{1, 2, \ldots, 9\}$, then by taking $j = 10 - k$ we see that j also lies in $\{1, 2, \ldots, 9\}$ and moreover

$$z_k \cdot z_j = z_{10} = 1 \tag{3}$$

So (P) is a true statement. It also says that $z_j = z_k^{-1}$. Therefore the equation multiplying the equation $z_1 \cdot z = z_k$ by z_9, we have $(z_9 \cdot z_1) \cdot z =$

$z_9 \cdot z_k$, i.e. $z = z_9 \cdot z_k = z_{k+9}$ is a solution of the equation. As this holds for every k, (Q) is false.

For the remaining two parts, we use the fact that $z_1, z_2, \ldots, z_{10}$ are the distinct roots of the polynomial $z^{10} - 1 = 0$. Out of these z_{10} equals 1. So we can factorise $z^{10} - 1$ as

$$z^{10} - 1 = (z-1)(z-z_1)(z-z_2)\ldots(z-z_9) \tag{4}$$

Dividing by $z - 1$ we get

$$1 + z + z^2 + \ldots + z^9 = (z-z_1)(z-z_2)\ldots(z-z_9) \tag{5}$$

Although in dividing by $z - 1$ we need $z \neq 1$, (5) is an equality of two polynomials in z and hence is also true for $z = 1$. So putting $z = 1$, we have

$$10 = (1-z_1)(1-z_2)\ldots(1-z_9) \tag{6}$$

(Another way to arrive at this is to take the derivative of both the sides of (4) at 1. The complex derivatives obey the same rules as the real ones do.) By taking absolute values of both the sides we get that the ratio in (R) is 1.

Finally, we note that $\cos\left(\frac{2k\pi}{10}\right)$ is the real part of z_k, for $k = 1, 2, \ldots, 9$ and hence the sum in (S) is the real part of $z_1 + z_2 + \ldots + z_9$. To evaluate it, we need an expression for this sum. This is provided by the the general formula for the sum of the roots of a polynomial. We apply it to (4) and get

$$1 + z_1 + z_2 + \ldots + z_9 = 0 \tag{7}$$

whence

$$1 - (z_1 + z_2 + \ldots + z_9) = 2 \tag{8}$$

Equating the real parts of the two sides, we get

$$1 - \sum_{k=1}^{9} \cos\left(\frac{2\pi k}{10}\right) = 2 \tag{9}$$

A fairly simple problem for those who know the elementary properties of of the complex roots of unity.

CONCLUDING REMARKS

In the commentary to JEE (Advanced) 2013, a hope was expressed that unlike in the past, the candidates appearing for JEE (Advanced) have already undergone a successful testing with the same syllabus and that this fact would be reflected in the design of questions. Unfortunately this hope has been belied. As a result, some utterly trivial questions, which hardly deserve a place in the advanced test have been asked. By way of examples, we mention Q.47, 48, 52 and Q.56 of Paper -1. There are also questions where a sneak answer is possible because of the carelessness on the part of the paper-setters. Q.41 in Paper - 2 is one such.

This is not to say that there are no good problems. We especially mention Q. 51 of Paper 1 which reduces to the problem of finding the partitions of the integer 5 and Q.55 where a certain scalar triple product is needed in writing down the equations but its actual value is not needed in solving them! Q. 58 in Paper -1 stands out as a rare example of a problem which punishes unscrupulous students. Q.60 in Paper - 1 is a good problem testing elementary coordinate geometry. Q. 42 in Paper - 2 about the solution of a triangle is an excellent example of a question where the alternatives have been given in such a way that an intelligent candidate would be led to the appropriate trigonometric formula. Q.45 in Paper - 2 is an unusual question about complex roots of a real polynomial. Questions 47 and 58(P) in Paper -2 are examples where a judicious substitution simplifies the work considerably. Unfortunately, even after striking this idea, neither of these two problems can be done in the time allocated on a proportionate scale.

There are some questions where the knowledge of certain things which are at the JEE level but are not explicitly mentioned in the syllabus will give some candidates an unfair advantage. We mention the principle of inclusion and exclusion in Q.43, the hyperbolic sines and cosines in Q.46 and complex roots of unity in in Q.60, all in Paper - 2. To avoid such inequities, it would be better to include these topics explicitly in the JEE syllabus.

As in the past, the so-called ‘comprehension type’ questions in paper - 2 are quite laughable. Comprehension ought to mean the ability to read some piece of mathematical text and to answer questions regarding what it is all about, rather than the ability to solve problems. If the latter is the intended meaning of ‘comprehension’ then every question in mathematics could qualify as a comprehension type question, because in order to solve a problem, you have to comprehend it in the first place. A sample of what could be a good comprehension question is given in the comments on Q.51 and 52 in Paper - 2.

Another grossly unfair feature carried over from the past is the last four matching type questions in Paper - 2. Each of these questions involves four separate and often unrelated parts. The candidate gets 3 points if all the four parts are done correctly. Even a single mistake earns him one negative point and all his good work in the other three parts goes down the drain. It is doubtful if any good candidates get in by scoring on these matching type questions. It is time to conduct a statistical survey on this. (It is also high time to start the practice of time-testing of a question paper by asking one of the paper-setters to stay away completely from the paper-setting and then measuring how long he takes to answer the questions designed by others, down to the last detail. That will give a realistic idea of whether the time allocated is reasonable.)

As in 2013, there is considerable duplication. There are too many questions on functions defined by integrals. Also all the questions about vectors are about their dot and cross products. The trigonometric equations and inverse trigonometric functions have also got a disproportionately large representation. Several questions ask whether a given function is one-one and onto. All the three probability problems are of the simple type where one has to find the number of favourable cases. Both the questions on matrices involve the same idea, viz. the characterisation of the invertibility of a matrix in terms of its determinant.

The expected casualty of such duplications is the total or the near total omission of certain topics. There is no question on conditional probability. Number theory and inequalities make only a passing appearance in the solutions of Q. 54 of Paper - 1 and Q. 50 of Paper - 2 respectively. There are no binomial identities or geometric applications of vectors. There is only one question on conics (not counting questions on circles).

As in every year, there are many questions where the multiple choice format rewards an unscrupulous candidates. These have been commented on at the appropriate places. There is no point in blaming the examiners because this is an evil that can be corrected only by a policy decision to revert back to the conventional type examinations where a candidate has to justify his work.