Paper Specific Instructions

1. The examination is of 3 hours duration. There are a total of 60 questions carrying 100 marks. The entire paper is divided into three sections, **A**, **B** and **C**. All sections are compulsory. Questions in each section are of different types.

2. **Section – A** contains a total of 30 **Multiple Choice Questions (MCQ).** Each MCQ type question has four choices out of which only **one** choice is the correct answer. Questions Q.1 – Q.30 belong to this section and carry a total of 50 marks. Q.1 – Q.10 carry 1 mark each and Questions Q.11 – Q.30 carry 2 marks each.

3. **Section – B** contains a total of 10 **Multiple Select Questions (MSQ).** Each MSQ type question is similar to MCQ but with a difference that there may be **one or more than one** choice(s) that are correct out of the four given choices. The candidate gets full credit if he/she selects all the correct answers only and no wrong answers. Questions Q.31 – Q.40 belong to this section and carry 2 marks each with a total of 20 marks.

4. **Section – C** contains a total of 20 **Numerical Answer Type (NAT)** questions. For these NAT type questions, the answer is a real number which needs to be entered using the virtual keyboard on the monitor. No choices will be shown for these types of questions. Questions Q.41 – Q.60 belong to this section and carry a total of 30 marks. Q.41 – Q.50 carry 1 mark each and Questions Q.51 – Q.60 carry 2 marks each.

5. In all sections, questions not attempted will result in zero mark. In **Section – A** (MCQ), wrong answer will result in **NEGATIVE** marks. For all 1 mark questions, 1/3 marks will be deducted for each wrong answer. For all 2 marks questions, 2/3 marks will be deducted for each wrong answer. In **Section – B** (MSQ), there is **NO NEGATIVE** and **NO PARTIAL** marking provisions. There is **NO NEGATIVE** marking in **Section – C** (NAT) as well.

6. Only Virtual Scientific Calculator is allowed. Charts, graph sheets, tables, cellular phone or other electronic gadgets are **NOT** allowed in the examination hall.

7. The Scribble Pad will be provided for rough work.

Notation

$\mathbb{Z}_n$	Set of all residue classes modulo n
$X \setminus Y$	The set of elements from X which are not in Y
$\mathbb{N}$	The set of all natural numbers $1,2,3,\ldots$
$\mathbb{R}$	The set of all real numbers
S_n	Set of all permutations of the set $\{1,2,\ldots,n\}$
$GL_n(\mathbb{R})$	Set of all $n \times n$ invertible matrices with real entries
$\hat{\imath}, \hat{\jmath}, \hat{k}$	unit vectors having the directions of the positive x, y and z axes in a three dimensional rectangular coordinate system, respectively
M^T	Transpose of a matrix M

SECTION – A

MULTIPLE CHOICE QUESTIONS (MCQ)

Q. 1 – Q.10 carry one mark each.

Q.1 Consider the function $f(x,y) = 5 - 4\sin x + y^2$ for $0 < x < 2\pi$ and $y \in \mathbb{R}$. The set of critical points of $f(x,y)$ consists of

(A) a point of local maximum and a point of local minimum
(B) a point of local maximum and a saddle point
(C) a point of local maximum, a point of local minimum and a saddle point
(D) a point of local minimum and a saddle point

Q.2 Let $\varphi: \mathbb{R} \to \mathbb{R}$ be a differentiable function such that φ' is strictly increasing with $\varphi'(1) = 0$. Let α and β denote the minimum and maximum values of $\varphi(x)$ on the interval $[2,3]$, respectively. Then which one of the following is TRUE?

(A) $\beta = \varphi(3)$ (B) $\alpha = \varphi(2.5)$ (C) $\beta = \varphi(2.5)$ (D) $\alpha = \varphi(3)$

Q.3 The number of generators of the additive group $\mathbb{Z}_{36}$ is equal to

(A) 6 (B) 12 (C) 18 (D) 36

Q.4

$$\lim_{n\to\infty} \frac{\pi}{n} \sum_{k=1}^{n} \sin\left(\frac{\pi}{2} + \frac{5\pi}{2}\cdot\frac{k}{n}\right) =$$

(A) $\frac{2\pi}{5}$ (B) $\frac{5}{2}$ (C) $\frac{2}{5}$ (D) $\frac{5\pi}{2}$

Q.5 Let $f: \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. If $g(u,v) = f(u^2 - v^2)$, then

$$\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} =$$

(A) $4(u^2 - v^2)f''(u^2 - v^2)$
(B) $4(u^2 + v^2)f''(u^2 - v^2)$
(C) $2f'(u^2 - v^2) + 4(u^2 - v^2)f''(u^2 - v^2)$
(D) $2(u - v)^2 f''(u^2 - v^2)$

Q.6

$$\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx =$$

(A) $\frac{1+\cos 1}{2}$ (B) $1 - \cos 1$ (C) $1 + \cos 1$ (D) $\frac{1-\cos 1}{2}$

Q.7 Let $f_1(x), f_2(x), g_1(x), g_2(x)$ be differentiable functions on $\mathbb{R}$. Let $F(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ g_1(x) & g_2(x) \end{vmatrix}$ be the determinant of the matrix $\begin{bmatrix} f_1(x) & f_2(x) \\ g_1(x) & g_2(x) \end{bmatrix}$. Then $F'(x)$ is equal to

(A) $\begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1'(x) \\ f_2'(x) & g_2(x) \end{vmatrix}$

(B) $\begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1'(x) \\ f_2(x) & g_2'(x) \end{vmatrix}$

(C) $\begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1(x) & g_2(x) \end{vmatrix} - \begin{vmatrix} f_1(x) & g_1'(x) \\ f_2(x) & g_2'(x) \end{vmatrix}$

(D) $\begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1'(x) & g_2'(x) \end{vmatrix}$

Q.8 Let

$$f(x) = \frac{x + |x|(1+x)}{x} \sin\left(\frac{1}{x}\right), \qquad x \neq 0.$$

Write $L = \lim_{x \to 0^-} f(x)$ and $R = \lim_{x \to 0^+} f(x)$. Then which one of the following is TRUE?

(A) L exists but R does not exist

(B) L does not exist but R exists

(C) Both L and R exist

(D) Neither L nor R exists

Q.9 If $\lim_{T\to\infty} \int_0^T e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$, then

$$\lim_{T\to\infty} \int_0^T x^2 e^{-x^2}\, dx =$$

(A) $\frac{\sqrt{\pi}}{4}$ (B) $\frac{\sqrt{\pi}}{2}$ (C) $\sqrt{2\pi}$ (D) $2\sqrt{\pi}$

Q.10 If

$$f(x) = \begin{cases} 1 + x & \text{if } x < 0 \\ (1-x)(px+q) & \text{if } x \geq 0 \end{cases}$$

satisfies the assumptions of Rolle's theorem in the interval $[-1, 1]$, then the ordered pair (p, q) is

(A) $(2, -1)$ (B) $(-2, -1)$ (C) $(-2, 1)$ (D) $(2, 1)$

Q. 11 – Q. 30 carry two marks each.

Q.11 The flux of the vector field

$$\vec{F} = \left(2\pi x + \frac{2x^2y^2}{\pi}\right)\hat{\imath} + \left(2\pi xy - \frac{4y}{\pi}\right)\hat{\jmath}$$

along the outward normal, across the ellipse $x^2 + 16y^2 = 4$ is equal to

(A) $4\pi^2 - 2$ (B) $2\pi^2 - 4$ (C) $\pi^2 - 2$ (D) 2π

Q.12 Let $\mathcal{M}$ be the set of all invertible 5×5 matrices with entries 0 and 1. For each $M \in \mathcal{M}$, let $n_1(M)$ and $n_0(M)$ denote the number of 1's and 0's in M, respectively. Then

$$\min_{M \in \mathcal{M}} |n_1(M) - n_0(M)| =$$

(A) 1 (B) 3 (C) 5 (D) 15

Q.13 Let $M = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Then

$$\lim_{n \to \infty} M^n x$$

(A) does not exist (B) is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(C) is $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ (D) is $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Q.14 Let $\vec{F} = (3 + 2xy)\hat{\imath} + (x^2 - 3y^2)\hat{\jmath}$ and let L be the curve

$$\vec{r}(t) = e^t \sin t\, \hat{\imath} + e^t \cos t\, \hat{\jmath}, \qquad 0 \le t \le \pi.$$

Then

$$\int_L \vec{F} \cdot d\vec{r} =$$

(A) $e^{-3\pi} + 1$ (B) $e^{-6\pi} + 2$ (C) $e^{6\pi} + 2$ (D) $e^{3\pi} + 1$

Q.15 The line integral of the vector field

$$\vec{F} = zx\, \hat{\imath} + xy\, \hat{\jmath} + yz\, \hat{k}$$

along the boundary of the triangle with vertices (1,0,0), (0,1,0) and (0,0,1), oriented anti-clockwise, when viewed from the point (2,2,2), is

(A) $\frac{-1}{2}$ (B) -2 (C) $\frac{1}{2}$ (D) 2

Q.16 The area of the surface $z = \frac{xy}{3}$ intercepted by the cylinder $x^2 + y^2 \le 16$ lies in the interval

(A) $(20\pi, 22\pi]$ (B) $(22\pi, 24\pi]$ (C) $(24\pi, 26\pi]$ (D) $(26\pi, 28\pi]$

Q.17 For $a > 0, b > 0$, let $\vec{F} = \frac{x\hat{\jmath} - y\,\hat{\imath}}{b^2x^2+a^2y^2}$ be a planar vector field. Let

$$C = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = a^2 + b^2\}$$

be the circle oriented anti-clockwise. Then

$$\oint_C \vec{F} \cdot d\vec{r} =$$

(A) $\frac{2\pi}{ab}$ (B) 2π (C) $2\pi ab$ (D) 0

Q.18 The flux of $\vec{F} = y\,\hat{\imath} - x\hat{\jmath} + z^2\hat{k}$ along the outward normal, across the surface of the solid

$$\left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, 0 \le x \le 1,\ \ 0 \le y \le 1,\ \ 0 \le z \le \sqrt{2 - x^2 - y^2} \right\}$$

is equal to

(A) $\frac{2}{3}$ (B) $\frac{5}{3}$ (C) $\frac{8}{3}$ (D) $\frac{4}{3}$

Q.19 Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(2) = 2$ and

$$|f(x) - f(y)| \le 5(|x - y|)^{3/2}$$

for all $x \in \mathbb{R}, y \in \mathbb{R}$. Let $g(x) = x^3 f(x)$. Then $g'(2) =$

(A) 5 (B) $\frac{15}{2}$ (C) 12 (D) 24

Q.20 Let $f\colon \mathbb{R} \to [0, \infty)$ be a continuous function. Then which one of the following is NOT TRUE?

(A) There exists $x \in \mathbb{R}$ such that $f(x) = \frac{f(0)+f(1)}{2}$

(B) There exists $x \in \mathbb{R}$ such that $f(x) = \sqrt{f(-1)f(1)}$

(C) There exists $x \in \mathbb{R}$ such that $f(x) = \int_{-1}^{1} f(t)dt$

(D) There exists $x \in \mathbb{R}$ such that $f(x) = \int_{0}^{1} f(t)dt$

Q.21 The interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{1}{(-3)^{n+2}} \frac{(4x-12)^n}{n^2+1}$$

is

(A) $\frac{10}{4} \leq x < \frac{14}{4}$

(B) $\frac{9}{4} \leq x < \frac{15}{4}$

(C) $\frac{10}{4} \leq x \leq \frac{14}{4}$

(D) $\frac{9}{4} \leq x \leq \frac{15}{4}$

Q.22 Let $\mathcal{P}_3$ denote the real vector space of all polynomials with real coefficients of degree at most 3. Consider the map $T: \mathcal{P}_3 \to \mathcal{P}_3$ given by $T(p(x)) = p''(x) + p(x)$. Then

(A) T is neither one-one nor onto

(B) T is both one-one and onto

(C) T is one-one but not onto

(D) T is onto but not one-one

Q.23 Let $f(x,y) = \frac{x^2 y}{x^2+y^2}$ for $(x,y) \neq (0,0)$. Then

(A) $\frac{\partial f}{\partial x}$ and f are bounded

(B) $\frac{\partial f}{\partial x}$ is bounded and f is unbounded

(C) $\frac{\partial f}{\partial x}$ is unbounded and f is bounded

(D) $\frac{\partial f}{\partial x}$ and f are unbounded

Q.24 Let S be an infinite subset of $\mathbb{R}$ such that $S \setminus \{\alpha\}$ is compact for some $\alpha \in S$. Then which one of the following is TRUE?

(A) S is a connected set

(B) S contains no limit points

(C) S is a union of open intervals

(D) Every sequence in S has a subsequence converging to an element in S

Q.25

$$\sum_{n=1}^{\infty} \tan^{-1} \frac{2}{n^2} =$$

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$ (C) $\frac{3\pi}{4}$ (D) π

Q.26 Let $0 < a_1 < b_1$. For $n \geq 1$, define

$$a_{n+1} = \sqrt{a_n b_n} \text{ and } b_{n+1} = \frac{a_n + b_n}{2}.$$

Then which one of the following is NOT TRUE?

(A) Both $\{a_n\}$ and $\{b_n\}$ converge, but the limits are not equal

(B) Both $\{a_n\}$ and $\{b_n\}$ converge and the limits are equal

(C) $\{b_n\}$ is a decreasing sequence

(D) $\{a_n\}$ is an increasing sequence

Q.27

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{3}+\sqrt{6}} + \frac{1}{\sqrt{6}+\sqrt{9}} + \cdots + \frac{1}{\sqrt{3n}+\sqrt{3n+3}}\right) =$$

(A) $1+\sqrt{3}$ (B) $\sqrt{3}$ (C) $\frac{1}{\sqrt{3}}$ (D) $\frac{1}{1+\sqrt{3}}$

Q.28 Which one of the following is TRUE?

(A) Every sequence that has a convergent subsequence is a Cauchy sequence

(B) Every sequence that has a convergent subsequence is a bounded sequence

(C) The sequence $\{\sin n\}$ has a convergent subsequence

(D) The sequence $\left\{n \cos\frac{1}{n}\right\}$ has a convergent subsequence

Q.29 A particular integral of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^{2x}\sin x$$

is

(A) $\frac{e^{2x}}{10}(3\cos x - 2\sin x)$

(B) $-\frac{e^{2x}}{10}(3\cos x - 2\sin x)$

(C) $-\frac{e^{2x}}{5}(2\cos x + \sin x)$

(D) $\frac{e^{2x}}{5}(2\cos x - \sin x)$

Q.30 Let $y(x)$ be the solution of the differential equation

$$(xy + y + e^{-x})dx + (x + e^{-x})dy = 0$$

satisfying $y(0) = 1$. Then $y(-1)$ is equal to

(A) $\frac{e}{e-1}$ (B) $\frac{2e}{e-1}$ (C) $\frac{e}{1-e}$ (D) 0

SECTION - B

MULTIPLE SELECT QUESTIONS (MSQ)

Q. 31 – Q. 40 carry two marks each.

Q.31 For $\alpha, \beta \in \mathbb{R}$, define the map $\varphi_{\alpha,\beta}: \mathbb{R} \to \mathbb{R}$ by $\varphi_{\alpha,\beta}(x) = \alpha x + \beta$. Let

$$G = \{\varphi_{\alpha,\beta} \mid (\alpha,\beta) \in \mathbb{R}^2\}$$

For $f, g \in G$, define $g \circ f \in G$ by $(g \circ f)(x) = g(f(x))$. Then which of the following statements is/are TRUE?

(A) The binary operation $\circ$ is associative

(B) The binary operation $\circ$ is commutative

(C) For every $(\alpha,\beta) \in \mathbb{R}^2$, $\alpha \neq 0$ there exists $(a,b) \in \mathbb{R}^2$ such that $\varphi_{\alpha,\beta} \circ \varphi_{a,b} = \varphi_{1,0}$

(D) $(G, \circ)$ is a group

Q.32 The volume of the solid

$$\left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, 1 \le x \le 2, \quad 0 \le y \le \frac{2}{x}, \quad 0 \le z \le x\right\}$$

is expressible as

(A) $\int_1^2 \int_0^{2/x} \int_0^x dz\, dy\, dx$

(B) $\int_1^2 \int_0^x \int_0^{2/x} dy\, dz\, dx$

(C) $\int_0^2 \int_1^z \int_0^{2/x} dy\, dx\, dz$

(D) $\int_0^2 \int_{\max\{z,1\}}^2 \int_0^{2/x} dy\, dx\, dz$

Q.33 Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function. Then which of the following statements is/are TRUE?

(A) If f is differentiable at (0,0), then all directional derivatives of f exist at (0,0)

(B) If all directional derivatives of f exist at (0,0), then f is differentiable at (0,0)

(C) If all directional derivatives of f exist at (0,0), then f is continuous at (0,0)

(D) If the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous in a disc centered at (0,0), then f is differentiable at (0,0)

Q.34 If X and Y are $n \times n$ matrices with real entries, then which of the following is/are TRUE?

(A) If $P^{-1}XP$ is diagonal for some real invertible matrix P, then there exists a basis for $\mathbb{R}^n$ consisting of eigenvectors of X

(B) If X is diagonal with distinct diagonal entries and $XY = YX$, then Y is also diagonal

(C) If X^2 is diagonal, then X is diagonal

(D) If X is diagonal and $XY = YX$ for all Y, then $X = \lambda I$ for some $\lambda \in \mathbb{R}$

Q.35 Let G be a group of order 20 in which the conjugacy classes have sizes $1, 4, 5, 5, 5$. Then which of the following is/are TRUE?

(A) G contains a normal subgroup of order 5

(B) G contains a non-normal subgroup of order 5

(C) G contains a subgroup of order 10

(D) G contains a normal subgroup of order 4

Q.36 Let $\{x_n\}$ be a real sequence such that $7x_{n+1} = x_n^3 + 6$ for $n \geq 1$. Then which of the following statements is/are TRUE?

(A) If $x_1 = \frac{1}{2}$, then $\{x_n\}$ converges to 1

(B) If $x_1 = \frac{1}{2}$, then $\{x_n\}$ converges to 2

(C) If $x_1 = \frac{3}{2}$, then $\{x_n\}$ converges to 1

(D) If $x_1 = \frac{3}{2}$, then $\{x_n\}$ converges to -3

Q.37 Let S be the set of all rational numbers in $(0,1)$. Then which of the following statements is / are TRUE?

(A) S is a closed subset of $\mathbb{R}$

(B) S is not a closed subset of $\mathbb{R}$

(C) S is an open subset of $\mathbb{R}$

(D) Every $x \in (0,1)\setminus S$ is a limit point of S

Q.38 Let M be an $n \times n$ matrix with real entries such that $M^3 = I$. Suppose that $Mv \neq v$ for any non-zero vector v. Then which of the following statements is / are TRUE?

(A) M has real eigenvalues

(B) $M + M^{-1}$ has real eigenvalues

(C) n is divisible by 2

(D) n is divisible by 3

Q.39 Let $y(x)$ be the solution of the differential equation

$$\frac{dy}{dx} = (y-1)(y-3)$$

satisfying the condition $y(0) = 2$. Then which of the following is/are TRUE?

(A) The function $y(x)$ is not bounded above

(B) The function $y(x)$ is bounded

(C) $\lim_{x\to+\infty} y(x) = 1$

(D) $\lim_{x\to-\infty} y(x) = 3$

Q.40 Let $k, \ell \in \mathbb{R}$ be such that every solution of

$$\frac{d^2y}{dx^2} + 2k\frac{dy}{dx} + \ell y = 0$$

satisfies $\lim_{x\to\infty} y(x) = 0$. Then

(A) $3k^2 + \ell < 0$ and $k > 0$

(B) $k^2 + \ell > 0$ and $k < 0$

(C) $k^2 - \ell \leq 0$ and $k > 0$

(D) $k^2 - \ell > 0, k > 0$ and $\ell > 0$

SECTION – C
NUMERICAL ANSWER TYPE (NAT)

Q. 41 – Q. 50 carry one mark each.

Q.41 If the orthogonal trajectories of the family of ellipses $x^2 + 2y^2 = c_1,\ c_1 > 0$, are given by $y = c_2 x^\alpha,\ c_2 \in \mathbb{R}$, then $\alpha =$ __________

Q.42 Let G be a subgroup of $GL_2(\mathbb{R})$ generated by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$. Then the order of G is __________

Q.43 Consider the permutations $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 3 & 7 & 8 & 6 & 1 & 2 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 3 & 1 & 7 & 6 & 8 & 2 \end{pmatrix}$ in S_8. The number of $\eta \in S_8$ such that $\eta^{-1}\sigma\eta = \tau$ is equal to __________

Q.44 Let P be the point on the surface $z = \sqrt{x^2 + y^2}$ closest to the point $(4,2,0)$. Then the square of the distance between the origin and P is __________

Q.45

$$\left(\int_0^1 x^4(1-x)^5\,dx\right)^{-1} = \_\_\_\_\_\_\_\_$$

Q.46 Let $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Let M be the matrix whose columns are $v_1, v_2, 2v_1 - v_2, v_1 + 2v_2$ in that order. Then the number of linearly independent solutions of the homogeneous system of linear equations $Mx = 0$ is __________

Q.47 $$\frac{1}{2\pi}\left(\frac{\pi^3}{1!\,3}-\frac{\pi^5}{3!\,5}+\frac{\pi^7}{5!\,7}-\cdots+\frac{(-1)^{n-1}\pi^{2n+1}}{(2n-1)!\,(2n+1)}+\cdots\right) = \_\_\_\_\_\_\_\_$$

Q.48 Let P be a 7×7 matrix of rank 4 with real entries. Let $\boldsymbol{a} \in \mathbb{R}^7$ be a column vector. Then the rank of $P + \boldsymbol{a}\boldsymbol{a}^T$ is at least ___________

Q.49 For $x > 0$, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Then

$$\lim_{x\to 0^+} x\left(\left\lfloor\frac{1}{x}\right\rfloor + \left\lfloor\frac{2}{x}\right\rfloor + \cdots + \left\lfloor\frac{10}{x}\right\rfloor\right) = \_\_\_\_\_\_\_\_$$

Q.50 The number of subgroups of $\mathbb{Z}_7 \times \mathbb{Z}_7$ of order 7 is ___________

Q. 51 – Q. 60 carry two marks each.

Q.51 Let $y(x)$, $x > 0$ be the solution of the differential equation

$$x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 4y = 0$$

satisfying the conditions $y(1) = 1$ and $y'(1) = 0$. Then the value of $e^2y(e)$ is ___________

Q.52 Let T be the smallest positive real number such that the tangent to the helix

$$\cos t\ \hat{\imath} + \sin t\ \hat{\jmath} + \frac{t}{\sqrt{2}}\ \hat{k}$$

at $t = T$ is orthogonal to the tangent at $t = 0$. Then the line integral of $\vec{F} = x\hat{\jmath} - y\,\hat{\imath}$ along the section of the helix from $t = 0$ to $t = T$ is ___________

Q.53 Let $f(x) = \frac{\sin \pi x}{\pi \sin x}$, $x \in (0, \pi)$, and let $x_0 \in (0, \pi)$ be such that $f'(x_0) = 0$. Then

$$\left(f(x_0)\right)^2(1 + (\pi^2 - 1)\sin^2 x_0) = \_\_\_\_\_\_\_\_$$

Q.54 The maximum order of a permutation σ in the symmetric group S_{10} is ___________

Q.55 Let $a_n = \sqrt{n}$, $n \geq 1$, and let $s_n = a_1 + a_2 + \cdots + a_n$. Then

$$\lim_{n\to\infty}\left(\frac{a_n/s_n}{-\ln(1 - a_n/s_n)}\right) = \_\_\_\_\_\_\_\_$$

Q.56 For a real number x, define $\lceil x \rceil$ to be the smallest integer greater than or equal to x. Then

$$\int_0^1 \int_0^1 \int_0^1 (\lceil x \rceil + \lceil y \rceil + \lceil z \rceil)\ dx\, dy\, dz = \_\_\_\_\_\_\_\_$$

Q.57 For $x > 1$, let

$$f(x) = \int_1^x \left(\sqrt{\log t} - \frac{1}{2}\log\sqrt{t}\right) dt$$

The number of tangents to the curve $y = f(x)$ parallel to the line $x + y = 0$ is ___________

Q.58 Let $\alpha, \beta, \gamma, \delta$ be the eigenvalues of the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Then $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 =$ ___________

Q.59 The radius of convergence of the power series

$$\sum_0^\infty n!\, x^{n^2}$$

is ___________

Q.60 If

$$y(x) = \int_{\sqrt{x}}^{x} \frac{e^t}{t} dt, \quad x > 0$$

then $y'(1) =$ ___________

END OF THE QUESTION PAPER