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ENTRANCE EXAMINATION, 2013

Pre-Ph.D./Ph.D.
Mathematical Sciences

[Field of Study Code : MATP (160)]

Time Allowed : 3 hours

Maximum Marks : 70

INSTRUCTIONS FOR CANDIDATES

(i) All questions are compulsory.

(ii) For Section—A, the answers must be written in the space provided in the answer table. For Section—B, Section—C and Section—D, answers are to be written in the space given after each question. Answer written in any other place will not be evaluated. Additional pages are provided at the end for rough work.

(iii) For each question in Section—A, *exactly one* of the four choices [(a), (b), (c), (d)] is the correct answer. Each correct answer will be awarded +3 marks. Each wrong answer will be given −1 mark. If a question is not attempted, then no marks will be awarded for it.

(iv) Questions in Section—B have short answers and each question carries 2 marks.

(v) Answers to all the questions in Section—C and Section—D must be **justified with mathematical reasoning**, or else they will be considered **invalid**. Each question in Section—C carries 3 marks. The question in Section—D carries 6 marks.

(vi) In the following, the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, integers, rational numbers, real numbers and complex numbers, respectively. Subsets of $\mathbb{R}^n$ are assumed to have the usual topology unless mentioned otherwise. For $x \in \mathbb{C}$, $|x|$ denotes the absolute value of x.

(vii) The notation $|S|$ is used to denote the cardinality of a finite set S.

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SUBJECT . . . .
(Field of Study/Language)

FIELD OF STUDY CODE . . . .

NAME OF THE CANDIDATE . . . .

. . . .

REGISTRATION NO. | | | | | |

CENTRE OF EXAMINATION . . . .

. . . .

DATE. . . . .

....
(Signature of Candidate)

....
(Signature of Invigilator)

....
(Signature and Seal of Presiding Officer)

Not to be filled in by the candidate

Q. No.	Marks
B1	
B2	
Total—B	
C1	
C2	
C3	
C4	
Total—C	
D1	

Total of Section—A	
Total of Section—B	
Total of Section—C	
Total of Section—D	
Grand Total	

Answer table for Section—A

Question No.	Answer	Question No.	Answer
1.		**9.**	
2.		**10.**	
3.		**11.**	
4.		**12.**	
5.		**13.**	
6.		**14.**	
7.		**15.**	
8.		**16.**	

SECTION—A

1. Which of the following rings is a field?

   (a) $\mathbb{Z}/57\mathbb{Z}$

   (b) $(\mathbb{Z}/3\mathbb{Z})\times(\mathbb{Z}/3\mathbb{Z})$

   (c) $\mathbb{R}[x]/\langle x^2-2\rangle$

   (d) $\mathbb{Q}[x]/\langle x^2+2\rangle$

2. For a finite group G

   (a) there does not exist any group homomorphism $\varphi: G\to\mathbb{Z}$

   (b) there is a unique group homomorphism $\varphi: G\to\mathbb{Z}$

   (c) there are infinitely many group homomorphisms $\varphi: G\to\mathbb{Z}$

   (d) there are exactly $|G|$ group homomorphisms $\varphi: G\to\mathbb{Z}$

3. Let R be a subring of $\mathbb{C}$ containing $\mathbb{Q}$. Suppose $\pi, \sqrt{3}\in R$. Which of the following is ***not*** necessarily true?

   (a) $\sqrt{3}/\pi\in R$

   (b) $\pi/\sqrt{3}\in R$

   (c) $[(\pi+1)^2-(\pi-1)^2]/(\pi\sqrt{3})\in R$

   (d) $(\sqrt{3}\pi^2-7)/(\sqrt{3}+1)\in R$

4. Let X be a set and let B and C be some fixed subsets of X. If for any subset A of X, $A\subseteq C$ implies $A\subseteq B$, which of the following statements is true?

   (a) $C\subsetneq B$

   (b) $B\subsetneq C$

   (c) $C\subseteq B$

   (d) $B\subseteq C$

5. Let $f : X \to Y$ be a surjective map. Which of the following is necessarily true? (In the following, Id_S stands for the identity map on the set S)

(a) There exists $g : Y \to X$ such that $g \circ f = \mathrm{Id}_X$

(b) There exists a unique $g : Y \to X$ such that $g \circ f = \mathrm{Id}_X$

(c) There exists $g : Y \to X$ such that $f \circ g = \mathrm{Id}_Y$

(d) There exists a unique $g : Y \to X$ such that $f \circ g = \mathrm{Id}_Y$

6. Let ~ be some equivalence relation on $\mathbb{R}$. We are told that under this relation, $r \sim (r+1)$ for every $r \in \mathbb{R}$. We can now definitely conclude that

(a) the number of equivalence classes is infinite

(b) the number of equivalence classes is finite

(c) $(-\pi) \sim \pi$

(d) $(\pi - \frac{7}{2}) \sim (\pi + \frac{7}{2})$

7. Let V be a non-trivial inner product space over $\mathbb{R}$. For vectors $v, w \in V$, we say $v \sim w$ if $\langle v, w \rangle = 0$. Then the relation ~ is

(a) symmetric but neither reflexive nor transitive

(b) transitive but neither reflexive nor symmetric

(c) an equivalence relation (reflexive, symmetric and transitive)

(d) symmetric and transitive, but not reflexive

8. Let $A \in SL_3(\mathbb{R})$ be a matrix such that $Av = v$ for some $v \neq 0$ in $\mathbb{R}^3$. Which of the following statements about A is necessarily true?

(a) A is a rotation

(b) A is the identity map

(c) A is diagonalizable

(d) None of the above

9. A box contains 4 blue and 3 green balls. Two balls are drawn out together at random from the box. What is the probability that the two balls are of different colours?

(a) 5/7

(b) 4/7

(c) 3/7

(d) 2/7

10. Which of the following is a complex analytic (holomorphic) function on the complex plane $\{x + iy \mid x,\ y \in \mathbb{R},\ i = \sqrt{-1}\}$?

(a) $3(x^2 - y^2) + 2ixy$

(b) $(x^3 - 3xy^2 - 3x) - i(y^3 - 3x^2y - 3y)$

(c) $(x^3 - 3xy^2 + 3x) - i(y^3 - 3x^2y - 3y)$

(d) $(x^3 + xy^2 + 3x) + i(y^3 + x^2y + 3y)$

11. For a complex analytic (holomorphic) function f on $\mathbf{C}$, consider the following conditions :

[C1] $(\mathrm{Re}\, f)(z) > 0$

[C2] $|f(z)| \in \mathbb{Z}$ for all $z \in \mathbf{C}$

[C3] $f(z) = i$ if $z = 1 + \frac{1}{n} + i$ for all $n \in \mathbb{N}$, where $i = \sqrt{-1}$

Which of the above conditions implies/imply that f is a constant function?

(a) All of [C1], [C2] and [C3]

(b) Both [C2] and [C3], but not [C1]

(c) Only [C2]

(d) Only [C3]

12. The series $\sum_{n=0}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$ is

(a) absolutely convergent

(b) divergent

(c) conditionally convergent

(d) bounded but not convergent

13. For any pair of non-negative real numbers x, y, consider the following inequalities :

[I1] $\sqrt{x^2 + y^2} \geq \frac{1}{\sqrt{2}}(x + y)$

[I2] $\sqrt{x^2 + y^2} \leq x + y$

Which of the following is true?

(a) Only [I1] holds

(b) Only [I2] holds

(c) Neither [I1] nor [I2] holds

(d) Both [I1] and [I2] hold

14. Consider two maps $d_i : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$ defined as follows :

$$d_1((x_1, y_1), (x_2, y_2)) := |x_1 - x_2| + 3|y_1 - y_2|$$

$$d_2((x_1, y_1), (x_2, y_2)) := \frac{1}{3} \max\{|x_1 - x_2|, |y_1 - y_2|\}$$

Which of the following is true?

(a) d_1 is a metric on $\mathbb{R}^2$, but d_2 is not

(b) d_2 is a metric on $\mathbb{R}^2$, but d_1 is not

(c) Both d_1 and d_2 are metrics on $\mathbb{R}^2$

(d) Neither d_1 nor d_2 is a metric on $\mathbb{R}^2$

15. Consider the following statements about the closed interval $X = [0, 1]$:

[S1] Every infinite sequence in X has a limit point.

[S2] X has a subset which is connected but not path connected.

[S3] X has the finite intersection property.

[S4] X is a complete metric space.

Which of the above statements about X are true?

(a) Only [S1] and [S3]

(b) Only [S1], [S3] and [S4]

(c) Only [S2] and [S4]

(d) All of the above

16. Let $A_1 \supseteq A_2 \supseteq \cdots$ be a countable family of nonempty connected subsets of $\mathbf{R}^2$. Suppose $A := \bigcap_{n \geq 1} A_n$ is a nonempty set. Which of the following statements is necessarily true?

(a) A is always connected

(b) A is connected if each A_n is path connected

(c) A is connected if each A_n is closed

(d) A is connected if each A_n is compact

SECTION—B

B1. Prove or disprove :

The multiplicative groups $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ and $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ are isomorphic.

B2. Find with reason the flaw, if any, in the following sequence of arguments :

$$\frac{(-1)}{(64)} = \frac{1}{(-64)}$$

Step 1. $\Rightarrow \dfrac{\sqrt{-1}}{\sqrt{64}} = \dfrac{\sqrt{1}}{\sqrt{-64}}$, square roots taken on both sides

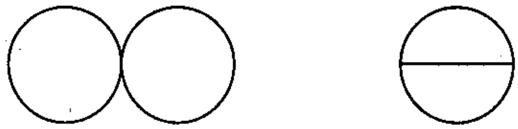
Step 2. $\Rightarrow \dfrac{i}{8} = \dfrac{1}{8i}$, $\sqrt{-1} = i$ is used

Step 3. $\Rightarrow \dfrac{i}{1} = \dfrac{1}{i}$, $\dfrac{1}{8}$ cancelled from both sides

Step 4. $\Rightarrow i^2 = 1$, multiplied by i on both sides

SECTION—C

C1. Prove that the figure of eight and the figure of theta (as shown below) are not homeomorphic as subsets of $\mathbb{R}^2$:



C2. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 f(x)\, x^n\, dx = 0$ for all $n \geq 0$. Prove that $f \equiv 0$.

C3. Let X and Y be two finite sets and $f : X \to Y$ be a map. Prove that

$$|X| = \sum_{y \in Y} |f^{-1}(y)|$$

C4. For a fixed $n \in \mathbb{N}$, let $X = \{1, 2, \cdots, n\}$. Let μ be a measure on X defined by $\mu(\{a\}) = n - a$ for every $a \in X$. Find a non-constant real-valued function f on X such that $\int_X f \, d\mu = 0$.

SECTION—D

D1. Let $E = \mathbb{R}^{[3]}[x]$ be the real vector space of real polynomials of degree less than or equal to 3, with an inner product defined by

$$\langle P, Q \rangle := \int_{-1}^{1} P(t)\, Q(t)\, dt$$

Consider the map $\sigma : E \to E$ defined by $(\sigma P)(x) = P(-x)$ for all $P \in E$. Prove that σ is a linear operator on E and that $\langle \sigma P, \sigma Q \rangle = \langle P, Q \rangle$. Find the eigenvalues and eigenvectors of σ. Is σ diagonalizable?