ENTRANCE TEST FOR ADMISSION 2001

Integrated Ph.D

Mathematical Sciences

Day & Date	:	Sunday 29th April 2001
Time	:	1.30 p.m. to 4.30 p.m.

INDIAN INSTITUTE OF SCIENCE
BANGALORE

INSTRUCTIONS

- The question paper is in two parts Part A and Part B. Part A carries 30 marks and Part B carries 70 marks.
- Part A comprises 30 multiple choice questions each carrying 1 mark. Four possible answers are provided for each question. Select the correct answer by marking (√) against (a), (b), (c) or (d) on the answer script exactly as given below.
  For example, Question: $2+2=$ Answer: (a) 0 (b) 2 (√) 4 (d) 8.
  Answer all questions from Part A.
- Part B comprises 10 questions Answer any 7 questions. Each question carries 10 marks.
- All answers must be written in the answer book and *not on the question paper.*

MATHEMATICAL SCIENCES

Part A

1. The numbers $2^{800}, 3^{600}, 5^{400}, 6^{200}$ listed in the increasing order are
   (a) $2^{800}, 3^{600}, 5^{400}, 6^{200}$
   (b) $6^{200}, 2^{800}, 3^{600}, 5^{400}$
   (c) $6^{200}, 2^{800}, 5^{400}, 3^{600}$
   (d) $2^{800}, 5^{400}, 3^{600}, 6^{200}$

2. The point $(3, 4)$ in the xy-plane is reflected w.r.t the x-axis and then rotated through 90 degrees in the clockwise direction in the plane about the origin. The final position of the point is
   (a) $(3, -4)$
   (b) $(4, -3)$
   (c) $(-3, -4)$
   (d) $(-4, -3)$.

3. The maximum value of

$$10 - \sqrt{3\cos\theta - 4\sin\theta + 9}$$

   for $0 \leq \theta \leq 2\pi$ is
   (a) 8
   (b) 7
   (c) 10
   (d) $10 - \sqrt{14}$.

4. The derivative w.r.t. x of the product

$$+ x)(1 \quad x^2)(1 + x^4)(1 + x^8) \quad \cdot (1 + x^{2^n}$$

   at $x = 0$ is
   (a) 0
   (b) 1
   (c) n
   (d) $n +$

5. If z is a complex number for which $|z-3-4i| \leq 2$ then the maximum value of $|z|$ is

(a) 2

(b) 5

(c) 7

(d) 9.

6. If $I = \int_0^1 e^x dx$, then which of the following is true?

(a) $I < 1$

(b) $1 < I < 2$

(c) $2 < I < e$

(d) $I > e$.

7. Let f be the real function defined by

$$f(x) = \begin{matrix} ax+b & \text{if } x < -1; \\ x^2+1 & \text{if } -1 \leq x \leq 1 \\ -ax+b & \text{if } x > 1, \end{matrix}$$

where a, b are real numbers. If f is continuous on the real line then the product ab is equal to

(a) 2

(b) -4

(c) -2

(d) 0.

8. A heavy ball tied to a string spins around in a circle. While the ball is spinning, the length of the string is slowly halved. The angular frequency of rotation of the ball is

a) halved

b) doubled

c) quadrupled

d) unchanged

9. Unpolarized light passes through three polarizing filters. The axis of the second one is at an angle of $+30^\circ$ with respect to the first, and the axis of the third is at an angle $+30^\circ$ with respect to the second. The fraction of the original intensity that emerges from the third polarizer is

a) 9/32

b) 3/8

c) 2/9

d) 1/8

A violin string that is 22cm long and weighs 0.8g has a fundamental frequency of 960Hz. The speed of sound in air is 320m/s. The wavelength of the sound waves (in air) emitted when the string vibrates at its fundamental frequency is

a) 22cm

b) 33cm

c) 44cm

d) 88cm

11. Two large metal spheres, A and B, are near each other. The electrostatic force between them is attractive. Of the three possibilities:

i) the two spheres are oppositely charged

ii) one sphere is charged and the other is uncharged

iii) both spheres are uncharged

a) Only case i) is possible

b) Cases i) and ii) are possible, but not iii)

c) All three cases are possible

d) It depends on the size of the spheres compared to their separation.

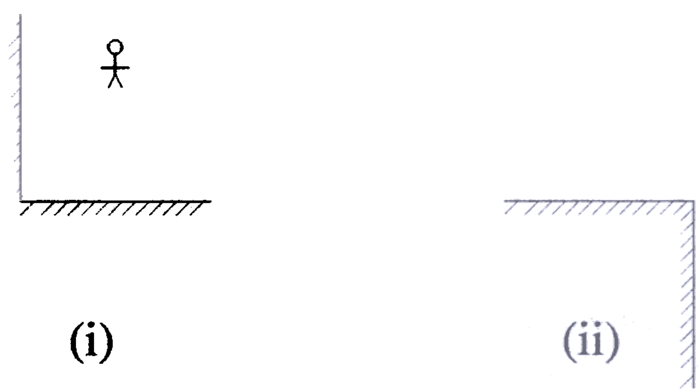

Figure 1:

12. An object is placed between two mirrors at right angles to each other as shown. How many images are formed by the mirrors in each case?

a) 3 and 2

b) 3 and 3

c) 2 and 2

d) 3 and 0

13. A resistor, inductor and a capacitor are connected in series to an ac voltage source $v(t) = V \cos[2\pi\nu t]$. The peak voltages across the three elements are V_R, V_L and V_C.

a) V_R, V_L and V_C must be less than V.

b) V_R must be less than V, but V_L and V_C need not.

c) At any instant, the voltage across the resistor and the voltage from the source must have the same sign.

d) At any instant, the voltage across the resistor must be smaller in magnitude than the voltage from the source.

14. A slab of ice at 0°C is placed in a beaker of water at 0°C. (Take the melting point of ice to be 0°C.) Ignore heat exchange with the surroundings (air, etc.).

a) Some of the ice will melt to water if there is more water.

b) Some of the ice will melt and some of the water will also freeze.

c) Both the water and the ice will remain unchanged.

d) There is not enough information to decide between these.

15. Two spheres of radius r_1 and r_2, and at temperatures T_1 and T_2, are placed in vacuum. The first sphere is a blackbody. The second sphere may absorb more heat from the first than it radiates out if

a) $T_1 = T_2$, but r_1 is sufficiently large compared to r_2.

b) $T_1 = T_2$, but the second sphere is painted, with a colour matching the peak of the radiation from the first.

c) $T_1 > T_2$.

d) None of the above.

16. The pH of 10^{-10} molar solution of HCl is:

a) 10

b) 7

c) 4

d) 1

17. The molecular weight of $MgCl_2$ determined from elevation of boiling point experiment is (atomic masses of Mg and Cl are 24 and 35.5 respectively):

a) 47.5

b) 95.0

c) 63.4

d) 31.7

18. In a monoatomic body-centered cubic lattice with lattice constant a, the closest distance of approach between the atoms is:

a) a

b) $a\sqrt{2}$

c) $a\sqrt{3}/2$

d) $a/2$

19. The maximum number of electrons in an atom that can possess a principal quantum number of 4 is:

a) 8

b) 14

c) 18

d) 32

20. The empirical formula of the inorganic compound whose molecular structure most resembles that of benzene:

a) HBS

b) $PNCl_2$

c) SN

d) BNH_2

21 Aldol condensation is carried out under:

a) acidic conditions

b) basic conditions

c) neutral conditions

d) pyrolytic conditions

22. Enolisation involves:

a) resonance

b) complexation

c) tautomerisation

d) aromatisation

23. In DNA, the G-C base pairs are stronger than A-T base pairs because of

a) their partial double bond character

b) the presence of an additional hydrogen bond

c) hydrophobic effect

d) their covalent nature

24. Erythrocytes when placed in a hypotonic solution will
   a) shrink
   b) burst
   c) first shrink and then burst
   d) not show any effect

25. A protein has 3 glutamic acid and 4 lysine residues. It has no other charged residues. The pI of the protein is likely to be
   a) 3
   b) 4
   c) 7
   d) 8

26. The sequence of which of the following is used to establish phylogenetic relationships between organisms ?
   a) DNA Polymerase protein
   b) Actin gene
   c) Ribosomal gene
   d) Hexokinase gene

27. PKU is one of the best known hereditary disorders in amino acid metabolism. The defect is attributed to a lesion in one of the following enzymatic activities
   a) Phenylalanine ammonia lyase
   b) Phenylalanine hydroxylase
   c) Tyrosine hydroxylase
   d) Phenylalanine transaminase

28. Which of the following have the highest basal metabolic rate ?
   a) Blue Whale
   b) Cheetah
   c) Humming Bird
   d) Eagle

29. The place where an organism lives is known as its
   a) home range
   b) biome
   c) habitat
   d) community

30. Analysis of paleoclimatological data indicate that environments during the last 100,000 years

a) have essentially the same as they are now

b) have been consistently warming

c) have been consistently cooling

d) have fluctuated repeatedly from warm to cold

MATHEMATICAL SCIENCES

PART B

1. a) Show that the real function $f(x) = x|x|$ is differentiable everywhere on the real line.

   b) Let a, b be two non-zero complex numbers. If $az + b\bar{z} = 0$ represents a straight line in the plane then show that $|a| = |b|$. (Here $z = x + iy$ in the plane.)

2. Let a, b, c be three complex numbers such that

$$a^2 + b^2 + c^2 = a^3 + b^3 + c^3 = a^4 + b^4 + c^4 = 0.$$

   Show that $a = b = c = 0$.

3. a) Let ρ be a non-trivial relation on a non-empty set A (i.e., there exist $a, b \in A$ such that $a \rho b$ holds). If ρ is symmetric and transitive then show that there exists a non-empty set $B \subseteq A$ such that ρ is an equivalence relation on B.

   b) Let A be a non-empty finite set. If $f : A \to A$ is a bijection (i.e., one-to-one and onto) and $a \in A$ then show that there exists $n \geq 1$ such that $f^{(n)}(a) = a$. [Here $f^{(1)} \equiv f$ and for $n \geq 2$, $f^{(n)}(x) = f(f^{(n-1)}(x))$.]

4. Let '$*$' be a binary operation on a non-empty set S. If

$$x * y = y^n * x,$$

   for some positive integer n (≥ 2), then show that

   (i) $x^n = x^{n^2}$ for all $x \in S$ and

   (ii) $x * y = y * x$ for all $x, y \in S$.

5. Let

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$$

   be two real matrices. For $1 \leq i \leq 3$, let P_i be the plane given by $a_i x + b_i y + c_i z + d_i = 0$. Show that $P_1 \cap P_2 \cap P_3$ is a line if and only if A and B have the same rank and this common rank is equal to 2.

6. Let F, F' be the foci of an ellipse and P a point on the ellipse. Show that PF and PF' are equally inclined to the tangent at P to the ellipse.

7. Evaluate

$$\int_0^1 x f''(x) dx,$$

if

$$f(x) \quad \int_0^x t e^{-t^2} dt.$$

8. Find the solution of the system

$$\frac{dy_1}{dt} \quad 1 - \frac{1}{y_2}$$

$$\frac{dy_2}{dt} - \frac{1}{y_1 - t},$$

$$y_1(0) = y_2(0) = 1$$

9. Compute approximately the value of π using Simpson's rule (with four equal subintervals of the interval $[0, 1]$) on the integral

$$\int_0^1 \frac{dx}{1 + x^2}.$$

10. a) Find the number of positive integers n such that $1 \le n \le 2000$ and $\gcd(2000, n) = 40$.

 b) Find the number of positive integers m such that $1 \le m \le 2000$ and $\operatorname{lcm}(250, m) = 2000$.